Lecture Notes

Introduction to Stochastic Analysis

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1. Introduction to Pathwise Itô-Calculus

1.1. Preparation

Definition 1.1.1. \( X : [0, \infty] \to \mathbb{R} \) has bounded variation, if for all \( t \geq 0 \)

\[
\text{var}_t X := \sup_{\tau_n} \sum_{t_i^{(n)} \leq t} \left| X \left( t_{i+1}^{(n)} \right) - X \left( t_i^{(n)} \right) \right| < \infty, \tag{1.1.1}
\]

where \( \tau_n \) is a partition \( 0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_N^{(n)} < \infty \).

Notation: Since in Stochastics a process \( X \) also depends on \( \omega \), we write \( X_t \) and \( X_t(\omega) \) instead of \( X(t) \) and \( X(t)(\omega) \) respectively.

As a reference to this field see [dB03]. Note that \( \text{var}_t X \) defined by (1.1.1) is independent of the choice of \( (\tau_n)_{n \in \mathbb{N}} \).

Definition 1.1.2. Suppose \( X \) to be right-continuous and of bounded variation. Let \( f \in C(\mathbb{R}) \) and \( (\tau_n)_{n \in \mathbb{N}} \) be a sequence of partitions, whose mesh

\[
|\tau_n| := \sup_{1 \leq i \leq N_n} \left( t_{i+1}^{(n)} - t_i^{(n)} \right)
\]

converges to zero as \( n \to \infty \) and \( t_{N_n} \to \infty \). Then, since \( X \) is of bounded variation, there exists the Lebesgue-Stieltjes-Integral defined by

\[
\int_0^t f_s \, dX_s := \lim_{n \to \infty} \sum_{t_i^{(n)} \leq s \leq t_{i+1}^{(n)}} f_{t_i^{(n)}} \cdot \left( X_{s}^{(n)} - X_{t_i^{(n)}} \right). \tag{1.1.2}
\]

Remark 1.1.3. Note that for continuous \( X \) this definition is independent of the choice of \( (\tau_n)_{n \in \mathbb{N}} \):

\[
\left| \lim_{n \to \infty} \sum_{t_i^{(n)} \leq s} f_{t_i^{(n)}} \left( X_{s}^{(n)} - X_{t_i^{(n)}} \right) - \lim_{n \to \infty} \sum_{s_i^{(n)} \leq s} f_{s_i^{(n)}} \left( X_{s}^{(n)} - X_{s_i^{(n)}} \right) \right|
\]

\[
\leq \sup_{s \in [0,t]} |f_s| \lim_{n \to \infty} \sum_{t_i^{(n)} \leq s} \sum_{s_i^{(n)} \leq t} \left| (X_{s}^{(n)} - X_{t_i^{(n)}}) - (X_{s_i^{(n)}} - X_{s_i^{(n)}}) \right|
\]

\[
= \sup_{s \in [0,t]} |f_s| \cdot |(X_t - X_0) - (X_t - X_0)| = 0.
\]
1. Introduction to Pathwise Itô-Calculus

1.1.1. Quadratic Variation of Brownian Motion

We know (cf. [Röc06]) that a typical path of Brownian motion $X$ on $\mathbb{R}^1$ is of unbounded variation, since for its quadratic variation $\langle X \rangle_t = t$ (see 1.1.4(i) below). Nonetheless, one wants to define

$$\int_0^t f_s \, dX_s$$

for a typical path of Brownian motion $X$. More generally, we want to do this for every continuous path with continuous quadratic variation $t \mapsto \langle X \rangle_t$.

$$(X_t)_{t \geq 0}$$

is a (continuous) $\mathbb{R}$-valued Brownian motion on $(\Omega, \mathcal{F}, P)$ if

i. The increments $X_t - X_s$ are independent and $N(0, t - s)$ distributed ($t > s$).

ii. $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$.

**Theorem 1.1.4.** Let $(X_t)_{t \geq 0}$ be a (continuous) Brownian motion and $(\tau_n)_{n \in \mathbb{N}}$ a sequence of subdivisions with $|\tau_n| \xrightarrow[n \to \infty]{} 0$, $\tau_n \subset \tau_{n+1}$, $t^{(n)}_N \xrightarrow[n \to \infty]{} \infty$. Then, for all $t \geq 0$

i. $\sum_{i(n) \leq \tau_n} \left( X_{i+1} - X_i \right)^2 \xrightarrow[n \to \infty]{} t$ P-a.s.

(Here the zero sets could depend on $t$.)

ii. Moreover,

$$P\left( \sum_{i(n) \leq \tau_n} \left( X_{i+1} - X_i \right)^2 \xrightarrow[n \to \infty]{} t, \forall t \geq 0 \right) = 1.$$

**Proof.**

i. See [Röc06, Part II, Chapter III, Proposition 4.3].

ii. Exercise (“sandwich argument”, notice that zero set in (i) depends on $t$).

1.2. Quadratic Variation and Itô’s Formula

Fix a continuous and real-valued function $t \mapsto X_t$ on $[0, \infty]$ (in short $(X_t)_{t \geq 0}$) with an existing sequence of subdivisions $(\tau_n)$ with $|\tau_n| \xrightarrow[n \to \infty]{} 0$ and $t^{(n)}_N \xrightarrow[n \to \infty]{} \infty$ such that the quadratic variation (along $(\tau_n)$)

$$\langle X \rangle_t := \lim_{n \to \infty} \sum_{i(n) \leq \tau_n} \left( X_{i+1} - X_i \right)^2, \quad t \geq 0, \quad \text{(1.2.3)}$$

exists for all $t \geq 0$ and such that $t \mapsto \langle X \rangle_t$ is continuous on $[0, \infty)$. Note that $\langle X \rangle_t$ in (1.2.3) could depend upon the choice of $(\tau_n)_{n \geq 1}$ in contrary to the limit in (1.1.2) (cf. [RY99, (2.3)-(2.5) p. 27/28]). By definition it is obvious that $t \mapsto \langle X \rangle_t$ is increasing.

**Example:** For a typical path of a Brownian motion $X$ and $\tau_n \subset \tau_{n+1}$ we have $\langle X \rangle_t = t$ for all $t$. 

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1.2. Quadratic Variation and Itô’s Formula

Remark 1.2.1.  

i. If \((X_t)_{t \geq 0}\) is of bounded variation, then

\[
\sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right)^2 \leq \max_{t_i^{(n)} \leq t} \left| X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right| \sum_{t_i^{(n)} \leq t} \left| X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right| \to 0, \\
\text{uniformly for large } n
\]

hence, \(\langle X \rangle \equiv 0\).

Therefore, \(\langle X \rangle \neq 0\) implies that \((X_t)_{t \geq 0}\) is not of bounded variation and the Lebesgue-Stieltjes-Integral cannot be defined in the usual way.

ii. If \(t \mapsto \langle X \rangle_t\) is increasing, continuous and \(\langle X \rangle_0 = 0\), hence, \(t \mapsto \langle X \rangle_t\) is a distribution function of a measure \(\mu\) (i.e. \(d\mu = d\langle X \rangle_t\) on \([0, \infty), B([0, \infty])\)), then (1.2.3) is equivalent to: The distribution function of

\[
\mu_n := \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right)^2 \delta_{t_i}
\]

converges pointwise to

\[
F(t) := \mu(]-\infty, t]) = \int 1_{]-\infty, t]} \, d\mu.
\]

But, since for continuous \(\langle X \rangle_t\)

\[
\mu(\{t\}) = \lim_{n \to \infty} \mu(]-\infty, t]) - \mu(]-\infty, t - \frac{1}{n}] = \lim_{n \to \infty} \left( \langle X \rangle_t - \langle X \rangle_{t - \frac{1}{n}} \right) = 0,
\]

we have that \(\mu_n \to \mu\) weakly by Portemanteau.

iii. Note that if \(X\) is an increasing function, then \(X\) is always of bounded variation:

\[
\text{var}_t X = \sup_{\tau_n} \sum_{t_i^{(n)} \leq t} \left| X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right| = \sup_{\tau_n} \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right) = X_t - X_0 < \infty.
\]

Therefore, we can define the Itô-integral with respect to \(\langle X \rangle\) as a Lebesgue-Stieltjes-Integral.

Lemma 1.2.2 (Calculating integrals with respect to \(d\langle X \rangle_s\)). Let \(g \in C([0, \infty))\). Then

\[
\sum_{t_i^{(n)} \leq t} g(t_i) \left( X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}} \right)^2 \to \int_0^t g(s) \, d\langle X \rangle_s.
\]

Proof. The left hand side is equal to \(\int g1_{[0,t]} \, d\mu_n\), whereas the right hand side equals \(\int g1_{[0,t]} \, d\mu\). But the integrand \(g1_{[0,t]}\) is \(\mu\)-a.e. continuous and bounded. Hence, convergence follows by the Portemanteau theorem and Remark 1.2.1 (ii).

Theorem 1.2.3 (Pathwise Itô-formula): Let \(F \in C^2(\mathbb{R})\). Then for all \(t \geq 0\) the Itô-Formula is given by

\[
F(X_t) - F(X_0) = \int_0^t F'(X_s) \, dX_s + \frac{1}{2} \int_0^t F''(X_s) \, d\langle X \rangle_s,
\]

(1.2.4)
1. Introduction to Pathwise Itô-Calculus

where

$$\int_0^t F'(X_s) \, dX_s := \lim_{n \to \infty} \sum_{\frac{i}{n} \leq t, \frac{i}{n} < t} F' \left( X_{\frac{i}{n}} \right) \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right).$$

\(\int_0^t F'(X_s) \, dX_s\) is called the (pathwise) Itô-Integral and depends on \((\tau_n)_{n \in \mathbb{N}}\).

**Proof.** Consider \((\tau_n)_{n \in \mathbb{N}}\) such that \(\langle X \rangle_t\) exists along \((\tau_n)_{n \in \mathbb{N}}\). We apply the Taylor formula on \(F\). Hence, for all \(n \in \mathbb{N}\) there exist \(\theta_i^{(n)} \in [0,1]\) such that

$$\sum_{\frac{i}{n} \leq \tau_n} F \left( X_{\frac{i}{n}} \right) - F \left( X_{\frac{i}{n}} \right) \xrightarrow{n \to \infty} F(X_t) - F(X_0)$$

$$= \sum_{\frac{i}{n} \leq \tau_n} F' \left( X_{\frac{i}{n}} \right) \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right) + \sum_{\frac{i}{n} \leq \tau_n} \frac{1}{2} F'' \left( X_{\frac{i}{n}} \right) \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right)^2$$

$$+ \sum_{\frac{i}{n} \leq \tau_n} \frac{1}{2} \left[ F'' \left( X_{\frac{i}{n}} + \theta_i^{(n)} \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right) \right) - F'' \left( X_{\frac{i}{n}} \right) \right] \cdot \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right)^2$$

$$=: S$$

Since \(F''\) is locally uniformly continuous and \(X\) is uniformly continuous,

$$F'' \left( X_{\frac{i}{n}} + \theta_i^{(n)} \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right) \right) - F'' \left( X_{\frac{i}{n}} \right) < \varepsilon$$

holds uniformly in \(i\) for \(n\) big enough. Therefore,

$$S \leq \varepsilon \sum_{\frac{i}{n} \leq \tau_n} \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right)^2 \xrightarrow{n \to \infty} 0,$$

which finishes the proof. \(\square\)

**Remark 1.2.4.** If \(\langle X \rangle_t \equiv 0\) (e.g. \((X_t)_{t \geq 0}\) is of bounded variation), then we are in the classical case:

$$F(X_t) - F(X_0) = \int_0^t F'(X_s) \, dX_s$$

is an ordinary Lebesgue-Stieltjes-Integral. (If \(X_t = t\), it is the “Fundamental theorem of calculus”). We introduce a different notation to (1.2.4)

$$dF(X) = F'(X) \, dX + \frac{1}{2} F''(X) \, d\langle X \rangle$$

(1.2.5)

in contrast to the classical case (i.e. \(\langle X \rangle_t \equiv 0\), where

$$dF(X) = F'(X) \, dX.$$  

(1.2.6)
Example 1.2.5. Consider the differential equation

\[ dX^n = nX^{n-1} \, dX \quad \text{for } n \in \mathbb{N} \text{ fixed.} \]

If \( (X) = 0 \), then a solution is \( X^n \).

If \( (X) \neq 0 \), this is not a solution, since by Itô

\[ dX^n = nX^{n-1} \, dX + \frac{1}{2} n(n-1)X^{n-2} \, d\langle X \rangle. \]

We would like to find a function \( h_n : \mathbb{R} \to \mathbb{R} \) such that

\[ dh_n(X) = nh_{n-1}(X) \, dX \]

in the general case \( (X) \neq 0 \). Later (cf. Example 1.3.12(ii) below), we shall see that the \( n \)-th Hermite polynomial \( h_n \) will provide a solution to this problem.

Remark 1.2.6. We know that, if \( f \in C^1(\mathbb{R}) \), then the Itô-integral

\[ \int_0^t f(X_s) \, dX_s, \quad t \geq 0, \]

is (well-)defined. (Simply take \( F \) as a primitive of \( f \), i.e. \( F' = f \), and apply Itô formula.)

Definition 1.2.7 (\( \alpha \)-Integral). More generally we define for \( \alpha \in [0, 1] \) and \( f \in C^1(\mathbb{R}) \)

\[ \alpha-\int_0^t f(X_s) \, dX_s := \lim_{n \to \infty} \sum_{\substack{t_{(n)}^i \leq t \leq t_{(n)}^{i+1} \in \mathbb{N} \cap t_{(n)}^i \leq t}} f(X_{t_{(n)}^i} + \alpha (X_{t_{(n)}^{i+1}} - X_{t_{(n)}^i}))) \cdot (X_{t_{(n)}^{i+1}} - X_{t_{(n)}^i}) \quad (1.2.7) \]

Claim: This limit exists and

\[ \alpha-\int_0^t f(X_s) \, dX_s = \int_0^t f(X_s) \, dX_s + \alpha \int_0^t f'(X_s) \, d\langle X \rangle_s. \quad (1.2.8) \]

Proof. Exercise (Compare “\( \alpha \)-sum” with “0-sum” (Itô-Integral) and use the mean-value-theorem for \( f \)). \( \square \)

Special cases:

\( \alpha = 0 \): “Itô-integral”

\( \alpha = 1 \): “Backward Itô-integral”.

\( \alpha = \frac{1}{2} \): “Stratonovich-Fisk-Integral”

Notation: \( \oint : = \int \ldots \circ \, dX_s := \alpha-\int \ldots \, dX_s \). Hence

\[ \oint_0^t f(X_s) \, dX_s \left( = \frac{1}{2} - \int_0^t f(X_s) \, dX_s \right) = \int_0^t f(X_s) \, dX_s + \frac{1}{2} \int_0^t f'(X_s) \, d\langle X \rangle_s \]

and we have by Itô the Stratonovich-formula

\[ F(X_t) - F(X_0) = \oint_0^t F'(X_s) \, dX_s \left( = \int_0^t F'(X_s) \, dX_s \right). \quad (1.2.9) \]

Remark 1.2.8. i. An advantage of the Itô-integral is (see Section 1.4 below) that, if \( X \) is a martingale, then, again, \( \oint f(X_s) \, dX_s \) is a martingale.

ii. In the Stratonovich-formula one only has to deal with derivatives of first order and, therefore, it can be used for manifold-valued \( X \).
1. Introduction to Pathwise Itô-Calculus

1.2.1. Supplement on the Quadratic Variation

Lemma 1.2.9.  

i. Let \( F \in C^1(\mathbb{R}) \). Then \( t \mapsto F(X_t) \) has (finite) quadratic variation (along fixed \( (\tau_n)_{n \in \mathbb{N}} \))

\[
\langle F(X) \rangle_t = \int_0^t (F'(X_s))^2 \, d\langle X \rangle_s \quad \text{(automatically continuous in } t)\]

ii. If \( M_t := X_t + A_t, t \geq 0 \), for some \( t \mapsto A_t \) continuous and \( \langle A \rangle \equiv 0 \) (again \( \langle A \rangle \) calculated along \( (\tau_n) \)), then

\[
\langle M \rangle_t = \langle X \rangle_t.
\]

iii. The Itô-integral \( t \mapsto \int_0^t f(X_s) \, dX_s =: M_t \) with \( f \in C^1(\mathbb{R}) \), has quadratic variation (along \( (\tau_n) \)) and

\[
\langle M \rangle_t = \left\langle \int_0^t f(X_s) \, dX_s \right\rangle_t = \int_0^t f(X_s)^2 \, d\langle X \rangle_s.
\]

Proof.  

i. We first apply Taylor up to order 1, then take the square on both sides and finally apply the Binomial formula to get

\[
\sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} \left( F(X_{i+1}^{(n)}) - F(X_i^{(n)}) \right)^2
\]

\[
= \sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} F'(X_i^{(n)})^2 \left( X_{i+1}^{(n)} - X_i^{(n)} \right)^2
\]

\[
\left( \sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} \int_0^t (F'(X))^2 \, d\langle X \rangle_s \right. \text{ by Lemma 1.2.2}
\]

\[
+ \sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} \left( F'(X_{i+1}^{(n)}) + \theta_{i+1}^{(n)} \left( X_{i+1}^{(n)} - X_i^{(n)} \right) \right) \left( F'(X_i^{(n)}) \right) \left( X_{i+1}^{(n)} - X_i^{(n)} \right)^2
\]

\[
< \varepsilon \text{ for large } n, \text{ since } F' \text{ is continuous on the compact set } \{ X_{i+1}^{(n)} \mid i \in \mathbb{N}, \varepsilon \} \text{ uniformly in } i
\]

\[
+ 2 \sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} F'(X_i^{(n)}) \left( X_{i+1}^{(n)} - X_i^{(n)} \right)
\]

\[
\cdot \sum_{\substack{i, n \in \mathbb{N} \\text{ such that } \frac{n}{i} \leq t}} \left( F'(X_{i+1}^{(n)}) + \theta_i^{(n)} \left( X_{i+1}^{(n)} - X_i^{(n)} \right) \right) \left( X_{i+1}^{(n)} - X_i^{(n)} \right).
\]

Since the second term goes to zero as \( n \to \infty \), so, by Cauchy-Schwartz, the third does.
1.3. d-Dimensional Itô-Formula and Covariation

ii. \[
\sum_{t_i^{(n)} \leq t} \left( M_{t_i^{(n)}} - M_{t_i} \right)^2 = \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_i} \right)^2 + \sum_{t_i^{(n)} \leq t} \left( A_{t_i^{(n)}} - A_{t_i} \right)^2 + 2 \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_i} \right) \left( A_{t_i^{(n)}} - A_{t_i} \right) \text{n}\to\infty 0 \text{ by assumption}\]
\[
= \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_i} \right)^2 \text{n}\to\infty 0 \text{ by Cauchy-Schwartz}\]

iii. Let \( F \in C^2(\mathbb{R}) \) such that \( F' = f \) and apply Itô to get
\[
M_t = F(X_t) - \left( F(X_0) + \frac{1}{2} \int_0^t F''(X_s) \, d\langle X \rangle_s \right) =: A_t.
\]

But \( A_t \) can be written as a difference of increasing functions. Therefore, \( A_t \) is of bounded variation, hence, \( \langle A \rangle = 0 \). Thus, by (ii) and (i)
\[
\langle M \rangle = \langle F(X) \rangle_t = \int_0^t (F'(X_s))^2 \, d\langle X \rangle_s.
\]

\( \Box \)

1.3. d-Dimensional Itô-Formula and Covariation

Fix \( X, Y : [0, \infty) \to \mathbb{R} \) continuous with bounded quadratic variation \( \langle X \rangle, \langle Y \rangle \) (along the same \( (\tau_n)_{n\in\mathbb{N}} \) (cf. [RY99, (2.3)-(2.5) (p.27/28)]).

Definition 1.3.1. If
\[
\langle X, Y \rangle_t := \lim_{n\to\infty} \sum_{t_i^{(n)} \leq t} \left( X_{t_i^{(n)}} - X_{t_i} \right) \left( Y_{t_i^{(n)}} - Y_{t_i} \right), \quad t \geq 0,
\]
exists, then it is called the covariation of \( X \) and \( Y \) (along \( (\tau_n) \)).

Lemma 1.3.2. The following assertions are equivalent:

i. \( \langle X, Y \rangle \) exists and is continuous.

ii. \( \langle X + Y \rangle \) exists and is continuous. In this case the Polarization identity holds:
\[
\langle X, Y \rangle = \frac{1}{2} \left( (X + Y) - \langle X \rangle - \langle Y \rangle \right).
\]
In particular, \( \langle X, Y \rangle \) is the distribution function of a signed measure on \( \mathbb{R}_+ \)
\[
d\langle X, Y \rangle = \frac{1}{2} \, d(X + Y) - \frac{1}{2} \, d(X) - \frac{1}{2} \, d(Y).
\]
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Furthermore, if \( \langle X, Y \rangle \) exists, we have

\[
|\langle X, Y \rangle| \leq \langle X \rangle^{\frac{1}{2}} \langle Y \rangle^{\frac{1}{2}}.
\]

Proof. Exercise. \( \square \)

**Remark 1.3.3.** \(|\langle X, Y \rangle| \leq \langle X \rangle^{\frac{1}{2}} \langle Y \rangle^{\frac{1}{2}}\) is a special case of the Kunita-Watanabe-inequality (cf. 2.2.11 below). By this inequality we get an estimation of \( d\langle X, Y \rangle \) by \( d\langle X \rangle \) and \( d\langle Y \rangle \).

**Example 1.3.4.**

i. Let \( (X_t)_{t \geq 0}, (Y_t)_{t \geq 0} \) be independent Brownian motions on \((\Omega, \mathcal{F}, P)\).

Then there exists \( \langle X, Y \rangle(\omega) \) for \( P\)-a.e. \( \omega \in \Omega \) and

\[
\langle X, Y \rangle(\omega) = 0 \quad P\text{-a.e. } \omega \in \Omega.
\]

Proof. We know that

\[
Z_t := \frac{1}{\sqrt{2}}(X_t + Y_t), \quad t \geq 0,
\]

is a Brownian motion. Hence, (cf. Proposition 1.1.4(i)) \( \langle Z \rangle_t = t \), so there exists \( \langle X + Y \rangle_t = 2\langle Z \rangle_t = 2t \). Then it follows by Lemma 1.3.2 applied to \( P\)-a.e. \( \omega \in \Omega \) that

\[
\langle X, Y \rangle = \frac{1}{2}(\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle) = 0.
\]

\( \square \)

ii. Let \( f, g \in C(\mathbb{R}) \) and

\[
Y_t := \int_0^t f(X_s) \, d\langle X \rangle_s, \quad Z_t := \int_0^t g(X_s) \, d\langle X \rangle_s.
\]

Then (again with respect to our \((\tau_n)\)) there exists

\[
\langle Y, Z \rangle_t = \int_0^t f(X_s)g(X_s) \, dX_s.
\]

Proof. By 1.2.9(iii) the quadratic variation of

\[
Y_t + Z_t = \int_0^t (f + g)(X_s) \, dX_s
\]

along our \((\tau_n)\) exists. Hence, by the polarization identity, Lemma 1.2.9(iii) and Lemma 1.3.2 we get

\[
2\langle Y, Z \rangle = \langle Y + Z \rangle - \langle Y \rangle - \langle Z \rangle
\]

\[
= 2 \int f(X_s)g(X_s) \, d\langle X \rangle_s + \int f(X_s)^2 \, d\langle X \rangle_s + \int g(X_s)^2 \, d\langle X \rangle_s - \langle Y \rangle - \langle Z \rangle
\]

\[
= 2 \int f(X_s)g(X_s) \, d\langle X \rangle_s.
\]

\( \square \)
Proposition 1.3.5 (Itô product rule). Let $X, Y$ be as above such that there exists $\langle X, Y \rangle$ (with respect to $(\tau_n)$) and is continuous in $t \geq 0$. If there exists either

$$
\lim_{n \to \infty} \sum_{i \in \tau_n, i \leq t} X_{i(n)} (Y_{i+1(n)} - Y_{i(n)}) =: \int_0^t X_s \, dY_s
$$

or

$$
\lim_{n \to \infty} \sum_{i \in \tau_n, i \leq t} Y_{i(n)} (X_{i+1(n)} - X_{i(n)}) =: \int_0^t Y_s \, dX_s,
$$

then both of these limits exist and

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t.
$$

**Proof.** We have

$$X_t Y_t = \frac{1}{2} \left((X_t + Y_t)^2 - X_t^2 - Y_t^2\right).$$

Furthermore, by Lemma 1.3.2 $\langle X + Y \rangle$ exists and is continuous, since by Itô with $F(X) = \frac{1}{2} X^2$ we know that

$$X_t Y_t = \frac{1}{2} \left((X_0 + Y_0)^2 - X_0^2 - Y_0^2\right) + \int_0^t (X + Y)_s \, d(X + Y)_s
$$

$$- \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s + \frac{1}{2} \left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t\right).
$$

By definition we have

$$
\int_0^t (X + Y)_s \, d(X + Y)_s
$$

$$= \lim_{n \to \infty} \sum_{i \in \tau_n, i \leq t} \left( X_{i(n)} + Y_{i(n)} \right) \left( X_{i+1(n)} + Y_{i+1(n)} - X_{i(n)} - Y_{i(n)} \right)
$$

$$= X_{i(n)} \left( Y_{i+1(n)} - Y_{i(n)} \right) + Y_{i(n)} \left( X_{i+1(n)} - X_{i(n)} \right)
$$

$$+ X_{i(n)} \left( Y_{i(n)} - Y_{i+1(n)} \right) + Y_{i(n)} \left( X_{i(n)} - X_{i+1(n)} \right).
$$

Therefore, we get

$$X_t Y_t = X_0 Y_0 + \lim_{n \to \infty} \sum_{i \in \tau_n, i \leq t} \left( X_{i(n)} (Y_{i+1(n)} - Y_{i(n)}) + Y_{i(n)} (X_{i+1(n)} - Y_{i(n)}) \right) + \langle X, Y \rangle_t,
$$

which implies the assertion.

\[ \square \]

Remark 1.3.6. If $X$ or $Y$ has bounded variation, e.g. $X$, then there already exists $\int Y_s \, dX_s$ and all assumptions in Proposition 1.3.5 are fulfilled. In this case $\langle X, Y \rangle = 0$ and, by substituting $dY_s = Y'_s ds$, we are in the classical case of integration by parts:

$$X_t Y_t - X_0 Y_0 = \int X_s Y'_s \, ds + \int Y_s X'_s \, ds.$$
Example 1.3.7. Suppose $t \mapsto Y_t$ is of bounded variation, hence, $\langle Y \rangle \equiv 0$ and $\langle X, Y \rangle = 0$ (by Hölder). Then by Proposition 1.3.5

$$\int_0^t Y_s \, dX_s = - \int_0^t X_s \, dY_s + X_t Y_t - X_0 Y_0.$$ 

Here, we can define the left hand side by the right hand side since $\int_0^t X_s \, dY_s$ is a usual Lebesgue-Stieltjes integral. This approach was used by Paley-Wiener to define stochastic integrals, if $X$ is a Brownian motion:

Let $X(\omega)$ be a typical Brownian path (hence, $X_0(\omega) = 0$) and $h(= Y_s)$ continuous, of bounded variation and independent of $\omega$ with $h(1) = 0$. Define

$$\int_0^1 h(s) \, dX_s(\omega) := - \int_0^1 h(s) \, dX_s.$$ 

One can show that

$$E \left[ \left( \int_0^1 h(s) \, dX_s \right)^2 \right] = \int_0^1 h(s)^2 \, ds,$$

hence,

$$\mathcal{L}^2([0, 1], ds) \to \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \quad \quad h \mapsto \int h(s) \, dX_s(\omega)$$

is an isometry. It is first defined for a dense subset of functions $h$ in $\mathcal{L}^2([0, 1], ds)$, and then extended to the closure, i.e. for all $h \in \mathcal{L}^2([0, 1], ds)$, by this isometry.

Fix now $X_t = (X^1_t, \ldots, X^d_t) : [0, \infty) \to \mathbb{R}^d$ continuous with continuous $\langle X^i \rangle, \langle X^i, X^j \rangle$, $i, j \in \{1, \ldots, d\}$, $i \neq j$ (along our $(\tau_n)$).

Proposition 1.3.8 (d-dimensional Itô-formula). Let $F \in C^2(\mathbb{R}^d)$. Then with $(\cdot, \cdot)_t$ as Euklidean inner product on $\mathbb{R}$

$$F(X_t) - F(X_0) = \int_0^t (\nabla F(X_s), dX_s) + \frac{1}{2} \int_0^t \sum_{k,l=1}^d \frac{\partial^2 F}{\partial x_k \partial x_l} (X_s) \, d\langle X^k, X^l \rangle_s, \quad (1.3.10)$$

where

$$\int_0^t (\nabla F(X_s), dX_s) := \lim_{n \to \infty} \sum_{\substack{i^{(n)} \leq \tau_n \leq t \atop l^{(n)} \leq t}} \left( \nabla F(X_{l^{(n)}}), X_{l^{(n)} + 1} - X_{l^{(n)}} \right)$$

is called multidimensional Itô-integral.

Proof. Obviously, we have

$$\sum_{\substack{i^{(n)} \leq \tau_n \leq t \atop l^{(n)} \leq i}} F(X_{i^{(n)} + 1}) - F(X_{l^{(n)}}) \xrightarrow{n \to \infty} F(X_t) - F(X_0).$$
Furthermore, by $d$-dimensional Taylor formula we obtain

\[
\sum_{l_i^{(n)} \leq t} F\left( X_{l_i}^{(n)} \right) - F\left( X_{l_i}^{(n)} \right) = \sum_{l_i^{(n)} \leq t} \left( \nabla F\left( X_{l_i}^{(n)} \right) , X_{l_i+1}^{(n)} - X_{l_i}^{(n)} \right) + \frac{1}{2} \sum_{l_i^{(n)} \leq t} \left( A\left( X_{l_i}^{(n)} \right) \left( X_{l_i+1}^{(n)} - X_{l_i}^{(n)} \right) , \left( X_{l_i+1}^{(n)} - X_{l_i}^{(n)} \right) \right) =: S
\]

where

\[
A(x) = \left( \frac{\partial^2}{\partial x_i \partial x_j} F(x) \right)_{i,j}.
\]

The third summand vanishes analogously to the 1-dimensional case. Moreover, since we can interchange the sums, we have by polarization

\[
S = \sum_{j,k=1}^{d} \sum_{l_i^{(n)} \leq t} \frac{\partial^2 F}{\partial x_j \partial x_k} \left( X_{l_i}^{(n)} \right) \left( X_{l_i+1}^{(j)} - X_{l_i}^{(j)} \right) \left( X_{l_i+1}^{(k)} - X_{l_i}^{(k)} \right)
\]

\[
= \sum_{j,k=1}^{d} \sum_{l_i^{(n)} \leq t} \frac{\partial^2 F}{\partial x_j \partial x_k} \left( X_{l_i}^{(n)} \right)
\]

\[
\cdot \frac{1}{2} \left( \left( X_{l_i+1}^{(j)} + X_{l_i+1}^{(k)} \right) - \left( X_{l_i}^{(j)} + X_{l_i}^{(k)} \right) \right)^2 - \left( X_{l_i+1}^{(j)} - X_{l_i}^{(j)} \right)^2 - \left( X_{l_i+1}^{(k)} - X_{l_i}^{(k)} \right)^2
\]

\[
\xrightarrow{n \to \infty} \frac{1}{2} \sum_{j,k=1}^{d} \left( \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k} (X_s) \, d\langle X^{(j)} + X^{(k)} \rangle_s - \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k} (X_s) \, d\langle X^{(j)} \rangle_s \right.
\]

\[
- \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k} (X_s) \, d\langle X^{(k)} \rangle_s
\]

\[
= \int_0^t \sum_{k,l=1}^{d} \frac{\partial^2 F}{\partial x_k \partial x_l} (X_s) \, d\langle X^k, X^l \rangle_s, \quad \text{P-a.s.}
\]
1. Introduction to Pathwise Itô-Calculus

Remark 1.3.9.  

i. Of course, we have defined

$$\int_0^t (\nabla F(X_s), dX_s) := \lim_{n \to \infty} \sum_{k=1}^d \sum_{(l(n), l_i(n)) \leq t} \frac{\partial F}{\partial x_k} \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right),$$

but we cannot interchange \(\sum_{k=1}^d\) with the limit, since we do not know whether for every \(k\) the limit exists.

ii. For \(F \in C^1(\mathbb{R}^d)\) the function \(t \mapsto F(X_t)\) has continuous quadratic variation and

$$\langle F(X) \rangle_t = \sum_{k,l=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s) \frac{\partial F}{\partial x_l}(X_s) \, d\langle X^k, X^l \rangle_s.$$

Proof. By Taylor

$$\sum_{(l(n), l_i(n)) \leq t} \left( F \left( X_{l(n)}^{t_i(n)} \right) - F \left( X_{l_i(n)}^{t_i(n)} \right) \right)^2$$

$$= \sum_{k,l=1}^d \sum_{(l(n), l_i(n)) \leq t} \frac{\partial F}{\partial x_k} \left( X_{l(n)}^{t_i(n)} \right) \frac{\partial F}{\partial x_l} \left( X_{l_i(n)}^{t_i(n)} \right) \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right)$$

$$- \frac{1}{2} \sum_{k,l=1}^d \int_0^t \frac{\partial F}{\partial x_k} \left( X_s \right) \frac{\partial F}{\partial x_l} \left( X_s \right) d\langle X^k, X^l \rangle_s \text{ by polarization}$$

$$+ \sum_{k,l=1}^d \sum_{(l(n), l_i(n)) \leq t} \left( \frac{\partial F}{\partial x_k} \left( X_{l(n)}^{t_i(n)} + \theta^n \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right) \right) \frac{\partial F}{\partial x_l} \left( X_{l(n)}^{t_i(n)} \right) \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right)$$

$$\leq \varepsilon \text{ uniformly in } t \text{ for } n \text{ big enough}$$

$$+ \sum_{k,l=1}^d \sum_{(l(n), l_i(n)) \leq t} \left( \frac{\partial F}{\partial x_k} \left( X_{l(n)}^{t_i(n)} \right) \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right) \right)$$

$$+ 2 \sum_{k,l=1}^d \sum_{(l(n), l_i(n)) \leq t} \left( \frac{\partial F}{\partial x_k} \left( X_{l(n)}^{t_i(n)} + \theta^n \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right) \right) \frac{\partial F}{\partial x_l} \left( X_{l(n)}^{t_i(n)} \right) \left( X_{l(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right)$$

$$\cdot \sum_{k,l=1}^d \frac{\partial F}{\partial x_l} \left( X_{l_i(n)}^{t_i(n)} \right) \left( X_{l_i(n)}^{t_i(n)} - X_{l_i(n)}^{t_i(n)} \right).$$

By Cauchy-Schwartz the last two double sums converge to 0 as \(n\) goes to \(\infty\). \(\square\)

iii. Since

$$\frac{1}{2} \int_0^t \sum_{k,l=1}^d \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s) \, d\langle X^k, X^l \rangle_s$$

is of bounded variation, its quadratic variation is 0. But then, by 1.3.8 and 1.2.9 (ii), we can conclude that

$$\left\langle \int_0^t (\nabla F(X_s), dX_s) \right\rangle_t = \langle F(X) - F(X_0) \rangle_t = \langle F(X) \rangle_t \overset{(ii)}{=} \sum_{k,l=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s) \frac{\partial F}{\partial x_l}(X_s) \, d\langle X^k, X^l \rangle_s.$$
1.3.1. Important special cases

Itô’s product rule in $d$ dimensions

Let $X, Y$ be continuous with existing continuous $\langle X \rangle, \langle Y \rangle, \langle X, Y \rangle$. Then

$$X_tX_t = X_0Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t.$$  

Proof. Applying Itô’s formula with $F(X, Y) = X \cdot Y$ is completely wrong since by (2-dimensional) Itô-formula

$$\int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s$$

only exists as a sum, whereas single components don’t need to exist. But this proof holds if $X, Y$ are (semi)-martingales. \hfill \Box

Brownian motion in $\mathbb{R}^d$ and Laplace-operator $\Delta$

The components of a $d$-dimensional Brownian motion are independent, hence, (by 1.3.4(i))

$$\langle X^k, X^l \rangle_t(\omega) = \delta_{kl} \cdot t \quad \text{("covariation reflects independence")},$$

which implies by Itô for $F \in C^2(\mathbb{R})$

$$F(X_t) - F(X_0) = \int_0^t (\nabla F(X_s), dX_s) + \frac{1}{2} \int_0^t \Delta F(X_s) \, ds.$$

In particular, if $F$ is harmonic (i.e. $\Delta F = 0$), then

$$F(X_t) = F(0) + \int_0^t (\nabla F(X_s), dX_s),$$

which is an Itô-integral of a Brownian motion. Hence, a harmonic function preserves the martingale property since the Itô-integral again is a martingale (see Section 1.4 below).

Itô-formula for time dependent functions

Proposition 1.3.10. Let $F \in C^2(\mathbb{R}^2)$ and $X : [0, \infty) \to \mathbb{R}$ be continuous with continuous $\langle X \rangle$ (along $(\tau_n)_{n \in \mathbb{N}}$). Then

$$F(X_t, \langle X \rangle_t) = F(X_0, 0) + \int_0^t \frac{\partial F}{\partial x}(X_s, \langle X \rangle_s) \, dX_s + \int_0^t \left( \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(X_s, \langle X \rangle_s) + \frac{\partial F}{\partial y}(X_s, \langle X \rangle_s) \right) d\langle X \rangle_s.$$

Proof. Apply Itô for $d = 2$ with $(X_t, \langle X \rangle_t)$ to get

$$F(X_t, \langle X \rangle_t) - F(X_0, 0)$$

$$= \int_0^t \frac{\partial F}{\partial x}(X_s, \langle X \rangle_s) \, dX_s + \int_0^t \frac{\partial F}{\partial y}(X_s, \langle X \rangle_s) \, d\langle X \rangle_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, \langle X \rangle_s) \, d\langle X \rangle_s$$

$$+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial y^2}(X_s, \langle X \rangle_s) \, d\langle X \rangle_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x \partial y}(X_s, \langle X \rangle_s) \, d\langle X \rangle_s.$$

$$= S = 0$$

since $\langle X \rangle_s$ is of bounded variation and by 1.3.7

The second summand of $S$ exists since $\langle X \rangle_t$ is of bounded variation (along $(\tau_n)_{n \in \mathbb{N}}$). As the whole sum $S$ exists by Itô ($d = 2$), so does the first summand. \hfill \Box
1. Introduction to Pathwise Itô-Calculus

Remark 1.3.11.  
i. If $X_t$ is a Brownian motion, then $\langle X \rangle_t$ represents the time, i.e.

$$F(X_t, \langle X \rangle_t) = F(X_t, t).$$

Therefore, in this respect $\langle X \rangle_t$ is also called inner clock of $X_t$.

ii. If $F$ is a solution to the backward heat equation, i.e.

$$\frac{1}{2} \frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial t} = 0,$$

then we have by above (for certain $X$ as above)

$$F(X_t, \langle X \rangle_t) = F(X_0, 0) + \int_0^t \frac{\partial F}{\partial x}(X_s, \langle X \rangle_s) \, dX_s = F(X_0, 0) + \text{“Itô-integral”}.$$

Later we shall see that the Itô-integral is a local martingale, if $X$ is one. Therefore, every solution to the backward heat equation provides a local martingale!

Example 1.3.12.  
i. Let $F(x, t) := \exp(\alpha x - \frac{1}{2} \alpha^2 t)$. Then $F$ solves the backward heat equation. Thus, if $X_0 = 0$, then

$$G_t := F(X_t, \langle X \rangle_t) = \exp(\alpha X_t - \frac{1}{2} \alpha^2 \langle X \rangle_t)$$

solves the differential equation

$$G_0 = 1,$$

$$dG = \alpha G \, dX,$$

i.e.

$$G_t = 1 + \int_0^t \alpha G_s \, dX_s.$$

Recall, if $\langle X \rangle \equiv 0$, then $G_t = \exp(\alpha X_t)$. Hence, in the Itô-calculus $G_t$ as above is the right analog to the exponential function $\exp(\alpha t)$ in the usual case.

Application to Brownian motion:
For $X$ with $\langle X \rangle_t = t$

$$G_t(\omega) := G_0 e^{\beta t} \exp(\alpha X_t(\omega) - \frac{1}{2} \alpha^2 t), \quad t \geq 0,$$

solves the linear SDE

$$dG = \alpha G \, dX + \beta G \, dt$$

because of Itô’s product rule.

Classical ($\alpha = 0$):

$$dG = \beta G \, dt \quad \Rightarrow \quad G_t = G_0 e^{\beta t}.$$ 

Here, for $\beta > 0$ $G_t$ tends to $\infty$ as $t \to \infty$.

Stochastic ($\alpha \neq 0$):

$$G_t = G_0 \exp(\alpha X_t + \left(\beta - \frac{1}{2} \alpha^2\right) t).$$

By law of iterated logarithm

$$\frac{X_t}{t} \xrightarrow{t \to \infty} 0, \ a.s.,$$
1.3. d-Dimensional Itô-Formula and Covariation

hence, for large $t$

$$\left| \frac{X_t}{t} \right| < \frac{1}{2} \left( \frac{1}{2} \alpha^2 - \beta \right).$$

Thus, for $\beta < \frac{1}{2} \alpha^2$

$$G_t \leq G_0 e^{-\left(\frac{1}{2} \alpha^2 - \beta\right) t} \overset{t \to \infty}{\longrightarrow} 0, \quad (a.s. \text{ pathwise stable}).$$

But $G_t$ is not uniformly integrable, since by the Residue theorem

$$E[e^{\alpha X_t}] = e^{\frac{1}{2} \alpha^2 t},$$

and therefore,

$$E[G_t] = G_0 e^{\beta t} E[e^{\alpha X_t}] e^{-\frac{1}{2} \alpha^2 t} = G_0 e^{\beta t} \to \infty, \quad \text{if } \beta > 0 \quad (\text{unstable in the mean}).$$

ii. “Hermite- polynomials” (cf. Example 1.2.5):

Define $h_n(x,t)$ by

$$e^{-\alpha x - \frac{1}{2} \alpha^2 t} = \sum_{n=0}^\infty \frac{\alpha^n}{n!} h_n(x,t), \quad (1.3.11)$$

where the left hand side is analytic in $\alpha$, i.e.

$$h_n(x,t) = \frac{\partial^n}{\partial \alpha^n} \left( e^{\alpha x - \frac{1}{2} \alpha^2 t} \right) \bigg|_{\alpha = 0} = e^{\alpha x} \sum_{k=0}^n \binom{n}{k} x^k \frac{\partial^{n-k}}{\partial \alpha^{n-k}} \left( e^{-\frac{1}{2} \alpha^2 t} \right). \quad (1.3.12)$$

Recall, that for all $n \in \mathbb{N}$

$$H_n(\cdot,t) := \frac{1}{\sqrt{n!}} h_n(\cdot,t)$$

is an orthonormal basis of $L^2(\mathbb{R}, N(0,t))$.

Proof. Clearly, (by monotone classes) $\text{span}\{H_n, n \in \mathbb{N}\}$ is a dense subset of $L^2(\mathbb{R}, N(0,t))$.

Hence, it is sufficient to show that it is an ONS:

$$\sum_{n,m} \frac{\alpha^n}{n!} h_n(x,t) \frac{\beta^m}{m!} h_m(x,t) = e^{(\alpha + \beta)x - \frac{1}{2}(\alpha^2 + \beta^2)t} = \exp((\alpha + \beta)x) e^{-\frac{1}{2}(\alpha + \beta)^2 t} e^{\alpha \beta t}. \quad (1.3.11)$$

By integration with $N(0,t)$ in $X$, since sums interchange with integration and

$$\int e^{(\alpha + \beta)x} N(0,t) \, dx = e^{\frac{1}{2}(\alpha + \beta)^2 t},$$

we get by (1.3.12) that

$$\sum_{n,m} \alpha^n \beta^m \int \frac{1}{n!} h_n(x,t) \frac{1}{m!} h_m(x,t) N(0,t) \, dx = e^{\alpha \beta t} = \sum_n \frac{\alpha^n \beta^n}{n!}, \quad \forall \alpha, \beta.$$
1. Introduction to Pathwise Itô-Calculus

Additionally, for $n \geq 1$ we have

$$\frac{\partial}{\partial x} h_n(\cdot, t) \overset{(1.3.11)}{=} n \cdot h_{n-1}(\cdot, t).$$

Therefore, if $X, \langle X \rangle$ are continuous and $X_0 = 0$, (since $h_n(0, 0) = 0$), we get

$$h_n(X_t, \langle X \rangle_t) = \int_0^t \frac{\partial}{\partial x} h_n(X_{t_1}, \langle X \rangle_{t_2}) \, dX_{t_1} = \int_0^t h_{n-1}(X_{t_1}, \langle X \rangle_{t_2}) \, dX_{t_1}$$

$$= \ldots = n! \int_0^t dX_{t_1} \int_0^{t_1} dX_{t_2} \ldots \int_0^{t_{n-1}} dX_{t_n}.$$
1.4. Itô-Integrals as (Local) Martingales

Recall: If \((\mathcal{F}_t)_{t \geq 0}\) is not right-continuous, then define

\[
\mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s, \quad t \geq 0.
\]

Then \((\mathcal{F}_{t+})_{t \geq 0}\) is right-continuous.

\(X\) is called \textit{continuous local martingale} (up to \(T\)) if there exist a localising sequence \(T_1 \leq T_2 \leq \ldots \leq T\) on \(\{T > 0\}\), i.e. they are \((\mathcal{F}_t)\)-stopping times such that

1. \((X_{t \wedge T_n})_{t \geq 0}\) is a martingale for all \(n \in \mathbb{N}\),
2. \(\sup_{n \in \mathbb{N}} T_n = T\) \(P\text{-a.s.}\).

Let \(X_t(\omega), 0 \leq t \leq T(\omega)\) (on \(\{T > 0\}\)), be a stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) with continuous paths, adapted to a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\), i.e.

\[
\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s.
\]

Assume in addition, that \(X\) has a continuous quadratic variation \(\langle X \rangle_t \) along \((\tau_n)\) for \(P\text{-a.e. } \omega \in \Omega\).

Remark 1.4.1. Later we shall see that for any \((\tau_n)\) there exists \((\tau_{n_k})\) such that \(X\) has continuous quadratic variation \(\langle X \rangle_t \) along \((\tau_{n_k})\) for \(P\text{-a.e. } \omega \in \Omega\).

Proposition 1.4.2. Let \(f \in C^2(G \times \mathbb{R}_+), G \subset \mathbb{R}^1\) open (or \(f \in C^1(G)\)). Assume there exists a compact set \(K \subset G\) such that \(X_0(\omega) \subset K\) for \(P\text{-a.e. } \omega \in \Omega\).

Then the Itô-integral

\[
M_t := \int_0^t f(X_s(\omega), \langle X \rangle_s(\omega)) \, dX_s(\omega), \quad 0 \leq t < T(\omega) \text{ on } \{T > 0\}
\]

is a continuous local martingale up to

\[
S := \inf\{t > 0 | X_t \not\in G\} \wedge T.
\]

Proof. Step 1: Assume that \(T = \infty\), \(X_t\) is a bounded martingale, \(G = \mathbb{R}^1\) and \(f\) is bounded. Then \(M_t(\omega)\) is defined for all \(t\), since \(S = +\infty\).

Claim: \((M_t)_{t \geq 0}\) is a continuous martingale.

To see this define for \(n \in \mathbb{N}\) fixed

\[
M^{(n)}_t := \sum_{\substack{t^{(n)}_i \leq t < t^{(n)}_{i+1} \in \tau_n \quad \text{such that} \quad t^{(n)}_k \leq t \leq t^{(n)}_{k+1}, \quad t^{(n)}_i \leq s \leq t^{(n)}_{i+1} \text{. Then, we have} \quad M^{(n)}_t = M^{(n)}_{t_k} \quad \text{and}}}
\]

\[
M^{(n)}_{t_k} := \sum_{l^{(n)}_j \in \tau_n \quad \text{such that} \quad l^{(n)}_i \leq s \leq l^{(n)}_{i+1}} f \left( X^{(n)}_{l^{(n)}_i}, X^{(n)}_{l^{(n)}_{i+1}} \right) \left( X^{(n)}_{l^{(n)}_{i+1}} - X^{(n)}_{l^{(n)}_i} \right).
\]
1. Introduction to Pathwise Itô-Calculus

\[ M_s^{(n)} = M_t^{(n)} \] and therefore,

\[ E[M_t^{(n)} - M_s^{(n)} | \mathcal{F}_s] = E[M_t^{(n)} - M_t^{(n)} | \mathcal{F}_s] \]

\[ = \sum_{i=t+1}^k E \left[ f \left( X_{t_i}^{(n)}, \langle X \rangle_{t_i}^{(n)} \right) \left( X_{t_i+1}^{(n)} - X_{t_i}^{(n)} \right) | \mathcal{F}_{t_i}^{(n)} \right] | \mathcal{F}_s \]

\[ = \sum_{i=t+1}^k f \left( X_{t_i}^{(n)}, \langle X \rangle_{t_i}^{(n)} \right) E \left[ \left( X_{t_i+1}^{(n)} - X_{t_i}^{(n)} \right) | \mathcal{F}_{t_i}^{(n)} \right] | \mathcal{F}_s \]

\[ = 0 \text{ since } X_t \text{ is a martingale} \]

By Proposition 2.1.4 below it is sufficient to show that \( M_t^{(n)} \to M_t \) in \( L^1 \). We know that by definition,

\[ M_t^{(n)} \to M_t, \quad P\text{-a.s. } \forall t \geq 0. \]

Furthermore, for all \( t \geq 0 \),

\[ E \left[ \left( M_t^{(n)} \right)^2 \right] = \sum_{i \in \tau_n, t_i^{(n)} \leq t} E \left[ f^2 \left( X_{t_i}^{(n)}, \langle X \rangle_{t_i}^{(n)} \right) \left( X_{t_i+1}^{(n)} - X_{t_i}^{(n)} \right)^2 \right], \]

since all terms, which are not on the diagonal, vanish by martingale property of \( X_t \). Furthermore,

\[ E \left[ \left( M_t^{(n)} \right)^2 \right] \leq \sup \frac{1}{r^2} \sum_{i \in \tau_n, t_i^{(n)} \leq t} E \left[ X_{t_i+1}^{2} - 2X_{t_i}^{(n)}X_{t_i}^{(n)} + X_{t_i}^{2} | \mathcal{F}_{t_i}^{(n)} \right] \]

\[ = \sup \frac{1}{r^2} \sum_{i \in \tau_n, t_i^{(n)} \leq t} E \left[ X_{t_i+1}^{2} - X_{t_i}^{2} \right] \]

\[ = \sup_r \frac{1}{r^2} E \left[ \frac{X_{t_i}^{2}}{r^2} - X_0^2 \right]. \]

Hence, \( \sup_n E \left[ \left( M_t^{(n)} \right)^2 \right] < \infty \) and \( \left( M_t^{(n)} \right) \) \( n \in \mathbb{N} \) is uniformly integrable. By the generalized Lebesgue dominated convergence theorem the claim follows, i.e. \( (M_t)_{t \geq 0} \) is a martingale.

It still remains to show that \( M_t \) has \( P\text{-a.s.} \) continuous sample paths:

In order to see that consider

\[ \tilde{M}_t^{(n)} := \sum_{i=0}^n f \left( X_{t_i}^{(n)}, \langle X \rangle_{t_i}^{(n)} \right) \left( X_{t_i+1}^{(n)} - X_{t_i}^{(n)} \right). \]

Then as above one shows that \( \tilde{M}_t(\omega) \) is a martingale. Note that \( \left( \tilde{M}_t^{(n)} \right) \) \( n \in \mathbb{N} \) has \( P\text{-a.s.} \) continuous sample paths and that

\[ M_t^{(n)} - \tilde{M}_t^{(n)} \nrightarrow 0 \quad P\text{-a.s.} \]

and in \( L^q \) (by the same argument as above). Hence, \( \tilde{M}_t^{(n)} \nrightarrow M_t \) in \( L^q \) for all \( q \in [1, 2) \). Then by Doob’s maximal inequality one can show that \( \left( M_t \right)_{t \geq 0} \) has \( P\text{-a.s.} \) continuous sample paths.
1.4. Itô-Integrals as (Local) Martingales

(cf. below Proposition 2.1.4).

**Step 2** ("Localization by stopping times"): Let \( (T_n) \) be a localizing sequence for \( X \). For \( n \in \mathbb{N} \) define

\[
\bar{T}_n := \inf\{t > 0 | \langle X \rangle_t > n\} \land T
\]

and

\[
S_n := T_n \land \sigma_{G_n} \land \bar{T}_n,
\]

where \( G_n \not\subset G, G_n \) relatively compact and open and \( G_n \subset G_{n+1} \) for all \( n \in \mathbb{N} \). Without loss of generality assume that \( K \subset G_n \). Then \( \sup_n \sigma_{G_n} = \sigma_G \) and \( S_n \) are stopping times. Furthermore, \( \bar{T}_n \not\subset T \), hence, \( S_n \not\subset T \land \sigma_G \). By optional stopping \( (X_{t \land S_n})_{t \geq 0} \) is a continuous martingale taking values in \( G_n \), because for \( n \geq N(K) \), where \( N(K) \in \mathbb{N} \) such that \( G_n \supset K \) for all \( n \geq N(K) \), hence, \( (X_{t \land S_n})_{t \geq 0} \) is bounded in \((t, \omega)\). Furthermore, (exercise)

\[
M_{t \land S_n}(\omega) = \int_0^t f(X_{s\land S_n}(\omega), \langle X \rangle_{s\land S_n}(\omega)) \, dX_{s\land S_n}(\omega)(\omega). \quad (1.4.13)
\]

Hence, we can take \( \chi_n \in C^2_0(G \times \mathbb{R}_+) \) such that

\[
\chi_n = 1 \text{ on } G_n \times [0,n].
\]

Then we can replace \( f \) in (1.4.13) by \( \chi_n f \in C^2_0(G \times \mathbb{R}_+) \). Therefore, the representation for \( (M_{t \land S_n})_{t \geq 0} \) in (1.4.13) and Step 1 imply that \( (M_{t \land S_n})_{t \geq 0} \) is a continuous martingale.

**Corollary 1.4.3.** Let \( X \) be a continuous local martingale up to \( T \) with continuous quadratic variation (later proved to always be the case). Then

i. \( X^2 - \langle X \rangle \) is a continuous local martingale up to \( T \).

ii. If \( \langle X \rangle = 0 \) (which is particularly true if \( X \) has bounded variation), then for \( P \)-a.s. \( \omega \in \Omega \)

\[
X_t(\omega) = X_0(\omega) \quad \forall t \in [0, T(\omega)]. \quad (1.4.14)
\]

**Proof.** Without loss of generality assume that \( X_0 \equiv 0 \). (Otherwise consider \( X_t - X_0, t \leq T \).)

i. By Itô

\[
X_t^2 = 2 \cdot \int_0^t X_s \, dX_s + \langle X \rangle_t \quad \text{on } \{t < T\}.
\]

But the first term on the right hand side is a continuous martingale up to \( T \) by Proposition 1.4.2.

ii. By (i) it also follows that \( X^2 \) is a continuous local martingale up to \( T \) if \( \langle X \rangle = 0 \). Hence, if \( T_n \not\subset T \) is a localising sequence, then

\[
E[X^2_{t\land T_n}] = 0 \quad \forall t \geq 0.
\]

Therefore, \( 1_{\{t < T\}}X_t = 0 \) \( P \)-a.s. \( \forall t \geq 0 \), (with zero set depending on \( t \)). Hence, \( P[X_{t\land T} = 0 \land \forall t \in \mathbb{Q}] = 1 \), and by \( P \)-a.s. continuity in \( t \) it follows that \( X_t = 0 \) on \( \{t < T\} \) for all \( t \geq 0 \) \( P \)-a.s.

Therefore, the interesting cases are of type \( \langle X \rangle \neq 0 \), where the Itô integral occurs.
1. Introduction to Pathwise Itô-Calculus

**Proposition 1.4.4** (d-dimensional version of Proposition 1.4.2). Let \( X = (X^1, \ldots, X^d) \) with \( X^1, \ldots, X^d \) continuous local martingales up to \( T \) such that \((X_i, X_j)\) exist for all \( 1 \leq i, j \leq d \) and are continuous up to \( T \). Let \( F \in C^2(D), D \subset \mathbb{R}^d \), open, with \( X_0(\omega) \subset K \) for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \) for some compact \( K \subset \mathbb{R}^d \). Then

\[
M_t := \int_0^t (\nabla F(X_s), dX_s)_{\mathbb{R}^d} \quad \text{(d-dimensional Itô-integral)}
\]
is a continuous local martingale up to

\[
T \wedge \inf\{t > 0|X_t \notin D\}.
\]

**Proof.** Exercise (Proceed as for Proposition 1.4.2).

**Remark 1.4.5.** By Remark 1.3.9 (iii) for \( M \) as in Proposition 1.4.4 we have

\[
\langle M \rangle_t = \sum_{k,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s) \frac{\partial F}{\partial x_l}(X_s) \, d\langle X^k, X^l \rangle_s \quad \text{on} \{t < \bar{T}\}.
\]

Hence, if \( \bar{T} \equiv \infty \), then \( M \) is a Brownian motion. This is a consequence of Corollary 1.4.7 below.

**Proposition 1.4.6.** Let \( M \) be a continuous martingale up to \( T \) (with continuous \( \langle M \rangle \)) and let \( T_0 < T \) be a stopping time such that

\[
E[\langle M \rangle_{T_0}] < \infty.
\]

Then

\[
E[M_{T_0}] = E[M_0]
\]
and \( (M_{T_0 \wedge T_{n \wedge t}})_{t \geq 0} \) is a square integrable martingale.

Note that \( T_0 \) doesn’t need to be bounded.

**Proof.** Without loss of generality assume \( M_0 \equiv 0 \). (Otherwise consider \( M_t - M_0 \).) Let \( T_n \nearrow T \) be a localising sequence for \( M \) and \( \int_0^T M_s \, dM_s \). We may assume \( T_n \leq n \). (Otherwise we replace it by \( T \wedge n \).) Then \( (M_{T_0 \wedge T_n \wedge t})_{t \geq 0} \) is a martingale such that

\[
M_{T_0 \wedge T_n \wedge t} \to M_{T_0 \wedge t} \quad \text{P-a.s.} \forall t \geq 0
\]
and

\[
E \left[ M_{T_0 \wedge T_n \wedge t}^2 \right] \overset{\text{Cor. 1.4.3}}{=} E[\langle M \rangle_{T_0 \wedge T_n \wedge t}] \leq E[\langle M \rangle_{T_0}] < \infty.
\]

Hence, \( \{M_{T_0 \wedge T_n \wedge t} | n \in \mathbb{N}, t \geq 0\} \) is uniformly integrable. In particular, by Lebesgue

\[
M_{T_0 \wedge T_n \wedge t} \to M_{T_0 \wedge t} \quad \text{in} \mathcal{L}^1 \forall t \geq 0.
\]

Therefore, \( (M_{T_0 \wedge t})_{t \geq 0} \) is a martingale. In addition,

\[
E \left[ M_{T_0 \wedge t}^2 \right] \leq \liminf_{n \to \infty} E \left[ M_{T_0 \wedge T_n \wedge t}^2 \right] \leq E[\langle M \rangle_{T_0}] < \infty \quad \forall t \geq 0.
\]

So, \( (M_{T_0 \wedge t})_{t \geq 0} \) is square integrable (with uniformly integrable bounded \( \mathcal{L}^2 \)-norm). Furthermore, it is easy to show that

\[
E[M_{T_0}] = \lim_{n \to \infty} E \left[ M_{T_0 \wedge T_n} \right] = 0 = E[M_0].
\]
Corollary 1.4.7. Let $M$ be a continuous local martingale up to $T \equiv \infty$ (with continuous $\langle M \rangle$) such that

$$E[\langle M \rangle_t] < \infty \quad \forall t \geq 0.$$ 

Then $M$ is a square integrable martingale.

Finally, we want to investigate the following proposition, which can be considered as the core statement of the Itô formula.

Proposition 1.4.8. Let $T$ be a stopping time and $X,A$ continuous processes with continuous $\langle X \rangle$ and $\langle A \rangle = 0$ up to $T$. Assume $A_0 = 0$. Then the following are equivalent:

i. $X$ is a local martingale up to $T$ with $\langle X \rangle = A$.

ii. For all $\alpha \geq 0$ is

$$G_t^\alpha := \exp[\alpha X_t - \frac{1}{2} \alpha^2 A_t] \quad \text{for} \quad t \text{ up to } T$$

a local martingale up to $T$.

Proof. (i) $\Rightarrow$ (ii): Without loss of generality $X_0 \equiv 0$. Then by Itô-formula for time independent functions (cf. Proposition 1.3.11)

$$G_t^\alpha = G_0^\alpha + \int_0^t \alpha G_s^\alpha \, dX_s + 0$$

is a local martingale up to $T$ by Proposition 1.4.2 for $f(X_t, \langle X \rangle_t) = \exp(\alpha X_t - \frac{1}{2} \alpha^2 \langle X \rangle_t)$. In the general case consider $e^{-\alpha X_0} G_t^\alpha$ and apply above to get that $(e^{-\alpha X_0} G_t^\alpha)_{t \geq 0}$ is a local martingale up to $T$. Multiplying by $e^{\alpha X_0}$ implies the assertion.

(ii) $\Rightarrow$ (i): Without loss of generality assume $X_0 = 0$. (Otherwise consider $e^{-\alpha X_0} G_t$ for $t \leq T$. This is a local martingale up to $T$ since $e^{-\alpha X_0}$ is $\mathcal{F}_t$-measurable. Therefore, in the special case ($X_0 = 0$) we get that $X_t - X_0$ is a local martingale up to $T$, hence, so is $X$.) By Itô-formula we get

$$X_t = \frac{1}{\alpha} \left( \log G_t^\alpha - \log G_0^\alpha \right) + \frac{1}{2} \alpha A_t$$

$$= \lim_{\alpha \to 0} \left( \int_0^t \frac{1}{G_s^\alpha} \, dG_s^\alpha - \frac{1}{2} \int_0^t \frac{1}{(G_s^\alpha)^2} \, d\langle G^\alpha \rangle_s \right) + \frac{1}{2} \alpha A_t$$

$$\overset{1.3.9(ii)}{=} \lim_{\alpha \to 0} \left( \int_0^t \frac{1}{G_s^\alpha} \, dG_s^\alpha - \frac{1}{2} \int_0^t \frac{1}{(G_s^\alpha)^2} (\alpha G_s^\alpha)^2 \, d\langle G \rangle_s \right) + \frac{1}{2} \alpha A_t$$

$$= \frac{1}{\alpha} \int_0^t \frac{1}{G_s^\alpha} \, dG_s^\alpha + \frac{\alpha}{2} (A_t - \langle X \rangle_t) \quad \forall \alpha \in \mathbb{R} \setminus \{0\}.$$ 

Here, $\frac{1}{\alpha} \int_0^t \frac{1}{G_s^\alpha} \, dG_s^\alpha$ is a local martingale up to $T$ by Proposition 1.4.2 with $G \in (0, \infty)$. Consider $\alpha \neq \alpha'$ and take the difference of the two corresponding equalities to get

$$0 = M + \frac{\alpha - \alpha'}{2} (A_t - \langle X \rangle_t),$$

where $M$ is a local martingale up to $T$. By assumption and by Corollary 1.4.3(ii) $A_t - \langle X \rangle_t = A_0 - \langle X \rangle_0 = 0$ on \{ $t < T$ \} P-a.s. \qed
1. Introduction to Pathwise Itô-Calculus

1.5. Levy’s Characterization of Brownian Motion

Let $X = (X_t)_{t \geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ with continuous sample paths (and continuous quadratic variation $(X)$).

**Proposition 1.5.1** (Levy). Assume that $X$ is a continuous local martingale up to $\infty$ with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$ such that $X$ is $(\mathcal{F}_t)$-adapted. If $\langle X \rangle_t = t$ holds for all $t \geq 0$, then $X$ is a Brownian motion.

**Remark 1.5.2.** Every continuous martingale can be transferred by time change into a Brownian motion provided its quadratic variation is strictly increasing (see later). Therefore, Brownian motion is the only local martingale, where the quadratic variation can be considered as “proper time”.

**Proof of 1.5.1.** By Corollary 1.4.7 $X$ is already a martingale. We apply Itô-formula to

$$F(x) = e^{iuX} = \cos(ux) + is(ux), \forall x \in \mathbb{R}, \forall u \in \mathbb{R}.$$ 

Then, for all $s < t$

$$e^{iuX_{t \wedge T_n}} - e^{iuX_{s \wedge T_n}} = \int_0^{t \wedge T_n} iue^{iuX_r} \, dX_r - \int_0^{s \wedge T_n} iue^{iuX_r} \, dX_r + \frac{1}{2} \int_{s \wedge T_n}^{t \wedge T_n} (-u^2)e^{iuX_r} \, d\langle X \rangle_r,$$

where $T_n \nearrow \infty$ are stopping times such that

$$\left(\int_0^{t \wedge T_n} iue^{iuX_r} \, dX_r\right)_{t \geq 0}$$

is a martingale for all $n$. Now take $E[\cdot | \mathcal{F}_s]$ of the equality above:

$$E[e^{iuX_{t \wedge T_n}} - e^{iuX_{s \wedge T_n}} | \mathcal{F}_s] = \frac{1}{2} E\left[\int_{s \wedge T_n}^{t \wedge T_n} (-u^2)e^{iuX_r} \, dr | \mathcal{F}_s\right].$$

Hence, letting $n \to \infty$ by path-continuity we obtain

$$E\left[\int_s^t (-u^2)e^{iuX_r} \, dr | \mathcal{F}_s\right] = E[e^{iuX_t} - e^{iuX_s} | \mathcal{F}_s].$$

Multiplication by $e^{-iuX_s}$ yields

$$E[e^{iu(X_t - X_s)} | \mathcal{F}_s] - 1 = \frac{1}{2} E\left[\int_s^t (-u^2)e^{iu(X_r - X_s)} \, dr | \mathcal{F}_s\right].$$

Therefore, for all $A \in \mathcal{F}_s$

$$E[e^{iu(X_t - X_s)}, A] = P(A) \overset{\text{Fubini}}{=} -\frac{1}{2} u^2 \int_s^t E[e^{iu(X_r - X_s)}, A] \, dr.$$

Thus, $\varphi \in C^1$ (since the right hand side is $C^1$) and by differentiation

$$\dot{\varphi}(t) = \frac{1}{2} u^2 \varphi(t) \quad \forall t \geq s.$$ 

Solving this equation we get

$$\varphi(t) = C \cdot e^{\frac{1}{2} u^2 t}.$$
Substituting $t$ by $s$ yields

$$P(A) = \varphi(s),$$

which implies

$$C = P(A)e^{-\frac{1}{2}u^2 s},$$

hence,

$$E[e^{iu(X_t - X_s)}; A] = e^{-\frac{1}{2}u^2(t-s)}P(A) \quad \forall A \in \mathcal{F}_s, \quad (1.5.15)$$

Taking $A := \Omega$ implies that $X_t - X_s$ is $N(0, (t - s))$ distributed by uniqueness of the Fourier-transformation. By monotone class argument, (1.5.15) implies that $X_t - X_s$ is independent of $\mathcal{F}_s$ (Exercise). In particular,

$$X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_2} - X_{t_1}$$

are independent for all $0 \leq t_1 < t_2 < \ldots < t_n < \infty$. \qed
1. Introduction to Pathwise Itô-Calculus
2. (Semi-)Martingales and Stochastic Integration

In this chapter we want to define 
\[ \int g(t, \cdot) \, dM_t, \]
where \( g \) is defined “more general” and \( M_t \) is an arbitrary semimartingale. This is more general than so far, since we could only define 
\[ \int f(t, \mu_t) \, dM_t, \quad \forall f \in C^1. \]

2.1. Review of Some Facts from Martingale Theory

Fix a probability space \((\Omega, \mathcal{F}, P)\) and let \((\mathcal{F}_t)_{t \geq 0}\) be a right-continuous filtration. (Sometimes assume in addition that \( N \in \mathcal{F} \), \( P(N) = 0 \Rightarrow N \in \mathcal{F}_0 \).) These conditions are also called “usual conditions”.

**Proposition 2.1.1.** Let \((M_t)_{t \geq 0}\) be a martingale. Then there exists a version 
\[ (\tilde{M}_t)_{t \geq 0} \] 
(i.e. \( M_t = \tilde{M}_t \) on \( N(t)^c \) such that \( P(N(t)) = 0 \)), such that \( t \mapsto \tilde{M}_t(\omega) \) is càdlàg (i.e. is right-continuous and has left limits) for all \( \omega \in \Omega \).

**Proof.** Doob’s upcrossing lemma (cf. [vWW90]).

**Proposition 2.1.2** (Optional stopping theorem). Let \((M_t)_{t \geq 0}\) be a martingale and \( T \) be a stopping time. Then \((M_{t \wedge T})_{t \geq 0}\) is a martingale.

**Proof.** See [Rôc06, Proposition VIII.4.2, Corollary VIII.4.3].

**Proposition 2.1.3** (Doob’s inequality). Let \( p > 1 \). Then 
\[ \left\| \sup_{s \leq t} |M_s| \right\|_p \leq \frac{p}{p-1} \|M_t\|_p, \]

where \( \| \cdot \|_p = \| \cdot \|_{L^p} \). In particular, if 
\[ M^* := \sup_{t \geq 0} |M_t|, \]

then 
\[ \|M^*\|_p \leq \frac{p}{p-1} \sup_{t \geq 0} \|M_t\|_p. \]

**Proposition 2.1.4.** Let \( \left( M^{(n)}_t \right)_{t \geq 0}, n \in \mathbb{N}, \) be a sequence of martingales such that 
\[ M^{(n)}_t \xrightarrow{n \to \infty} M_t \quad \text{in } L^p, \forall t \geq 0, \forall p \geq 1. \]

Then
2. (Semi-)Martingales and Stochastic Integration

i. \((M_t)_{t \geq 0}\) is again a martingale in \(L^p\).

ii. If \(p > 1\) and \((\mathcal{F}_t)_{t \geq 0}\) such that all \(P\)-zero sets in \(\mathcal{F}_t\) are in \(\mathcal{F}_0\) and each \(\left( M^{(n)}_t \right)_{t \geq 0}\) has \(P\)-a.s. (right-) continuous ((càdlàg)) sample paths, then \((M_t)_{t \geq 0}\) has a (right-) continuous ((càdlàg)) \((\mathcal{F}_t\))-adapted version and

\[
M^{(n)}_t \xrightarrow{n \to \infty} M_t
\]

locally uniformly in \(t\) and in \(L^p\) and has a locally uniformly in \(t\) \(P\)-a.s. convergent subsequence.

Proof. i. Obvious.

ii. Fix \(t > 0\). Since \(M^n - M^m\) is a martingale, we have by Doob

\[
\left\| \sup_{s \leq t} |M^{(n)}_s - M^{(m)}_s| \right\|_p \leq \frac{p}{p-1} \left\| M^{(n)}_t - M^{(m)}_t \right\|_p.
\]

Since they are \(L^p\)-convergent, for some subsequence \((n_k)_{k \in \mathbb{N}}\)

\[
\sum_{k=1}^\infty \left\| \sup_{s \leq t} |M^{(n_{k+1})}_s - M^{(n_k)}_s| \right\|_p \leq \frac{p}{p-1} \sum_{k=1}^\infty \left\| M^{(n_{k+1})}_t - M^{(n_k)}_t \right\|_p < \infty.
\]

Therefore,

\[
P\left( \sum_{k=1}^\infty \sup_{s \leq t} \left| M^{(n_{k+1})}_s - M^{(n_k)}_s \right| < \infty \right) = 1.
\]

Hence, \(M^{(n_k)}_{t} \xrightarrow{k \to \infty} M_t\) uniformly on \([0,t]\) \(P\)-a.s. (cf. Proof of Riesz-Fischer!). Then up to a \(P\)-zero set \(M_s\) is \(\mathcal{F}_s\)-measurable, since so is each \(M^{(n_k)}_s\). But \(\mathcal{F}_s\) contains all \(P\)-zero sets in \(\mathcal{F}_0\). Therefore, \(M_s\) is \(\mathcal{F}_s\)-measurable.

\[\square\]

Alternative proof. Fix \(t \geq 0\). Consider the maps

\[
\Omega \ni \omega \mapsto \left( M^{(n)}_t(\omega) \right) \in (\mathcal{C}([0,t],[\mathbb{R}]),\|\cdot\|_\infty)
\]

and

\[
\int \left\| M^{(n)}(\omega) \right\|_\infty^p P(d\omega) < \infty.
\]

So, by assumption and Doob \(M^{(n)}\) is a Cauchy sequence in \(L^p(\Omega,\mathcal{F},P;\mathcal{C}([0,t],[\mathbb{R}]))\). Hence, \(M^{(n)} \to M_t\) in \(L^p(\Omega,\mathcal{F},P;\mathcal{C}([0,t],[\mathbb{R}]))\) by Riesz Fischer. But then also for all \(t M^{(n)}_t \to M_t\) in \(L^p\), hence, \(M_t = M_t\) \(P\)-a.s. So \(N\) is the required continuous version of \(M\).

\[\square\]

Remark 2.1.5 (Localization). Let \((M_t)_{t \geq 0}\) be a continuous local martingale (up to \(\infty\)) such that \(M_0 = 0\). Set

\[
R_n(\omega) := \inf \{ t > 0 | |M_t(\omega)| > n \} \xrightarrow{n \to \infty} \infty.
\]

Claim: For all \(n \in \mathbb{N}\) \((M_t \wedge R_n)_{t \geq 0}\) is a continuous bounded martingale.
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Proof. Let \((T_k)_{k \in \mathbb{N}}\) be a localizing sequence for \(M\), such that \(T_k \leq k\). (Otherwise consider \(T_k \wedge k\).) Then for \(s \leq t\) and \(n\) fixed

\[
E[M_{t \wedge R_n \wedge T_k} \wedge A_s] = E[M_{s \wedge R_n \wedge T_k} \wedge A_s] \quad \forall A_s \in \mathcal{F}_s.
\]

By Lebesgue, for all \(A_s \in \mathcal{F}_s\),

\[
E[M_{t \wedge R_n \wedge T_k} \wedge A_s] \xrightarrow{k \to \infty} E[M_{t \wedge R_n} \wedge A_s]
\]

and, since \(|M_{t \wedge R_n \wedge T_k}| \leq n\) for fixed \(n\),

\[
E[M_{s \wedge R_n \wedge T_k} \wedge A_s] \xrightarrow{k \to \infty} E[M_{s \wedge R_n} \wedge A_s].
\]

So, we get

\[
E[M_{t \wedge R_n} \wedge A_s] = E[M_{s \wedge R_n} \wedge A_s].
\]

\(\square\)

2.2. Quadratic Variation and Covariation for Continuous Local Martingales

Let \((M_t)_{t \geq 0}\) be a càdlàg martingale.

(a) First possible approach to stochastic integration:

Assume initially \(M_t \in L^2\) for all \(t \geq 0\). (If \(M_t \not\in L^2\), then localize.) By Jensen’s inequality \((M_t^2)_{t \geq 0}\) is a submartingale. Then we can show (cf. Doob-Meyer decomposition) that there exists a unique adapted process \(\langle M \rangle_t\) with \(\langle M \rangle_0 = 0\), increasing, right-continuous, predictable (see below) such that

\[
\langle M_t^2 - \langle M \rangle_t \rangle_{t \geq 0} \tag{*}
\]

is a martingale. Then, \(\langle M \rangle\) is the variance process of \(M\), i.e.

\[
E[(M_t - M_s - E[M_t - M_s])^2 | \mathcal{F}_s] \quad \text{(conditioned variance of } M_t - M_s \text{ given } \mathcal{F}_s)
\]

\[
= E[M_t^2 - M_s^2 | \mathcal{F}_s] \tag{*} \xrightarrow{=} E[\langle M \rangle_t - \langle M \rangle_s | \mathcal{F}_s].
\]

In [Röc06] we proved Doob-Meyer decomposition in discrete time. In continuous time for càdlàg martingales this is difficult (cf. [Kry]).

We are going to take an alternative approach that we prove Doob-Meyer decomposition for continuous martingales. In the stochastic integration theory below, we shall, however, allow càdlàg martingales simply assuming Doob-Meyer without proof.

(b) Our approach:

Let \(M\) be a continuous martingale. We will construct the process \(\langle M \rangle\) such that it pathwise coincides \(P\)-a.s. with the quadratic variation of \(M\) along \((\tau_n)_{n \in \mathbb{N}}\) of the previous chapter.

(This is only possible for continuous \(M\)!

Let \(M\) be a continuous local martingale and let \((\tau_n)_{n \in \mathbb{N}}\) be a sequence of partitions of \([0, \infty)\) such that

\[
|\tau_n| \xrightarrow{n \to \infty} 0
\]

and

\[
t^{(n)}_{N_n} \xrightarrow{n \to \infty} \infty.
\]

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Definition 2.2.1. Let

\[ V_t^n := \sum_{s \leq \tau_n} (M_{s'} - M_s)^2, \quad n \in \mathbb{N}, \ t \geq 0 \]

be the quadratic variation of \(M\) along \(\tau_n\) on \([0,t]\). Here, \(s'\) denotes the successor of \(s\) in \(\tau_n\).

Note that \(t \mapsto V_t^n\) is \(P\)-a.s. continuous for all \(n \in \mathbb{N}\).

Remark 2.2.2. Since \((a - b)^2 = a^2 - 2ab - b^2\),

\[ V_t^{(n)} = M_t^2 - M_0^2 - 2 \sum_{s \in \tau_n} \left(M_{s'} - M_s\right)
\]

\[ = M_t^2 - M_0^2 - 2 \sum_{s \in \tau_n, s \leq t} \left(M_{s'} - M_s\right). \]

is local martingale

Proposition 2.2.3. Assume that \(\mathcal{F}_0\) contains all \(P\)-zero sets. Let \(M\) be a continuous (for all \(\omega \in \Omega\), local martingale (up to \(\infty\)). Then there exists a (unique) continuous increasing \(\mathcal{F}_t\)-adapted process \(\langle M \rangle\) with \(\langle M\rangle_0 = 0\) such that

i. \(\langle M \rangle_t = \lim_{n \to \infty} V_t^{(n)}\) in \(P\)-measure locally uniformly in \(t \geq 0\),

(Convergence is even in \(L^2(\Omega, \mathcal{F}, P; \mathbb{C}([0,t], \mathbb{R}))\) for all \(t \geq 0\), if \(M\) is bounded.)

ii. \(M^2 - \langle M \rangle\) is a continuous local martingale,

iii. if \(M_t \in L^2\) for all \(t \geq 0\) and if \(M_t\) is a martingale, then \(M_t^2 - \langle M \rangle_t\) is a martingale. In particular,

\[ E[M_t^2] = E[M_0^2] + E[\langle M \rangle_t] \quad \forall t \geq 0. \]

Proof. 

i. Case 1: \(M\) is bounded (i.e. \(\sup_{t \geq 0, \omega \in \Omega} |M_t(\omega)| < \infty\).

Claim: \((V_t^{(n)})_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^2\) for all \(t \geq 0\).

Proof. Take \(m, n\) large so that \(t \leq t_{N_n}^{(n)}, t \leq t_{N_m}^{(m)}\). Define

\[ V_t^{(m,n)} := M_t^2 - M_0^2 - 2 \sum_{s \in \tau_n \cup \tau_m} \left(M_{s'} - M_s\right) \]

\[ = V_t^{(n,m)}. \]

Here, \(s'\) denotes the successor of \(s\) in \(\tau_n \cup \tau_m\). Then for \(V_t^{(n)}\) as in Remark 2.2.2

\[ V_t^{(m,n)} - V_t^{(n)} = -2 \sum_{s \in \tau_n} \left( \left( \sum_{u \leq s < u'} M_u (M_{s'} - M_u) - M_u (M_{u'} - M_u) \right) - \left( M_{s'} - M_s \right) \right), \]

where \(u'\) is the successor in \(\tau_n\) and \(s'\) the one in \(\tau_n \cup \tau_m\). Using the fact that

\[ M_{u'} - M_u = \sum_{u \leq s < u'} (M_{s'} - M_s), \]

we conclude that

\[ V_t^{(m,n)} - V_t^{(n)} = -2 \sum_{u \in \tau_n} \sum_{u \leq s < u'} (M_u - M_s)(M_{s'} - M_s). \]
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Hence,

\[
E \left[ \left( V_{t}^{(m,n)} - V_{t}^{(m)} \right)^2 \right] = 4 E \left[ \sum_{u \in \tau_n} \sum_{u' \leq s < u} (M_s - M_u)^2 (M_{s' \wedge t} - M_{s \wedge t})^2 \right]
\]

Note that all occurring mixed terms are zero by the martingale property (exercise!). Furthermore,

\[
E \left[ \left( V_{t}^{(m,n)} - V_{t}^{(m)} \right)^2 \right] \leq 4 E \left[ \Delta_m \sum_{s \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^2 \right],
\]

where

\[
\Delta_m := \sup_n \sup_{s,u} \left\{ (M_s - M_u)^2 \bigg| s \in \tau_n \cup \tau_m, u \in \tau_n, u \leq s \leq u', s, u \leq t \right\}
\]

is bounded in \( \omega \), since \( M(\omega) \) is bounded. Then, by Cauchy-Schwarz

\[
E \left[ \left( V_{t}^{(m,n)} - V_{t}^{(m)} \right)^2 \right] \leq 4 \left( E \left[ \Delta^2_m \right] \right)^{1/2} \left\| \sum_{s \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^2 \right\|_2.
\]

By Lebesgue’s dominated convergence theorem,

\[
\left( E \left[ \Delta^2_m \right] \right)^{1/2} = \|\Delta_m\|^2_m \xrightarrow{m \to \infty} 0,
\]

since \( M(\omega) \) is uniformly continuous on \([0,t]\) for all \( t \) and \( (M_t)_{t \geq 0} \) is bounded in \((\omega, t)\). But for \( c := \sup_{t,\omega} |M_t(\omega)| \) and for all \( n, m \in \mathbb{N} \)

\[
E \left[ \left( \sum_{s \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^2 \right)^2 \right] = E \left[ \sum_{s \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^4 \right]
\]

\[
+ 2 E \left[ \sum_{s \in \tau_n \cup \tau_m} \sum_{u \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^2 (M_{u' \wedge t} - M_{u \wedge t})^2 \right]
\]

\[
\leq 4 c^2 E \left[ \sum_{s \in \tau_n \cup \tau_m} (M_{s' \wedge t} - M_{s \wedge t})^2 \right] \quad \overset{=: S_1}{=} \\
+ 2 \sum_{s \in \tau_n \cup \tau_m} \sum_{u \in \tau_n \cup \tau_m} E \left[ (M_{s' \wedge t} - M_{s \wedge t})^2 \cdot E[(M_{u' \wedge t} - M_{u \wedge t})^2 | \mathcal{F}_{u \wedge t}] \right] \quad \overset{=: S_2}{=}
\]

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and

\[ S_1 = \sum_{s \in \tau_n \cup \tau_m} E[M_{s'}^2 - 2M_{s'}M_{s\wedge t} + M_{s\wedge t}^2] \]
\[ = \sum_{s \in \tau_n \cup \tau_m} E[M_{s'}^2 + M_{s\wedge t}^2 - 2E[M_{s'}M_{s\wedge t}|\mathcal{F}_t]] \]
\[ = \sum_{s \in \tau_n \cup \tau_m} E[M_{s'}^2 + M_{s\wedge t}^2 - 2E[M_{s'}M_{s\wedge t}|\mathcal{F}_t]] \]
\[ = \sum_{s \in \tau_n \cup \tau_m} E[M_{s'}^2 - M_{s\wedge t}^2] = E[M_t^2 - M_0^2] \]

In addition,

\[ S_2 = \sum_{u \in \tau_n \cup \tau_m} \sum_{s \leq u} E[(M_{s'} - M_{s\wedge t})^2], \]
\[ = \sum_{u \in \tau_n \cup \tau_m} \sum_{s \leq u} E[(M_{s'} - M_{s\wedge t})^2] \cdot E[M_{s'}^2 - M_{s\wedge t}^2]|_{\mathcal{F}_u \wedge t} \]
\[ \leq \sum_{u \in \tau_n \cup \tau_m} \sum_{s \leq u} E[(M_{s'} - M_{s\wedge t})^2] \cdot (M_{s'}^2 - M_{s\wedge t}^2) \]
\[ = E[(M_{s'} - M_{s\wedge t})^2] \cdot (M_{s'}^2 - M_{s\wedge t}^2) \)
\[ \leq 2c^2 E[(M_{s'} - M_{s\wedge t})^2|\mathcal{F}_{s\wedge t}]] = 2c^2 E[M_{s'}^2 - M_{s\wedge t}^2]. \]

Thus,

\[ E \left( \sum_{s \in \tau_n \cup \tau_m} (M_{s'} - M_{s\wedge t})^2 \right)^2 \]
\[ \leq 4c^2 \sum_{s \in \tau_n \cup \tau_m} E[(M_{s'} - M_{s\wedge t})^2] + 4c^2 \sum_{s \in \tau_n \cup \tau_m} E[M_{s'}^2 - M_{s\wedge t}^2] \]
\[ \leq 8c^2 E[M_t^2 - M_0^2] \leq 16c^4. \]

Alltogether, we obtain for all \( n, m \) large enough

\[ \left( E \left[ (V_t^{(n)} - V_t^{(m)})^2 \right] \right)^{1/2} \leq E \left[ (V_t^{(n)} - V_t^{(m)})^2 \right]^{1/2} + \left( V_t^{(m,n)} - V_t^{(m)} \right)^2 \]
\[ \leq 2 \| \Delta_n \|_{1/2} 4c^2 + 2 \| \Delta_m \|_{1/2} 4c^2 n,m \rightarrow \infty 0 \]

and the claim is proved. \( \square \)

Let \( V_t := L^2|_{n \rightarrow \infty} V_t^{(n)} \), \( t \geq 0 \). Define

\[ Y_t^{(n)} := 2 \sum_{s \in \tau_n} M_s(M_{s'} - M_{s\wedge t}). \]

Then

\[ Y_t^{(n)} = M_t^2 - M_0^2 - V_t^{(n)} \xrightarrow{n \rightarrow \infty} M_t^2 - M_0^2 - V_t :=: Y_t \quad (\text{locally uniformly in } t \text{ in } L^2). \]

Hence, by Proposition 2.1.4 \( Y_t^{(n)} \) is a local martingale (cf. 2.2.2) and \( Y_t \) has a \( P \)-a.s. continuous version and, therefore, \( V_t \) has a continuous version \( \langle M \rangle \) and \( \langle M \rangle \) is \( (\mathcal{F}_t) \)-adapted since all \( P \)-zero sets in \( \mathcal{F} \) are in \( \mathcal{F}_0 \), hence, every \( \mathcal{F}_t \). Finally, we get

\[ E \left[ \sup_{s \leq t} \left| V_s^{(n)} - \langle M \rangle_s \right|^2 \right] \xrightarrow{n \rightarrow \infty} 0. \]
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Case 2: \( M \) as in the assertion.
Without loss of generality \( M_0 \equiv 0 \). (Otherwise consider \( (M_t - M_0)_{t \geq 0} \).) Let \( R_k \) be as in 2.1.5, i.e.
\[
R_k := \inf \{ t > 0 \mid |M_t| > k \}
\]
and therefore, \( R_k \) is a stopping time. Define
\[
M_t^k := M_{t \wedge R_k}, \quad t \geq 0.
\]
\( M_t^k \) is a bounded martingale. Hence, by case 1 there exists \( V_t^k := \langle M^k \rangle_t \). Let \( (V_t^k)^{(n)}, t > 0 \), be the corresponding approximations along \( (\tau_n) \), then there exists a subsequence \( (n_l)_{l \in \mathbb{N}} \) and \( \Omega_0 \in \mathcal{F}, P(\Omega_0) = 1 \) such that for all \( k \in \mathbb{N} \) (cf. Proposition 2.1.4(ii))
\[
(V_s^k)^{(n_l)}(\omega) \xrightarrow{l \to \infty} \langle M^k \rangle_s(\omega)
\]
locally uniformly for \( s \in [0, t] \) and for \( P\)-a.e. \( \omega \in \Omega_0 \). But
\[
1_{\{t \leq R_k\}} V_{t}^{(n_l)} = 1_{\{t \leq R_k\}} V_{t \wedge R_k}^{(n_l)} = 1_{\{t \leq R_k\}} (V_t^{k})^{(n_l)} \xrightarrow{l \to \infty} 1_{\{t < R_k\}} \langle M^k \rangle_t.
\]
Hence, we can (well-)define
\[
\langle M \rangle_t := \begin{cases} 
\langle M^k \rangle_t(\omega), & \text{with } k \in \mathbb{N} \text{ such that } \omega \in \{t \leq R_k\} \cap \Omega_0, \\
0, & \text{otherwise}
\end{cases}
\]
and is independent of \( k! \) Then \( \langle M \rangle \) is \( P\)-a.s. continuous and \( (\mathcal{F}_t) \)-adapted.
Furthermore, we fix \( k = k(\delta) \) large enough, such that
\[
P[\{R_k \geq t\}] = P[R_k < t] \leq \delta.
\]
Then
\[
P \left( \sup_{0 \leq s \leq t} \left| V_s^{(n)} - \langle M \rangle_s \right| \geq \varepsilon \right) \]
\[
= P \left( \sup_{0 \leq s \leq t} \left| V_s^{(n)} - \langle M \rangle_s \right| \geq \varepsilon, R_k \leq t \right) + P \left( \sup_{0 \leq s \leq t} \left| V_s^{(n)} - \langle M \rangle_s \right| \geq \varepsilon, R_k > t \right)
\]
\[
\leq P(\langle M^k \rangle_t \leq t) + P \left( \sup_{0 \leq s \leq t} \left| (V_s^k)^{(n_l)} - \langle M^k \rangle_s \right| \geq \varepsilon, R_k > t \right)
\]
\[
\leq P(\langle M^k \rangle_t \leq t) + P \left( \sup_{0 \leq s \leq t} \left| (V_s^k)^{(n_l)} - \langle M^k \rangle_s \right| \geq \varepsilon \right).
\]
But by the first part of the proof
\[
P \left( \sup_{0 \leq s \leq t} \left| (V_s^k)^{(n_l)} - \langle M^k \rangle_s \right| \geq \varepsilon \right) \xrightarrow{n \to \infty} 0.
\]
Therefore,
\[
\lim_{n \to \infty} P \left( \sup_{0 \leq s \leq t} \left| V_s^{(n)} - \langle M \rangle_s \right| \geq \varepsilon \right) \leq P[R_k \leq t] < \delta.
\]
It remains to show that \( t \mapsto \langle M \rangle_t \) is increasing \( P\)-a.s.. But for fixed \( t \geq 0 \) we have that
\[
V_t^{(n)} = \sum_{s \in \tau_n, s \leq t} (M_s - M_s')^2 + N_t^{(n)} \xrightarrow{n \to \infty} \langle M \rangle_t,
\]
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where the sum is increasing in \( t \) and

\[
N_t^{(n)} := (M_t - M_{\delta_t^{(n)}})^2 \xrightarrow{n \to \infty} 0
\]

and

\[
\delta_t^{(n)} := \sup\{s \in \tau_n | s \leq t\}.
\]

Hence, (i) is completely proved.

ii. Continuity is clear. We know, that \( P \)-a.s.

\[
M_t^2 - \langle M \rangle_t = (M_t^k)^2 - \langle M^k \rangle_t = Y_t + M_0^2
\]

is a martingale. Hence, \( M_t^2 - \langle M \rangle_t \) is a local martingale for \( T \) up to \( \bar{\infty} \).

(Alternative proof by using 1.4.3 (i).)

iii. Without loss of generality \( M_0 = 0 \). We have to show that

\[
E[\langle M \rangle_T] = E[M_T^2]
\]

for all bounded stopping times \( T \). By the monotone convergence theorem, we get that

\[
E[\langle M \rangle_T] = \lim_{n \to \infty} E[(\langle M \rangle_{T \wedge R_n})^2] = \lim_{n \to \infty} E[M_{T \wedge R_n}^2] \leq \lim_{n \to \infty} E[M_{T \wedge R_n}].
\]

On the other hand, by the submartingale property, \( \lim_{n \to \infty} E[M_{T \wedge R_n}^2] \leq E[M_T^2] \), so

\[
E[\langle M \rangle_T] = E[M_T^2].
\]

Finally, by assumption,

\[
E[M_T^2] \leq \left[ M_{\sup_{\omega \in \Omega} T(\omega)}^2 \right] < \infty,
\]

so that

\[
E[M_T^2 - \langle M \rangle_T] = 0.
\]

\( \Box \)

Remark 2.2.4.  

i. One can drop the assumption that “all \( P \)-zero sets in \( \mathcal{F} \) are in \( \mathcal{F}_0 \)”, but one only gets that the particular version of \( \langle M \rangle \) of \( V \) is only right-continuous and adapted. But it is continuous only for \( \omega \in N^c \), with a \( P \)-zero set \( N \in \mathcal{F}_0 \).

ii. Since convergence in probability implies \( P \)-a.s. convergence of a subsequence, it follows by Proposition 2.2.3(i) that for some subsequence \( (n_k)_k \in \mathbb{N} \)

\[
P(V_{s}^{(n_k)} \xrightarrow{k \to \infty} \langle M \rangle_s \text{ locally uniformly on } [0, t], \forall t \geq 0) = 1.
\]

iii. Since \( \langle M \rangle \) is exactly the pathwisely defined continuous quadratic variation process of \( M \) in Chapter I, we can apply all results form Chapter I for \( P \)-a.e. \( \omega \in \Omega \) fixed, i.e.

\[
\langle M(\omega) \rangle = \langle M \rangle(\omega).
\]

Corollary 2.2.5. Assume \( M_0 \equiv 0 \). Then

\[
M_t^2 - \langle M \rangle_t = 2 \int_0^t M_s \, dM_s.
\]

Hence, the continuous local martingale \( M_t^2 - \langle M \rangle_t \) is an \( \text{Itô-integral} \) (cf. Corollary 1.4.3(i)).
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**Corollary 2.2.6.** $P$-a.s. the paths of a continuous local martingale are either constant or of unbounded variation.

Proof. Apply Corollary 1.4.3(ii).

**Corollary 2.2.7.** $(M)$ is the unique increasing continuous adapted process such that $(M)_0 = 0$ and $M^2 - (M)$ is a continuous local martingale. In particular, $(M)$ is (up to a $P$-zero set in $\mathcal{F}$) independent of the chosen sequence of partitions $(\tau_n)$ in 2.2.1.

Proof. Let $A, B$ be two such processes. Then $M^2 - A, M^2 - B$ are continuous local martingales. Hence, $B - A$ is a continuous local martingale (with respect to $(\mathcal{F}_t)$ since $A, B$ are $(\mathcal{F}_t)$-adapted) of bounded variation. Therefore, by Corollary 2.2.6

$$B - A = c = B_0 - A_0 = 0 \quad P\text{-a.s..}$$

**Definition 2.2.8.** (cf. 1.3.1, 1.3.2) Let $M, N$ be continuous local martingales. Then define the covariation process of $M, N$ by

$$\langle M, N \rangle_t := \frac{1}{2} ((\langle M \rangle_t + \langle N \rangle_t) - \langle M \rangle_t - \langle N \rangle_t).$$

**Remark 2.2.9.** Since $M + N$ is a continuous local martingale, $(M + N)$ exists as a continuous process by Chapter I and polarization. Therefore,

$$\langle M, N \rangle = \lim_{n \to \infty} \sum_{s \in \tau_n} (M_{s^\land t} - M_{s^\land t})(N_{s^\land t} - N_{s^\land t}).$$

**Proposition 2.2.10.** Let $M, N$ be continuous local martingales. $(M, N)$ is uniquely determined by the following:

i. Its paths are ($P$-a.s.) continuous and of bounded variation and $(M, N)_0 = 0$.

ii. $M \cdot N - (M, N)$ is a continuous local martingale. In particular, $(M, N) \equiv 0$ if and only if $M \cdot N$ is a local martingale.

Proof. Analogous to the case $M = N$ in Corollary 2.2.7 (or use polarization.).

**Lemma 2.2.11.** Let $M, N$ be a continuous local martingale. Let $G(= G_s(\omega), s \geq 0, \omega \in \Omega)$ and $H$ be $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable. Then

$$\left| \int_0^t H_s(\omega) G_s(\omega) d\langle M, N \rangle_s(\omega) \right| \leq \left( \int_0^t H_s^2 d\langle M \rangle_s(\omega) \right)^{\frac{1}{2}} \left( \int_0^t G_s^2 d\langle N \rangle_s(\omega) \right)^{\frac{1}{2}}.$$

In particular, we obtain (by Cauchy) the Kunita-Watanabe inequality

$$E \left[ \left| \int_0^t H_s(\omega) G_s(\omega) d\langle M, N \rangle_s \right| \right] \leq E \left[ \int_0^t H_s^2 d\langle M \rangle_s \right]^{\frac{1}{2}} E \left[ \int_0^t G_s^2 d\langle N \rangle_s(\omega) \right]^{\frac{1}{2}}.$$

Proof. Exercise (cf. [RW87, Vol II, p.50]).
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2.3. Construction of stochastic integrals on Hilbert space

Fix a probability space \((\Omega, \mathcal{F}, P)\) with right-continuous filtration \((\mathcal{F}_t)\) such that all \(P\)-zero sets in \(\mathcal{F}_t\) are in \(\mathcal{F}_0\). We want to define

\[ H \cdot M := \int_0^H M_s \, ds \text{ as a martingale} \]

for càdlàg \((\mathcal{F}_t)\)-martingales and most general \(H\), where càdlàg means right-continuous for all \(\omega \in \Omega\) with left limits \(P\)-a.s. (The latter is automatic by [vWW90, p. 47, 3.25, 3.26].) \(M_s\) is called \textit{integrator process} and \(H_s\) \textit{integrand process}.

"Admissible integrators" are given by

\[ \mathcal{M}^2 := \mathcal{M}^2(\Omega, \mathcal{F}, P) := \left\{ M \mid M \text{ is a càdlàg martingale, } M_0 = 0, \|M\|^2 := \sup_{t > 0} E[M_t^2] < \infty \right\}. \]

By Proposition 2.2.3 we know that the Doob-Meyer decomposition holds, that is: For all \(M \in \mathcal{M}^2\) there exists a process \(\langle M \rangle\) such that it is the unique predictable, right-continuous increasing, adapted process and that \(M_0 \equiv 0\) and \(M^2 - \langle M \rangle\) is a martingale. We proved that this \(\langle M \rangle\) exists and is unique if \(M \in \mathcal{M}^2_c\). For the general case see [Kry] or [vWW90, p. 130, Cor. 6.6.3].

Remark 2.3.1. i. Define \(\mathcal{F}_\infty := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)\). Clearly, \(M_t \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)\) for all \(M \in \mathcal{M}^2\) and by the \(\mathcal{L}^2\)-martingale convergence theorem we have for any càdlàg martingale such that \(M_0 = 0\):

\[ M \in \mathcal{M}^2 \iff M \text{ is uniformly integrable and } \exists M_\infty := \lim_{t \to \infty} M_t \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P). \]

In this case

\[ M_t = E[M_\infty | \mathcal{F}_\infty], \quad t > 0. \]

So \((M_t)_{t \in [0, \infty]}\) is a martingale.

ii. Define with norm \(\|\cdot\|\) an inner product on \(\mathcal{M}^2\) by polarization:

\[ (M, N) := \frac{1}{4}(\|M + N\|^2 - \|M - N\|^2), \quad \forall M, N \in \mathcal{M}. \]

It remains to check that it is really an inner product. (This could be done by checking that \(\|\cdot\|\) on \(\mathcal{M}^2\) satisfies

\[ \|M + N\|^2 + \|M - N\|^2 = 2\|M\|^2 + 2\|N\|^2 \quad \text{(parallelogram identity)} \]

and using J. von Neumann’s theorem.)

iii. Let \(M \in \mathcal{M}^2\). Since \(\mathcal{M}^2\) is a submartingale and because of (i) we have

\[ E[M_\infty^2] = \lim_{t \to \infty} E[M_t^2] = \sup_{t \geq 0} E[M_t^2] = \|M\|^2. \]

Hence, for \(N \in \mathcal{M}^2\), by polarization

\[ E[M_\infty N_\infty] = \frac{1}{4} (E[M_\infty + N_\infty]^2 - E[M_\infty - N_\infty]^2) \]

\[ = \frac{1}{4} (\|M + N\|^2 - \|M - N\|^2) = (M, N). \]

Therefore, \((\cdot, \cdot)\) is really an inner product with corresponding norm \(\|\cdot\|\).
2.3. Construction of stochastic integrals on Hilbert space

**Proposition 2.3.2.**  
i. \( \mathcal{M}^2 \) is a Hilbert space and  
\[
\langle M, N \rangle = E[M_\infty N_\infty] = E[(M, N)_\infty],
\]  
where  
\[
(M, N)_\infty := \lim_{t \to \infty} (M, N)_t.
\]

ii. \( \mathcal{M}_c^2 \) is a closed subspace of \( \mathcal{M}^2 \).

**Proof.**  
i. \((\mathcal{M}^2, \| \cdot \|)\) is an inner product space (Pre-Hilbert space) by Remark 2.3.1. Furthermore, it is complete by Proposition 2.1.4(ii) and because the norm is complete in  
\( L^2(\Omega, \mathcal{F}, P; C([0, T], \mathbb{R})) \).

Finally,  
\[
\|M\|_2 = E[M_\infty^2] = \lim_{t \to \infty} E[M_t^2] = \lim_{t \to \infty} E[(M)_t] \overset{BL}\approx E[\lim_{t \to \infty} (M)_t] =: (M)_\infty.
\]

So, by polarization the last assertion follows.

ii. Apply Proposition 2.1.4(ii).

Now we define stochastic integrals with \( M \in \mathcal{M}^2 \) as integrators, but first for elementary functions:

**Definition 2.3.3.** Define \( \mathcal{E} \) to be the set of all processes \( H \) which are of the following form:  
For \( t \geq 0, \omega \in \Omega \)  
\[
H_t(\omega) := \sum_{i=0}^{n-1} h_{t_i}(\omega) 1_{[t_i, t_{i+1})}(t),
\]
where \( n \in \mathbb{N}, 0 = t_0 < t_1 < \ldots < t_n < \infty \) and \( h_{t_i} \) are \( \mathcal{F}_{t_i} \)-measurable bounded “elementary (predictable) adapted processes”. For \( M \in \mathcal{M}^2 \) and \( H \) as above define  
\[
\int_0^t H_s \, dM_s := \sum_{i=0}^{n-1} h_{t_i}(M_{t_{i+1} \wedge t} - M_{t_i \wedge t}).
\]

An easy exercise shows that this is independent of the representation of \( H \)! Set  
\[
(H \cdot M)_t := \int_0^t H_s \, dM_s, \quad t \geq 0.
\]

Then \( H \cdot M \) is called the **stochastic integral** of \( H \) with respect to \( M \).

**Lemma 2.3.4.** Let \( H \in \mathcal{E} \).

i. \( H \cdot M \in \mathcal{M}^2 \) and if \( M \in \mathcal{M}_c^2 \), then \( H \cdot M \in \mathcal{M}_c^2 \).

ii.  
\[
\langle H \cdot M \rangle_t = \int_0^t H_s^2 \, d\langle M \rangle_s \quad \forall t \geq 0
\]
\[
= \sum_{i=0}^{n-1} h_{t_i}^2 \langle (M)_{t_{i+1} \wedge t} - (M)_{t_i \wedge t} \rangle.
\]

In particular,  
\[
\|H \cdot M\|^2 = E \left[ \int_0^\infty H_s^2 \, d\langle M \rangle_s \right].
\]
2. (Semi-)Martingales and Stochastic Integration

Note that in this respect we always set
\[ \int_0^t H_s^2 \, d\langle M \rangle_s := \int_{[0,t]} H_s^2 \, d\langle M \rangle_s. \]

**Proof.**

i. By definition

- if \( M \) is a cadlag (continuous) martingale, then \( H \cdot M \) is càdlàg (continuous),
- \( H \cdot M \) is adapted,
- \( (H \cdot M)_0 = 0 \).

Furthermore, since \( M \) is a martingale,
\[
\sup_{t \geq 0} E[(H \cdot M)^2_t] = \sup_{t \geq 0} E \left[ \sum_{i=0}^{n-1} h_{t_i}^2 (M_{t_{i+1} \wedge t} - M_{t_i \wedge t})^2 \right] \\
= \sup_{t \geq 0} \sum_{i=0}^{n-1} E \left[ h_{t_i}^2 (M_{t_{i+1} \wedge t} - M_{t_i \wedge t})^2 \right] \\
\leq 2 \sum_{i=0}^{n-1} \sup_{1 \leq i \leq n} \|h_{t_i}\| \sup_{t \geq 0} E \left[ h_{t_i}^2 \left( M_{t_{i+1} \wedge t}^2 + M_{t_i \wedge t}^2 \right) \right] \\
\leq 2 \sum_{i=0}^{n-1} 2 \| M \| < \infty.
\]

It remains to show the martingale property of \( H \cdot M \):
Let \( T \) be a bounded stopping time. Then,
\[
E[(H \cdot M)_T] = \sum_{i} E \left[ h_{t_i} 1_{\{t_i < T\}} \left( M_{t_{i+1} \wedge T} - M_{t_i} \right) \right] \\
= \sum_{i} E \left[ h_{t_i} 1_{\{t_i < T\}} E \left| M_{t_{i+1} \wedge T} - M_{t_i} \right| \bigg| F_{t_i} \right] \\
= 0.
\]

Thus, \( H \cdot M \) is a martingale.

ii. By uniqueness of the Doob-Meyer decomposition it is enough to show that
\[
(H \cdot M)^2_t - \int_0^t H_s^2 \, d\langle M \rangle_s
\]
is a martingale. (Then \( (H \cdot M)_t = \int_0^t H_s^2 \, d\langle M \rangle_s \).) Let \( T \) be a bounded stopping time. Then, by defining
\[
\Delta_t M := M_{t+1 \wedge T} - M_t
\]
and since all mixed terms disappear, we get that

\[
E[(H \cdot M)^2] = \sum_{i,j} E \left[ h_{t_i} 1_{\{t_i < T\}} h_{t_j} 1_{\{t_j < T\}} (\Delta_i M)(\Delta_j M) \right] \\
= \sum_i E \left[ h_{t_i}^2 1_{\{t_i < T\}} E[(\Delta_i M)^2|\mathcal{F}_{t_i}] \right] \\
= \sum_i E \left[ h_{t_i}^2 1_{\{t_i < T\}} E[M_{t_{i+1}\wedge T}^2 - M_{t_i\wedge T}^2|\mathcal{F}_{t_i}] \right] \text{ on } \{t_i < T\} \\
= \sum_i E \left[ h_{t_i}^2 1_{\{t_i < T\}} E[(M)_{t_{i+1}\wedge T} - (M)_{t_i\wedge T}|\mathcal{F}_{t_i}] \right] \text{ on } \{t_i < T\} \\
= E \left[ \int_0^T H_s^2 \, d\langle M \rangle_s \right].
\]

In particular,

\[
\|H \cdot M\|^2 = \sup_t E[(H \cdot M)^2] = \sup_{t} E \left[ \left( \int_0^t H_s^2 \, d\langle M \rangle_s \right)^{B_{Levi}} \right] = E \left( \int_0^\infty H_s^2 \, d\langle M \rangle_s \right).
\]

We now want to consider the “Isometry property”:

Let \( \bar{\Omega} = \Omega \times (0, \infty) \), \( \bar{\omega} := (\omega, t) \), \( \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}((0, \infty)) \). \( \omega \mapsto (M)_t(\omega) \) is \( \mathcal{F} \)-measurable for fixed \( t \) and for all \( \omega \in \Omega \) \( t \mapsto (M)_t(\omega) \) is right-continuous, positive, increasing, \( (M)_0(\omega) = 0 \). Therefore, there exists a unique, positive measure

\[
d\langle M \rangle(\omega)dt = (M)(\omega, dt).
\]

Thus, \( (M)(\omega, dt) \) defines a transition kernel on \( \bar{\Omega} = \Omega \times (0, \infty) \) and it induces a measure

\[ P_M(d\bar{\omega}) := P(d\omega) \otimes (M)(\omega, dt) \]

on \( (\bar{\Omega}, \bar{\mathcal{F}}) \). Explicitly,

\[
P_M(A) = E \left[ \int_0^\infty 1_A(\cdot, t) \, d\langle M \rangle_t(\cdot) \right], \quad A \in \bar{\mathcal{F}}.
\]

Note that this is not a probability measure, but finite since \( M \in \mathcal{M}_c^2 \). In particular, if we denote

\[
E_M[\cdot] = \int \cdot \, dP_M,
\]

then

\[
E_M[H^2] = E \left[ \int_0^\infty H_s^2 \, d\langle M \rangle_s \right], \quad (2.3.1)
\]

which is the \( L^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M) \)-norm of \( H \in \mathcal{E} \). (\( H : \bar{\Omega} \to \mathbb{R}, H(\omega, t) = H_t(\omega) \).) Note that the map

\[
\mathcal{E} \to \mathcal{M}_c^2,
\]

\[
H \mapsto H \cdot M
\]

is obviously linear and by Lemmas 2.3.4(ii) and (2.3.1), an isometry from \( \mathcal{E} \subset L^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M) \) to \( \mathcal{M}_c^2, \| \cdot \| \)). Therefore, there exists a unique isometric extension on the closure of \( \mathcal{E} \) in \( L^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M) \) denoted by

\[
\bar{\mathcal{E}} := \mathcal{E}^M
\]

(which depends on \( M \)).
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Definition 2.3.5. For \( H \in \bar{\mathcal{E}}^M \), let \( H.M \) define the uniquely determined element in \( \mathcal{M}^2_{(c)} \) with

\[
\lim_{n \to \infty} \| H^n M - H.M \| = 0
\]

for every sequence \( (H^n)_{n \in \mathbb{N}} \subset \mathcal{E} \) which converges in \( L^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M) \) against \( H \).

Automatically, we have

\[
\| H.M \|^2 = EP_M[H^2].
\]

But we also have the analogon of 2.3.4(ii), namely:

Proposition 2.3.6. Let \( H \in \bar{\mathcal{E}}^M, M \in \mathcal{M}^2_{(c)} \), and therefore, \( H.M \in \mathcal{M}^2_{(c)} \). Then

\[
\langle H.M \rangle_t = \int_0^t H^2_s \, d\langle M \rangle_s, \quad t \geq 0.
\]

Proof. Let \( T \) be a bounded stopping time. Then

\[
E[(H.M)_T^2] = \lim_{n \to \infty} E[(H^n M)_T^2],
\]

since

\[
E[(H.M)_T - (H^n M)_T]^2 \leq E \left[ \sup_t (H.M - H^n M)_t^2 \right] \leq 4 \| H.M - H^n M \|^2 \to 0.
\]

Hence,

\[
E[(H.M)_T^2] = \lim_{n \to \infty} E[(H^n M)_T^2] = \lim_{n \to \infty} E \left[ \int_0^T (H^n)_s^2 \, d\langle M \rangle_s \right] = E \left[ \int_0^T H^2_s \, d\langle M \rangle_s \right],
\]

since \( H^n \to H \) in \( L^2(P_M) \), so, \( 1_{[0,T]} H^n \to 1_{[0,T]} H \). Therefore, \( (H.M)_T^2 - \int_0^T H^2_s \, d\langle M \rangle_s \) is a martingale and the assertion follows by the uniqueness of the Doob-Meyer decomposition.

In our next step we want to determine the size of \( \bar{\mathcal{E}}^M \). We want to characterize admissible integrands in dependence of \( M \).

Definition 2.3.7.

\[
\mathcal{P} := \sigma(H = (H_t)_{t \geq 0} \mid H \text{ is a left-continuous, adapted process})
\]

\[
\mathcal{P}_{exercise} := \sigma(H = (H_t)_{t \geq 0} \mid H \text{ is a continuous, adapted process})
\]

is called \( \sigma \)-algebra (on \( \Omega \times [0, \infty) \)) of predictable sets. A process \( H = (H_t)_{t \geq 0} \) is called predictable, if it is \( \mathcal{P} \)-measurable.

Let \( \mathcal{P}_M := \text{the completion of } \mathcal{P} \text{ with respect to } P_M \).
2.3. Construction of stochastic integrals on Hilbert space

Remark 2.3.8. Let \( \tau_n \) be a subdivision of \((0, \infty)\) and

\[
P_n := \sigma(A_t \times [t, t^{'}) \mid t \in \tau_n, t' \text{ is the successor of } t \text{ in } \tau_n, \text{ and } A_t \text{ is } \mathcal{F}_t \text{-measurable})
\]

\( P_n \) is called the \( \sigma \)-algebra of predictable rectangles (with respect to \( \tau_n \)).

i. If \( |\tau_n| \xrightarrow{n \to \infty} 0 \) and \( t_N n \xrightarrow{n \to \infty} \infty \), then

\[
P = \sigma\left( \bigcup_n P_n \right).
\]

ii. If \( H \in \mathcal{P}_n \)-measurable and bounded if and only if

\[
H_t(\omega) = \sum_{t_i \in \tau_n} h_{t_i}(\omega) 1_{[t_i, t_{i+1})}(t),
\]

where \( h_{t_i} \) is \( \mathcal{F}_{t_i} \)-measurable and bounded.

Proof. i. By definition we have \( P_n \subset \mathcal{P} \). Hence, \( \sigma\left( \bigcup_n P_n \right) \subset \mathcal{P} \). To show that \( \mathcal{P} \subset \sigma\left( \bigcup_n P_n \right) \), let \((H_t)_{t > 0}\) be a left-continuous adapted process. It suffices to show that \( H \) is \( \sigma\left( \bigcup_n P_n \right) \)-measurable. \( H \) is adapted and left-continuous. Therefore, for \( H \in \mathcal{P} \)

\[
H_t(\omega) = \lim_{n \to \infty} \sum_{s \in \tau_n} H_s(\omega) 1_{[s, s')}(t) \text{, where } H_s \text{ is } \mathcal{F}_s \text{-measurable and bounded.}
\]

Hence,

\[
P = \sigma\left( \bigcup_n P_n \right).
\]

ii. Obvious by a monotone class argument.

\( \square \)

Proposition 2.3.9.

\[
\mathcal{E}^M = \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M).
\]

In particular, for all \( H \in \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M) \) there exists

\[
H.M = \int H \, dM.
\]

(Note that any predictable process is adapted.)

In particular,

\[
\mathcal{M}^2_M := \{H.M \mid H \in \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M)\}
\]

is a closed subspace of \( \mathcal{M}^2 \).

Proof. Clearly, \( \mathcal{E}^M \subset \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M) \). To prove the dual inclusion, let \( H \in \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M) \). Without loss of generality \( H \geq 0 \). (Otherwise consider \( H^+ \) and \( H^- \).) Since for \( m \in \mathbb{N} \)

\[
H^{(m)} := (m \wedge H) \cdot 1_{\Omega \times [0, m]} \xrightarrow{m \to \infty} H,
\]

hence, \( H^{(m)} \) is in \( \mathcal{L}^2(\Omega, \mathcal{P}_M, P_M) \). So we may assume \( H \geq 0 \), bounded and \( H \) as \( H \cdot 1_{\Omega \times [0, m]} \) for some \( m \in \mathbb{N} \). Now let \( \tau_n \) be a sequence of partitions as in Remark 2.3.8(i). Then by the \( \mathcal{L}^p \)-martingale convergence theorem

\[
H_n := E_{P_M|\Omega \times [0, m]}[H|P_n] \xrightarrow{n \to \infty} E_{P_M|\Omega \times [0, m]} \left[ H \mid \sigma\left( \bigcup_n P_n \right) \right].
\]
2. (Semi-)Martingales and Stochastic Integration

By Remark 2.3.8(i) we get

\[
\lim_{n \to \infty} E_{P_M}[H|P_n] = E_{P_M|\Omega \times [0,m]}[H|\mathcal{P}] = E_{P_M|\Omega \times [0,m]}[H|P_M] = H.
\]

Since \( H_n \) is \( P_n \)-measurable, we have by Remark 2.3.8(ii), that

\[
H_n \in \mathcal{E}.
\]

Proposition 2.3.10. i. Let \( M \in \mathcal{M}_c^2 \) and, therefore, \( t \mapsto \langle M \rangle_t \) is continuous. Then

\[
\bar{\mathcal{E}} = L^2(\bar{\Omega}, \mathcal{O}_M, P_M) \quad \text{(2.3.9)}
\]

where the optional \( \sigma \)-algebra \( \mathcal{O} \) is defined by

\[
\mathcal{O} := \sigma(\{H \mid H \text{ is an adapted càdlàg process on } \Omega \times [0,\infty]\})
\]

and \( \mathcal{O}_M \) its completion with respect to \( P_M \).

ii. If \( M \in \mathcal{M}_c^2 \) and \( d\langle M \rangle \) is absolutely continuous with respect to Lebesgue measure \( dt \), then

\[
\bar{\mathcal{E}}^M = \{L^2(\Omega, \bar{\mathcal{F}}, P_M) \mid H \text{ has an adapted version}\}.
\]

Proof. i. We have that \( \mathcal{O} \supset \mathcal{P} \) (see exercises). Hence, also \( \mathcal{O}_M \supset \mathcal{P}_M \). It remains to prove \( \mathcal{O}_M \subset \mathcal{P}_M \), let \( H \) be càdlàg and adapted. Consider for \( \omega \in \Omega \)

\[
U_\omega := \{t \mid s \mapsto H_s(\omega) \text{ is discontinuous in } t\}.
\]

Then \( U_\omega \) is countable, since \( H \) is càdlàg (exercise). Define for \( \omega \in \Omega \)

\[
H_t^-(\omega) := \lim_{s \uparrow t} H_s(\omega) =: H_{t^-}(\omega).
\]

Then \( H^- \) is left-continuous. Hence, it is \( \mathcal{P} \)-measurable and

\[
U_\omega = \{t > 0 \mid H_t(\omega) \neq H_t^-(\omega)\}.
\]

Note that \( \{(\omega, t) \in \Omega \times [0,\infty) \mid H_t(\omega) \neq H_t^-(\omega)\} \in \mathcal{O} \subset \bar{\mathcal{F}} \). Since \( \langle M \rangle \) is continuous, we have

\[
\int_0^\infty 1_{U_\omega}(t) \, d\langle M \rangle_t = 0 \quad \text{for } P\text{-a.e. } \omega \in \Omega.
\]

But then by Fubini

\[
E_{P_M}[1_{\{H \neq H^\pm\}}] = \int \int_{[0,\infty]} 1_{u(\omega)}(t) \, d\langle M \rangle_t(\omega) P(d\omega) = 0.
\]

Hence, \( H \) is \( P\)-a.e. equal to a predictable process, therefore, \( H \) is \( \mathcal{P}_M \)-measurable and (i) is proved.

ii. One has to prove that \( \bar{\mathcal{F}} \subset \mathcal{P}_M \) (cf. [CW90, p.60] or [vWW90, p.124]).

\qed
Remark 2.3.11 (Extending stochastic integration via localization). Note that, if $M$ is a Brownian motion, then $M \notin \mathcal{M}^2$. Therefore, we have to stop. Define 

$$\mathcal{M}^2_{\text{loc}} := \{ M | \exists \text{ a sequence } T_n \not\to \infty \text{ of stopping times such that } M^{T_n} := M_{T_n \land \cdot} \in \mathcal{M}^2 \forall n \in \mathbb{N} \}.$$ 

Then for all $H \in L^2(\bar{\Omega}, \mathcal{F}, P)$

$$H.M = \int H \, dM$$

is defined via localization. We have to check that

- the definition is consistent on $\{ T_n = T_{n+1} \}$,

- $H.M \in \mathcal{M}^2_{\text{loc}}$ (We need Lemma 2.4.3 below).

Remark 2.3.12 (Semi-martingales as integrals). If $A$ is predictable and of bounded variation and $M \in \mathcal{M}^2_{\text{loc}}$ we can define for $H \in L^2(\bar{\Omega}, \mathcal{F}, P)$

$$\int H \, d(M + A) := \int H \, dM + \int H \, dA,$$

where $M + A$ is a semi-martingale and $\int H \, dA$ is a pathwise defined Lebesgue-Stieltjes-integral. Axiomatic considerations show that, if $\int H \, d(M + A)$ is required to have reasonable properties, then this cannot be generalized. As a conclusion, reasonable stochastic integrators are semi-martingales (Dellacherie-Mokobodzki-Bichteler) (cf. [vWW90]).
2. (Semi-)Martingales and Stochastic Integration

2.4. Characterization of $H \cdot M$ in $\mathcal{M}^2$

Fix $M \in \mathcal{M}^2$, $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, \mathcal{P}_M)$.

**Proposition 2.4.1.** $H \cdot M$ is the unique element $L \in \mathcal{M}^2$ such that

i. $d(L, N) = H \ d(M, N)$ $\forall N \in \mathcal{M}^2$,

that is

$$\langle L, N \rangle_t = \int_0^t H_s \ d\langle M, N \rangle_s := \int_{[0,t]} H_s \ d\langle M, N \rangle_s, \quad \forall M, N \in \mathcal{M}^2, \forall t > 0,$$

respectively, such that the following weaker property holds:

ii. $E[L_{\infty} N_{\infty}] = E\left[\int_0^\infty H_s \ d\langle M, N \rangle_s \right] = E\left[\int_{1}^{\infty} H_s \ d\langle M, N \rangle_s \right]$ for all $N \in \mathcal{M}^2$ (i.e., $N_{\infty} \in \mathcal{L}^2(\bar{\Omega}, \mathcal{F}_\infty, \mathcal{P})$).

**Remark 2.4.2.**

i. Because of $E[L_{\infty} N_{\infty}] = E[(L, N)_{\infty}]$ (cf. 2.3.2(i)), it follows that in 2.4.1, (i) implies (ii).

ii. By 2.4.1(i) particularly we have

$$d(H \cdot M, M) = H \ d(M, M) = H \ d(M),$$

hence,

$$d(H \cdot M) = d(H \cdot M, H \cdot M) = H \ d(M, H \cdot M) = H \ d(H \cdot M, M) = H^2 \ d(M).$$

That is (cf. 2.3.6)

$$\langle H \cdot M \rangle_t = \int_0^t H_s^2 \ d\langle M \rangle_s.$$  

**Proof of 2.4.1.** (a) Uniqueness if (ii) holds:

Let $L, L' \in \mathcal{M}^2$ such that both satisfy 2.4.1(ii). Then

$$E[(L_{\infty} - L'_{\infty}) N_{\infty}] = 0 \quad \forall N_{\infty} \in \mathcal{L}^2(\bar{\Omega}, \mathcal{F}_\infty, \mathcal{P}).$$

Hence, $L_{\infty} - L'_{\infty} \perp N_{\infty}$ in $\mathcal{L}^2$, so $L_{\infty} = L'_{\infty}$ $P$-a.s.. Therefore,

$$L_t = E[L_{\infty}|\mathcal{F}_t] = L'_t \quad P$-a.s..$$

(b) $L := H \cdot M$ satisfies (ii):

**Step 1:** Assume for fixed $s > 0$ that

$$H_t(\omega) := h_s(\omega)1_{[s,\infty)}(t),$$

where $h_s$ is bounded and $\mathcal{F}_s$ -measurable. Note that $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, \mathcal{P}_M)$. Therefore, by isometry there exists an $L = H \cdot M$ and we have

$$L_{\infty} = (H \cdot M)_{\infty} = \int_0^\infty h_s 1_{[s,\infty)}(t) \ dM_t$$

$$= \lim_{N \to \infty} \int_0^N h_s 1_{[s,\infty)}(t) \ dM_t$$

$$= \lim_{N \to \infty} h_s (M_N - M_s) = h_s (M_{\infty} - M_s)$$

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Thus, for all $N_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$,
\[
E[L_\infty N_\infty] = E[h_s(M_\infty - M_s)N_\infty]
\]
by Step 2:
\[
\lim_{n \to \infty} E[(H^{(n)})_\infty N_\infty] = E[H_\infty N_\infty] = E[h_s(M_\infty - M_s)N_\infty]
\]
by Step 2.

**Step 2:** Let $H \in \mathcal{E}$. Then (ii) is clear by Step 1 and linearity.

**Step 3:** (ii) holds for all $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$:

Let $H^{(n)} \in \mathcal{E}$, $n \in \mathbb{N}$, $H^{(n)} \to H$ in $\mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ ($= \bar{\mathcal{E})}$. Then $H^{(n)} \cdot M \to H \cdot M$ in $(\mathcal{M}^2, ||.||)$.

Therefore, by 2.3.1(iii) for all $N \in \mathcal{M}^2$
\[
E[(H \cdot M)_\infty N_\infty] = \lim_{n \to \infty} E[(H^{(n)} \cdot M)_\infty N_\infty]
\]
\[
\leq E\left[\int_0^\infty H_s \cdot d\langle M, N \rangle_s\right],
\]
because by Kunita-Watanabe-inequality (cf. 2.2.11) we have
\[
E\left[\int_0^\infty (H^{(n)} - H) \cdot 1 \; d\langle M, N \rangle_s\right] \leq \left(E\left[\int_0^\infty (H^{(n)} - H)^2 \; d\langle M \rangle_s\right]\right)^{\frac{1}{2}} \cdot \left(E[\langle N \rangle_\infty]\right)^{\frac{1}{2}} \quad n \to \infty.
\]
This proves (ii).

(c) $L := H \cdot M$ satisfies (i):

By Proposition 2.2.10 we have to show that $L_t N_t - \int_0^t H_s \cdot d\langle M, N \rangle_s$, $t \geq 0$, is a martingale.

Let $T$ be a bounded stopping time. Then, since $(L_t)_{t \in [0, \infty]}$ is a martingale, for $N^T_T := N_{T/\lambda}$
\[
\]
by (ii)
\[
\leq E\left[\int_0^\infty H_s \cdot d\langle M, N^T \rangle_s\right],
\]
2.4.3 below $\leq E\left[\int_0^T H_s \cdot d\langle M, N \rangle_s\right]$. 

\[\square\]

**Lemma 2.4.3.** Let $M, N \in \mathcal{M}^2$ and $T$ be a stopping time. Then
\[
\langle M, N^T \rangle_t = \langle M, N \rangle_{T/\lambda} \quad \forall t \geq 0.
\]

**Proof.** Exercise.  \[\square\]
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**Corollary 2.4.4.** Let $M \in \mathcal{M}^2$, $H_1, H_2 \in \mathcal{L}^2(\Omega, \mathcal{P}_M, \mathcal{P}_M)$ such that $H_1 \cdot H_2 \in \mathcal{L}^2(\Omega, \mathcal{P}_M, \mathcal{P}_M)$. Then

\[ H_1(H_2 \cdot M) = (H_1 \cdot H_2) \cdot M, \]

i.e.

\[ \int_0^\infty H_1(s) \, dH_2 \cdot M_s = \int_0^\infty H_1(s) \, d\left( \int_0^s H_2(r) \, dM_r \right) = \int_0^\infty H_1 H_2 \, dM. \]

**Proof.** Let $L := H_1(H_2 \cdot M)$. Then by 2.4.1 for all $N \in \mathcal{M}^2$

\[ d\langle L, N \rangle = H_1 \, d\langle H_2 \cdot M, N \rangle = H_1 H_2 \, d\langle M, N \rangle, \]

and again by 2.4.1,

\[ H_1(H_2 \cdot M) = L = (H_1 \cdot H_2) \cdot M. \]

**Corollary 2.4.5.** Let $T$ be a stopping time. Then $(H \cdot M)^T = H^{1[0,T]} \cdot M$, i.e.

\[ (H \cdot M)^T = \int_0^T H_s \cdot 1_{[0,T]}(s) \, dM_s. \]

In particular $(H \cdot M)^T \in \mathcal{M}_M^2$, hence, $\mathcal{M}_M^2$ is “stopping stable”, that is, $N \in \mathcal{M}_M^2$ implies $N^T \in \mathcal{M}_M^2$ for all stopping times $T$.

**Proof.** Let $N \in \mathcal{M}^2$. Then

\[
E \left[ \left( (H \cdot M)^T \right)_\infty \right]^{2.3.2(i)} = E \left[ \left( (H \cdot M)^T \right)_\infty \right]^{2.4.3} = E \left[ (H \cdot M)_T \right]^{2.4.1(i)} = E \left[ \int_0^\infty H_s \cdot 1_{[0,T]}(s) \, d\langle M, N \rangle_s \right].
\]

Hence, this assumption follows by 2.4.1(ii).

---

### 2.4.1. Orthogonality in $\mathcal{M}^2$

**Definition 2.4.6.** Let $M, N \in \mathcal{M}^2$.

i. $M, N$ are called weakly orthogonal if $E[M_\infty N_\infty] = 0$ (i.e. orthogonal in $(\mathcal{M}^2, \| \cdot \|)$).

ii. $M, N$ are called strongly orthogonal, denoted by $M \perp N$, if $\langle M, N \rangle = 0$.

**Remark 2.4.7.**

i. $M \perp N \Leftrightarrow M \cdot N$ is a martingale $\Leftrightarrow E[M_T N_T] = 0$ for all bounded stopping times $T$. If $M_t \xrightarrow{t \to \infty} M_\infty$ and $N_T \xrightarrow{T \to \infty} N_\infty$, then

\[ E[M_\infty N_\infty] = 0, \]

i.e. $M, N$ are weakly orthogonal.

ii. $M \perp N \Leftrightarrow \mathcal{M}_M^2 \perp \mathcal{M}_N^2$ (since by 2.4.1 $d\langle H \cdot M, \tilde{H} \cdot N \rangle = H \tilde{H} \, d\langle M, N \rangle$).

iii. Weak orthogonality in $\mathcal{M}^2 (\subseteq \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P))$ is just orthogonality in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$.
iv. Since \((\mathcal{M}^2, \|\cdot\|)\) is a Hilbert space and \(\mathcal{M}_M^2\) is a closed linear subspace of \(\mathcal{M}^2\), for all \(N \in \mathcal{M}^2\) there exists an \(H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, \mathcal{P}_M)\) and \(L \in \mathcal{M}^2\) such that

\[
N = H \cdot M + \underbrace{L}_{\in \mathcal{M}^2_M} \quad \in \mathcal{M}^2 \quad \text{and } L \text{ is weakly orthogonal to } \mathcal{M}_M^2.
\]

But in (iv) we have more because of the following proposition.

**Proposition 2.4.8.** Let \(M, L \in \mathcal{M}^2\), \(L\) weakly orthogonal to \(\mathcal{M}_M^2\). Then \(L \perp \mathcal{M}_M^2\) (since \(\mathcal{M}_M^2\) is stopping stable). Hence, \(L\) is weakly orthogonal to \(\mathcal{M}_M^2\) if and only if \(L \perp \mathcal{M}_M^2\).

**Proof.** Because of 2.4.7(i), it remains to show that \(E[L_TM_T^\infty] = 0\) for all bounded stopping times \(T\). But by 2.4.5 (stopping stability) we have \(M^T \in \mathcal{M}_M^2\). Hence,

\[
0 = E[L_\infty M^T_\infty] = E[L_\infty M_T^\infty] = E[L_TM_T^\infty].
\]

\(\square\)

**Corollary 2.4.9** (Kunita-Watanabe-decomposition). i. Let \(M, N \in \mathcal{M}^2\). Then there exist an unique \(L \in \mathcal{M}^2\) and an unique \(H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, \mathcal{P}_M)\) such that \(L \perp \mathcal{M}_M^2\) and \(N = H \cdot M + L\).

ii. Suppose \(\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)\) separable (e.g. true, if \(\mathcal{F}_\infty\) is countably generated). Then there exist \(M_i \in \mathcal{M}_C^2\), \(i \in \mathbb{N}\), such that

\[
M_i \perp M_j, \quad i \neq j,
\]

and for all \(N \in \mathcal{M}^2\) there exist \(H^i \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}, \mathcal{P}_M)\), \(i \in \mathbb{N}\), (uniquely determined by \(M_i\)) such that

\[
N = \sum_{i=1}^{\infty} H^i M_i,
\]

that is,

\[
\mathcal{M}^2 = \bigoplus_{i=1}^{\infty} \mathcal{M}_M^2 M_i
\]

and

\[
\mathcal{M}_M^2 M_i \perp \mathcal{M}_M^2 M_j, \quad \text{for } i \neq j.
\]

**Proof.** i. 2.4.7(iv) and 2.4.8.

ii. By assumption and 2.4.7(iii) \((\mathcal{M}^2, \|\cdot\|)\) is separable. So, we can apply (i) and Gram-Schmidt orthogonalization procedure (cf. [Wei87],[RS80]).

\(\square\)

**Corollary 2.4.10.** Let \(F \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)\), \(M \in \mathcal{M}^2\). Then there exist uniquely determined \(H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, \mathcal{P}_M)\) and \(L \in \mathcal{M}^2\), \(L \perp M\), such that, if \(\mathcal{F}_0 = \{\emptyset, \Omega\},\)

\[
F = E[F|\mathcal{F}_0] + \int_0^\infty H_s \, dM_s + L_\infty = E[F] + \int_0^\infty H_s \, dM_s + L_\infty.
\]

Here, \(\int_0^\infty H_s \, dM_s\) denotes the stochastic integral with respect to \(M\).
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Proof. Let $F_t := E[F | F_t] - E[F | F_0]$. Since $F_t$ is a martingale, by 2.1.1 there exists a càdlàg version $(\tilde{F}_t)_{t \geq 0}$ of $(F_t)_{t \geq 0}$ (i.e. $P[\{F_t = \tilde{F}_t\}] = 1 \forall t \geq 0$). Then $(\tilde{F}_t) \in \mathcal{M}^2$. Hence, by 2.4.9(i) there exist unique $L \in \mathcal{M}^2$, $L \perp M$, and $H \in L^2(\Omega, \mathcal{F}_t, P_M)$ such that

$$\tilde{F}_t = (H \cdot M)_t + L_t, \ t \geq 0.$$ 

In particular, (since $\tilde{F}_\infty = F - E[F | F_0]$) we have

$$F = E[F | F_0] + \int_0^\infty H_s \, dM_s + L_\infty.$$ 

\qed
2.5. Itô’s Representation Theorem

Let \((\Omega, \mathcal{F}, P)\) be a probability space with filtration \((\mathcal{F}_t)\) and let \((W_t)_{t \geq 0}\) be a (real-valued) Wiener process on \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\), that is, \((W_t)\) is a continuous \((\mathcal{F}_t)\)-adapted process such that, for all \(s \leq t\), \(W_t - W_s\) is independent of \(\mathcal{F}_s\) and \(N(0, t - s)\) distributed. Let

\[ F_t^W := \sigma\{W_s | 0 \leq s \leq t\} \]

and \(F_{t+}^W\) the right-continuous version of \(F_t^W\).

**Remark 2.5.1.** Compared with our definition of Brownian motion for a Wiener process the increment \(W_t - W_s\) is independent of the larger \(\sigma\)-algebra \(\mathcal{F}_s \supseteq F_t^W\). In this space of \((\mathcal{F}_t)\)-martingales belonging to \(L^2(\Omega, \mathcal{F}, P)\) the Kunita-Watanabe decomposition (cf. 4.9(i) with \(M = W\)) has a particularly simple form:

**Theorem 2.5.2** (Itô’s representation theorem). Let \(M = (M_t)_{t \geq 0} \subseteq L^2(\Omega, \mathcal{F}, P)\) be a right-continuous martingale with respect to \((F_{t+}^W)\) (hence, with respect to \(\sigma(F_{t+}^W, \sigma\{N \in \mathcal{F} | P(N) = 0\})\), \(t \geq 0\)). Then

\[ M_t = M_0 + \int_0^t H_s \, dW_s, \quad t \geq 0, P\cdot a.s., \]

where \(H\) is \((\mathcal{F}_t)\)-adapted and \(H \cdot 1_{[0,\tau]} \in L^2(\Omega, \tilde{\mathcal{F}}, P_W)\) for all \(\tau > 0\) (and through this it is uniquely determined). In particular, \(M\) has \(P\cdot a.s.\) continuous sample paths and \(M^2 = M^2_W\) (where \(M^2\) is defined as in the previous section with respect to \(\mathcal{F}_t = (F_{t+}^W)^P, t \geq 0\)) and

\[ M^2_W := \left\{ \left( \int_0^t H_s \, dW_s \right)_{t \geq 0} \bigg| H \cdot 1_{[0,\tau]} \in L^2(\Omega, \tilde{\mathcal{F}}, P_W) \quad \forall \tau \geq 0, \sup_{t \geq 0} \left\| \int_0^t H_s \, dW_s \right\|_{L^2} < \infty \right\}. \]

**Proof.** Without loss of generality \(M_0 = 0\). Fix \(k \in \mathbb{N}\) and set

\[ F_{k+}^W := (F_{k+}^W)^P, \quad M_t^{(k)} := M_{t \wedge k}, \quad W_t^{(k+1)} := W_{t \wedge (k+1)}, \quad t \geq 0. \]

Let \((M^n)_{n \in \mathbb{N}} \subseteq L^2(\Omega, \mathcal{F}_{k+}^W, P)\), such that

\[ \lim_{n \to \infty} M^n = M_k = M_{k}^{(k)} \quad \text{in} \quad L^2(\Omega, \mathcal{F}_{k+}^W, P), \]

and let \((M^n_t)_{t \geq 0}\) be a right-continuous modification of \((E[M^n_t | F_{t+}^W])_{t \geq 0}\) for fixed \(t\), (which always exists by 2.1.4) such that

\[ \|M_t^n\|_{L^2} \left( = \sup_{t, \omega} |M_t^n(\omega)| \right) \leq \sup_{\omega} |M^n(\omega)| < \infty. \]

In particular, \(M^n \in \mathcal{M}^2\) for all \(n\). Hence, by Kunita-Watanabe decomposition (cf. 2.4.9(i)) there exist \(H^n \in L^2(\Omega, \tilde{\mathcal{F}}, F_{m+}^{(n+1)}(W))\) adapted and \(N^n \in \mathcal{M}^2\), \(N^n \perp \mathcal{M}^2_{W^{(k+1)}}\), such that

\[ M^n = H^n W^{(k+1)} + N^n, \quad \forall n \in \mathbb{N}. \]

We would like to prove that \(N^n = 0\) for all \(n \in \mathbb{N}\). Fix \(n_0 \in \mathbb{N}\) and set \(N := N_{n_0}\).

**Step 1:** **Claim:** Let \(N \in \mathcal{M}^2\), \(N \perp W^{(k+1)}\). If \(N\) is bounded, then \(N_t = 0\) for all \(t < k + 1\).

**Proof of Claim.** Let \(c := \sup_{t, \omega} |N_t(\omega)| < \infty\). Define

\[ D := 1 + \frac{N_{\infty}}{2c}. \]
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Then $D \geq \frac{1}{2}$ and

$$E[D] = 1 + \frac{1}{2c} E[N_{\infty}] = 1 + \frac{1}{2c} E[N_0] = 1.$$  

Thus, $\tilde{P} := DP$ is a probability measure equivalent to $P$.

**Claim’**: $(W_t)_{t \leq k+1}$ is a Brownian motion under $\tilde{P}$.

Suppose claim is true. Then the finite dimensional distributions of $(W_t)_{t \leq k+1}$ under $P$ and $\tilde{P}$ are the same. Hence, by Radon-Nikodym, 

$$P = \tilde{P} \text{ on } \mathcal{F}_{k+1}^W.$$  

Therefore, $P = \tilde{P}$ on $\mathcal{F}_{t}^W$ for all $t < k+1$. Thus, $E[D|\mathcal{F}_{t}^W] = 1$ for all $t < k+1$. Hence,

$$N_t = E[N_{\infty}|\mathcal{F}_{t}^W] = 0, \quad \forall t < k+1.$$  

**Proof of Claim’**:

i. $(W_t^{(k+1)})$ is a martingale under $\tilde{P}$, because for all bounded stopping times $T$ we have 

$$E_{\tilde{P}}[W_T^{(k+1)}] = E_P[W_T^{(k+1)}D] = E_P[W_T^{(k+1)}] + \frac{1}{2c} E_P[W_T^{(k+1)}N_\infty].$$

By 2.4.7 $W_T^{k+1}N_T$ is a martingale, since $W^{k+1} \perp N$. Hence, 

$$A = E_P[W_T^{k+1}N_T] = E_P[W_0^{(k+1)}N_0] = 0.$$  

Thus, 

$$E_{\tilde{P}}[W_T^{(k+1)}] = 0.$$  

ii. For $P$-a.e. $\omega \in \Omega$ we have 

$$\langle W \rangle_T(\omega) = t \quad \forall t \geq 0.$$  

Hence, for $\tilde{P}$-a.e. $\omega \in \Omega$ 

$$\langle W \rangle_T(\omega) = t \quad \forall t \geq 0.$$  

Thus, 

$$\langle W_t^{(k+1)} \rangle = t \wedge (k + 1) \quad \forall t \geq 0.$$  

Therefore, by (i), (ii) and Levy’s characterization (cf. Proposition 1.5.1) $(W_t)_{t \geq k+1}$ is a Brownian motion under $P = DP$ (up to $k+1$).

**Step 2**: Define 

$$T_n := \inf \left\{ t > 0 \quad \left| \int_0^t H_s^{n_0} dW_s^{(n+1)} \right| > n \right\} \wedge (k + 1).$$  

Then $T_n$ are $(\mathcal{F}_{t}^N)$-stopping times such that $T_n \not\nearrow (k + 1)$ as $n \to \infty$. We know that $N_{T_n} (= N_{\Lambda T_n}) \perp M_{W^{(k+1)}}^2$ (cf. 2.4.3, 2.4.7(i) and (ii)). $(N_{\Lambda T_n} = \int_0^t 1_{[0,T_n]}(s) dN_s \overset{2.4.3}{=} \int_0^{t\wedge T_n} dN_s)$

Furthermore, 

$$|N_t^{T_n}| = |N_{t\wedge T_n}| \leq |M_t^{n_0}| + \left| \int_0^{t\wedge T_n} H_s^{n_0} dW_s^{(k+1)} \right| \leq \|M^{n_0}\|_{\infty} + n.$$
Hence,
\[
\sup_{t, \omega} |N_t^n(\omega)| \leq \|M^n_0\|_\infty + n.
\]

So, we can apply Step 1 to conclude that \( N_t^n = 0 \), for all \( t < k + 1 \) and for all \( n \in \mathbb{N} \). Letting \( n \to \infty \), we get \( N_t = 0 \) for all \( t < k + 1 \) \( \mathbb{P} \)-a.s. (since \( N \) is right-continuous and the zero set does not depend on \( t \)). Therefore, \( N^n = 0 \) for all \( n \) and \( t < k + 1 \). But for all \( n, m \in \mathbb{N} \)

\[
E_{P_{W^{k+1}}} \left[ (H^n \cdot 1_{[0,t]} - H^m \cdot 1_{[0,t]})^2 \right] = E_P \left[ \left( (H^nW^{k+1})_{k+1} - (H^mW^{k+1})_{k+1} \right)^2 \right] \\
= E \left( \int_0^{k+1} (H^n_s - H^m_s)^2 \, ds \right) \\
= E \left[ (M^n_{k+1} - M^m_{k+1})^2 \right] \\
= E \left[ (M^n_k - M^m_k)^2 \right] \xrightarrow{n,m \to \infty} 0.
\]

This is true, since \( P \)-a.e.

\[
M^n_k = E \left[ M^n |\mathcal{F}^W_{k+1} \right] = M^n \xrightarrow{n \to \infty} M_k \quad \text{in} \quad \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}).
\]

Therefore, \( (H^n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_{W^{k+1}}) \).

Hence, there exists \( H \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_{W^{k+1}}) \) such that \( H^n \to H \) in \( \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, \mathbb{P}) \). In particular, we can assume \( H \) to be adapted. Furthermore, for \( 0 \leq t \leq k + 1 \),

\[
M_t = \lim_{n \to \infty} M^n_t = \lim_{n \to \infty} (H^nW^{k+1})_t = (HW^{k+1})_t = \int_0^t H_s \, dW^{(k+1)}_s \quad \text{\( P \)-a.s.}\.
\]

Since \( M \) is a right-continuous \( (\mathcal{F}^W_{t+}) \)-martingale, by the following Lemma \( M \) has \( P \)-a.s. continuous sample paths.

**Lemma 2.5.3.** Every \( (\mathcal{F}^W_{t+}) \)-adapted local martingale \( (M_t) \in \mathcal{M}^2_{\text{loc}} \) is \( P \)-a.s. continuous.

**Proof.** Without loss of generality \( M \in \mathcal{M}^2 \) (localization). It suffices to consider the case

\[
M_t := E[F|\mathcal{F}^W_{t+}], \quad t \geq 0,
\]

where

\[
F := \prod_{i=1}^n f_i(W_{t_i}), \quad f_i \in \mathcal{C}_b(\mathbb{R}) \text{ uniformly continuous, } 0 \leq t_1 < \ldots < t_n < \infty \quad (\star)
\]

and to prove that \( (M_t) \) has a continuous modification. This is enough because \( M_t = E[M_\infty|\mathcal{F}^W_{t+}] \) (since \( M \in \mathcal{M}^2 \)) and \( F \) of type \( (\star) \) are dense in \( \mathcal{L}^2(\Omega, \mathcal{F}^\infty, \mathbb{P}) \).

(Exercise, by a monotone class argument: By 2.1.4 \( M \) has a continuous \( (\mathcal{F}^W_{t+})P \)-adapted version. Define

\[
V := \{ G \in \mathcal{L}^2(\Omega, \mathcal{F}^W, \mathbb{P}) | \exists F_n \text{ of type } (\star) \text{ such that } F_n \to G \text{ in } \mathcal{F}^W, \mathbb{P} \}
\]

and prove that \( V = \mathcal{L}^\infty(\Omega, \mathcal{F}^W, \mathbb{P}) \).)

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Let \( t \geq 0 \) and \( i \in \{1, \ldots, n\} \) such that \( t \in [t_i, t_{i+1}] \) where \( t_{n+1} := +\infty \). Then (cf. section of Markov property of Brownian motion),

\[
M_t = E[F|\mathcal{F}_{t+}^W] = \prod_{j \leq i} f_j(W_{t_j}) E\left[ \prod_{j > i} f_j(W_{t_j}) | \mathcal{F}_{t+}^W \right] = \prod_{j \leq i} f_j(W_{t_j}) E_{W_t} \left[ \prod_{j > i} f_j(W_{t_j-I}) \right] = \prod_{j \leq i} f_j(W_{t_j}) p_{t_{i+1}-} (f_{i+2}(p_{t_{i+1}-t_{i+2}} f_{i+2} \cdots p_{t_{n-1}-f_n}) \cdots)(\omega_t) \quad \text{P-a.s.,}
\]

where

\[
p_s f(x) := \frac{1}{\sqrt{2\pi s}} \int f(y) e^{-\frac{1}{2s}(x-y)^2} \, dy.
\]

So, we have to prove that \( p_{t_{i+1}-t} g(\omega_t) \) is P-a.s. continuous in \( t \). Set

\[
C_{b,u}(\mathbb{R}) := \{ \varphi \in C_b(\mathbb{R}) | \varphi \text{ uniformly continuous} \}.
\]

i. Let \( f \in C_{b,u}(\mathbb{R}) \). Take \( x, y \in \mathbb{R} \). Then

\[
|p_t f(x) - p_t f(y)| \leq \frac{1}{\sqrt{2\pi s}} \int |f(x+z) - f(y+z)| e^{-\frac{|z|^2}{2s}} \, dz.
\]

Let \( \varepsilon > 0 \). Then there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \). So, \( |p_t f(x) - p_t f(y)| \leq \varepsilon \) and \( p_s f \in C_{b,u}(\mathbb{R}) \).

ii. Let \( f \in C_{b,u}(\mathbb{R}) \). Then

\[
\lim_{s \to 0} \|p_s f - f\|_\infty = 0.
\]

**Proof.** Let \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \). Since

\[
f(x) = \frac{1}{\sqrt{2\pi s}} \int f(x) e^{-\frac{|y|^2}{2s}} \, dy
\]

we have

\[
|p_s f(x) - f(x)| \\
\leq \frac{1}{\sqrt{2\pi s}} \int |f(x+y) - f(y)| e^{-\frac{|y|^2}{2s}} \, dy \\
\leq \frac{1}{\sqrt{2\pi s}} \int_{|y| < \delta} |f(x+y) - f(x)| e^{-\frac{|y|^2}{2s}} \, dy + \frac{1}{\sqrt{2\pi s}} \int_{|y| > \delta} |f(x+y) - f(x)| e^{-\frac{|y|^2}{2s}} \, dy \\
\leq \varepsilon + 2 \|f\|_\infty \frac{1}{\sqrt{2\pi s}} \int_{|y| > \delta} e^{-\frac{|y|^2}{2s}} \, dy.
\]

Therefore,

\[
\limsup_{s \to 0} \|p_s f - f\|_\infty \leq \varepsilon + 2 \|f\|_\infty \limsup_{s \to 0} \frac{1}{\sqrt{2\pi s}} \int_{|y| > \delta} e^{-\frac{|y|^2}{2s}} \, dy \overset{y' = \frac{y}{\sqrt{s}}}{=} \varepsilon.
\]
2.5. Itô’s Representation Theorem

iii. 
\[ p_t(p_s f) = p_{t+s} f, \]
in short, 
\[ p_t p_s = p_{t+s}. \]

Proof. Exercise (by Fourier transform and by use of 
\[ p_s f(x) = (g_s * f)(x), \]
where \( g_s(z) = \frac{1}{\sqrt{2\pi s}} e^{-|z|^2/2s} \) and recall \( \hat{g}_s(\xi) = e^{-s\xi^2/2} \).

iv. For \( t < t_{i+1} \) we have 
\[ \lim_{s \searrow t} \| p_{t_{i+1} - s} g - p_{t_{i+1} - t} g \|_\infty = 0 \] (right-continuity).

Proof. Let \( h > 0 \) such that \( s := t + h < t_{i+1} \). Then
\[
\| p_{t_{i+1} - t - h} g - p_{t_{i+1} - t} g \|_\infty \overset{(iii)}{=} \| p_{t_{i+1} - t - h} (g - p_h g) \|_\infty \\
\leq \| p_{t_{i+1} - t - h} \|_\infty \cdot \| g - p_h g \|_\infty \\
\overset{p_{t_{i+1}} = 1}{\leq} \| g - p_h g \|_\infty \overset{s \to t}{\longrightarrow} 0 \quad \text{by (ii)}. \]

v. Assume \( t_i < t \) and let \( h > 0 \) such that \( s := t - h > t_i \). Then 
\[ \lim_{s \searrow t} \| p_{t_{i+1} - s} g - p_{t_{i+1} - t} g \|_\infty = 0. \]

Proof. Let \( h > 0 \) such that \( s := t - h > t_i \). Then
\[
\| p_{t_{i+1} - t + h} g - p_{t_{i+1} - t} g \|_\infty \overset{(iii)}{=} \| p_{t_{i+1} - t} (p_h g - g) \|_\infty \leq \| p_h g - g \|_\infty \overset{h \to 0}{\longrightarrow} 0. \]

vi. Consider \( t := t_i \):
Let \( h > 0 \)
\[
p_{t_i - (t_i - h)}(f_{i}p_{t_{i+1} - t_i}f_{i+1} \cdots p_{t_n - t_{n-1}}f_{n}) = p_h(f_{i}p_{t_{i+1} - t_i}f_{i+1} \cdots p_{t_n - t_{n-1}}f_{n}) \overset{h \to 0}{\longrightarrow} \tilde{g} \]
unequally in \( x \). Hence, also continuous in \( t = t_i \) from the left uniformly in \( x \).

Summary: The function \( G \) defined by
\[ G_t(x) := \prod_{j \leq i} \int f_j(x)p_{t_{i+1} - t}(f_{i+1}p_{t_{i+2} - t_{i+1}}f_{i+2} \cdots p_{t_n - t_{n-1}}f_{n}(x) \ldots) \quad \text{for } t \in [t_i, t_{i+1}] \]
is continuous in \( t \geq 0 \) uniformly in \( x \).

vii. \( t \mapsto G(t, \omega_t) \) is continuous \( P \)-a.s..
2. (Semi-)Martingales and Stochastic Integration

Proof. Take $|G(t, \omega_t) - G(s, \omega_s)| \leq |G(t, \omega_t) - G(s, \omega_t)| + |G(s, \omega_s) - G(s, \omega_s)|$. Fix $s > 0$. Then the right hand side is dominated by

$$\lim_{t \to s} \|G(t, \cdot) - G(s, \cdot)\|_\infty + \lim_{t \to s} \|G(s, \omega_t) - G(s, \omega_s)\|_\infty,$$

because $t \mapsto \omega_t$ is continuous and $G$ is continuous in $x$.

Corollary 2.5.4 (cf. 2.5.2). Let $F \in \mathcal{L}^2(\Omega, \mathcal{F}^{W}_{t_0+}, P)$, $t_0 > 0$ fixed. Such $F$ are sometimes called “Wiener functional”. Then there exists an $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P_{W^{t_0+1}})$, $(\mathcal{F}^W_t)$-adapted, such that

$$F - E[F] = \int_0^{t_0} H_s \, dW_s.$$

Proof. (cf. 2.4.10) Let $F^W_t$ be a càdlàg version of $E[F \mid \mathcal{F}^{W}_{t_0+}] - E[F \mid \mathcal{F}^{W}_{0+}]$, $t \geq 0$.

Then $(\mathcal{F}_t) \in \mathcal{M}^2$. Hence, by 2.5.2,

$$F - E[F \mid \mathcal{F}^{W}_{t_0+}] = E[F \mid \mathcal{F}^{W}_{t_0+}] - E[F \mid \mathcal{F}^{W}_{0+}] = F^W_{t_0} = \int_0^{t_0} H_s \, dW_s.$$

The uniqueness is also clear by martingale property.

Example 2.5.5 (Special case: the canonical Model). Let $(\Omega, \mathcal{F}, P)$ be the (classical) Wiener space, i.e. $\Omega = C([0,1], \mathbb{R})$, $\mathcal{F}$ its Borel $\sigma$-algebra and $P$ the Wiener measure. Define $X_t(\omega) := \omega(t)$, $t \in [0,1]$, $\omega \in \Omega$, hence $X$ is Brownian motion,

$$\mathcal{F}_t := \sigma(\{X_r \mid r \leq s \leq 1\}) P\text{-zero sets in } \mathcal{F},$$

and

$$\mathcal{F} = \bigcup_{0 < t \leq 1} \mathcal{F}_t.$$

Then, by 2.5.4

$$F \in \mathcal{L}^2(\Omega, \mathcal{F}, P) \Rightarrow F = E[F] + \int_0^1 H_s \, dX_s.$$

Example 2.5.6. Consider the situation of 2.5.5 the Wiener functional

$$F := \int_0^1 X_t \, dt.$$

Because of $E[F] = \int_0^1 E(X_t) \, dt = 0$,

$$F = \int_0^1 H_s \, dX_s$$

for some $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P_W)$.

Identification of $H$:

Let

$$M_t := E[F \mid \mathcal{F}_t] = \int_0^t E[X_s \mid \mathcal{F}_t] \, ds + \int_t^1 E[X_s \mid \mathcal{F}_t] \, ds = \int_0^t X_s \, ds + X_t(1-t) \in \mathcal{M}^2.$$

Claim: $H_t = 1 - t$. 

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Proof. In view of 2.4.1(i) it is enough to show
\[ \langle M, X \rangle = \int_0^t H_s \, d\langle X \rangle_s = \int_0^t (1 - s) \, ds, \]
since then, by \( M^2 \overset{2.5.2}{=} M^2 X \), we have \( M = H \cdot X \), in particular,
\[ M_1 = \int_0^1 X_t \, dt = F. \]
All processes are continuous with continuous \( \langle \cdot \rangle \). Therefore, we can consider everything pathwise and use the results of chapter 1. We have
\[ \langle M, X \rangle = \frac{1}{2} (\langle M + X \rangle - \langle M \rangle - \langle X \rangle). \]
Since
\[
M + X = \int_0^1 X_s \, ds + X \cdot A \\
= \int_0^1 X_s \, ds + X_t(1 - t) + X_s \\
= \int_0^1 X_s \, ds + X_t \cdot A_t,
\]
where \( A_t := (2 - t) \) is of bounded variation, it follows by Itô’s product rule
\[ X A = \int X \, dA + \int A \, dX. \]
Therefore,
\[
\langle XA \rangle_t = \left\langle \int_0^t A \, dX \right\rangle_t = \int_0^t A^2_s \, d\langle X \rangle_s = \int_0^t (2 - s)^2 \, ds.
\]
Thus,
\[ \langle M + X \rangle = \langle X \cdot A \rangle_t = \int_0^t (2 - s)^2 \, ds. \]
Likewise,
\[ \langle M \rangle = \langle X(1 - \cdot) \rangle_t = \int_0^t (1 - s)^2 \, d\langle X \rangle_s = \int_0^t (1 - s)^2 \, ds, \]
so,
\[ \langle M, X \rangle = \frac{1}{2} \left( \int_0^t (2 - s)^2 \, ds - \int_0^t (1 - s)^2 \, ds - t \right) = \int_0^t (1 - s) \, ds. \]
\[ \square \]
2. (Semi-)Martingales and Stochastic Integration
Chapter 3: Markov Processes

3.1. Markov Processes and Semigroups

Definition 3.1.1. A family \((p_t)_{t \geq 0}\) of transition kernels on a measurable space \((S, \mathcal{S})\) is called a semigroup of kernels if

\[ p_{t+s} = p_t p_s \quad \forall t, s \geq 0 \quad (\text{Chapman-Kolmogorov equation}), \]

i.e.

\[ p_t p_s(x, A) = \int p_t(x, dy) p_s(y, A). \]

If \(p_t : s \times S \to [0, 1]\), \(p_t(x, \cdot)\) is a measure on \(S\), \(x \mapsto p_t(x, A)\) is \(\mathcal{S}\)-measurable for all \(A \in \mathcal{S}\) and \(p_t(x, S) = 1\), \((p_t)_{t \geq 0}\) is called markovian and, if \(p_t(x, S) \leq 1\) for all \(x, t\), sub-markovian, respectively.

Definition 3.1.2. Let \((S, \mathcal{S})\) be a measurable space. A family of stochastic processes

\[ (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in S}) \]

with state space \((S, \mathcal{S})\) is called a Markov process, if

(M1) \(x \mapsto P_x(\Gamma)\) is \(\mathcal{S}\)-measurable for all \(\Gamma \in \mathcal{F}\),

(M2) there exists a filtration \((\mathcal{F}_t)_{t \geq 0}\) such that \((X_t)_{t \geq 0}\) is \(\mathcal{F}_t\)-adapted and

\[ P_x(X_{s+t} \in B | \mathcal{F}_s) = P_{X_s}(X_t \in B) \quad P_x\text{-a.e.} \quad \forall s, t \geq 0, B \in \mathcal{S}, x \in S \]

(Markov property with respect to \((\mathcal{F}_t)_{t \geq 0}\)).

The following theorem shows, at least if \((S, \mathcal{S})\) is polish, that Markovian semigroups and Markov processes are in correspondence to each other.

Theorem 3.1.3. i. Let \((S, \mathcal{S})\) be a measurable space and \(\mathcal{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in S})\) be a family of stochastic processes with state space \((S, \mathcal{S})\) and \(\mathcal{F} = \sigma((X_t)_{t \geq 0})\).

a) Suppose there exists a Markovian semigroup of kernels \((p_t)_{t \geq 0}\), such that, for all \(0 \leq t_0 < t_1 < \ldots < t_n\), \(f\) bounded and \(\mathcal{S}^{n+1}\) measurable, for all \(x \in S\),

\[ E_x[f(X_{t_0}, \ldots, X_{t_n})] = \int_S p_{t_0}(x, dx_0) \cdot \int_p p_{t_n-t_{n-1}}(x_{n-1}, dx_{n-1}) f(x_0, \ldots, x_n) \quad (3.1.1) \]

Then \(M\) is a Markov process with respect to \(\mathcal{F}_t = \sigma(X_s | s \leq t), t \geq 0\).

b) Suppose \(\mathcal{M}\) is a Markov process and set

\[ p_t f(x) := E_x(f(X_t)) \quad x \in S, f \text{ bounded, } S \text{-measurable, } t \geq 0. \quad (3.1.2) \]

Then \((p_t)_{t \geq 0}\) is a Markovian semigroup of kernels on \((S, \mathcal{S})\) and we have (3.1.1).

ii. If \((S, \mathcal{S})\) is Polish and \((p_t)_{t \geq 0}\) is a Markovian semigroup of kernels on \((S, \mathcal{S})\). Then there exists a Markov process \(\mathcal{M}\) with (3.1.2) and \(\Omega = S^{[0, \infty)}\).
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Proof.  

i.  

(a) (M1) follows by “monotone classes” from (3.1.1), since \( \mathcal{F} = \sigma(\{X_t | t \geq 0\}) \). For (M2) we show a stronger fact:

\[
E_x[f(X_{t_1}, \ldots, X_{t_n + s}), \Gamma] = E_x[E_{X_s}[f(X_{t_1}, \ldots, X_{t_m})], \Gamma] 
\]  

(3.1.3)

for all \( x \in S \), bounded and \( S^n \)-measurable \( f \), \( 0 \leq s_0 < s_1 < \ldots < s_m = s \) and \( \Gamma \in \mathcal{F}, s \geq 0 \). By monotone classes (applied to \( \Gamma \)) this follows from

\[
E_x[f(X_{t_1}, \ldots, X_{t_m})g(X_{s_0}, \ldots, X_{s_m})] = E_x[E_{X_s}[f(X_{t_1}, \ldots, X_{t_m})]g(X_{s_0}, \ldots, X_{s_m})]
\]

for all bounded \( S^{m+1} \)-measurable \( g \), \( 0 \leq s_0 < s_1 < \ldots < s_m = s \). By (3.1.1) the left hand side is equal to

\[
\int p_{s_0}(x, dx_0) \int p_{s_m-s_m-1}(x_{m-1}, dx_m) \ldots \int p_{t_1}(x_1, dy_1) f(y_1, \ldots, y_n) f(x_0, x_1, \ldots, x_m) = E_x[E_{X_{s_0}}[f(X_{t_1}, \ldots, X_{t_m})]g(X_{s_0}, X_{s_1}, \ldots, X_{s_m})].
\]

ii. It is clear that (3.1.2) defines a Markov kernel for all \( t \geq 0 \). Furthermore,

\[
p_{t+s} f(x) = E_x[f(X_{t+s})] = E_x[E_x[f(X_{t+s})|\mathcal{F}_t]] 
\]

\[
\overset{M.P.}{=} E_x[E_{X_t}[f(X_t)]] = E_x[p_t(f)(X_t)] = p_s(p_t) f(x)
\]

Let \( f = f_0 \otimes f_1 \otimes \ldots \otimes f_n \), \( f_i \) bounded and \( \mathcal{S} \)-measurable. Then

\[
E_x[f_0(X_{t_0}) f_1(X_{t_1}) \ldots f_n(X_{t_n})] = E_x[f_0(X_{t_0}) f_1(X_{t_1}) \ldots f_{n-1}(X_{t_{n-1}}) E_x[f_n(X_{t_{n-1}+t_n})|\mathcal{F}_{t_{n-1}}]] \overset{M.P.}{=} E_x[f_0(X_{t_0}) f_1(X_{t_1}) \ldots f_{n-1}(X_{t_{n-1}}) E_{X_{t_{n-1}}}[f_n(X_{t_{n-1}+t_n})]]
\]

(3.1.2)

\[
\overset{\text{Ind.Hyp.}}{=} \int \ldots \int p_{t_0}(x, dx_0) p_{t_1-t_0}(x_0, dx_1) p_{t_2-t_1-t_0}(x_{n-2}, dx_{n-1}) f_0(x_0) f_1(x_1) \ldots f_{n-1}(x_{n-2}) f_n(x_{n-1})
\]

By monotone classes this equality extends to all \( f : S^n \to \mathbb{R} \) bounded, measurable. Hence, (3.1.1) holds.

\[\square\]

Remark 3.1.4. The Markov property is the core argument for the proof above, i.e. for \( f = 1_A, A \in \mathcal{S} \)

\[
E_x[1_A(X_{t+s})] = P_x[X_{t+s} \in A] \overset{M.P.}{=} E_x[1_A(X_t)]P_x(d\omega) = \int P_{X_{t+s}}(\omega)[X_t \in A]P_x(d\omega).
\]
Now consider the “canonical model”:
Let $S$ be a topological space, $S$ the Borel-$\sigma$-algebra and $\Omega \subset S^{[0, \infty]}$ (e.g. $\Omega = S^{[0, \infty]}$ or $\Omega = C([0, \infty[, S)$ or $\Omega$ are all bounded continuous paths in $S$, hence, $S^{[0, \infty]} = \{ f : [0, \infty[ \to S$ and continuous $\}$) and define

$$X_t(\omega) := \omega(t), \quad t \geq 0, \ \omega \in \Omega,$$

$$\mathcal{F} := \sigma(X_t|t \geq 0),$$

$$\mathcal{F}_t^0 := \sigma(X_s|s \leq t) \quad \text{("past")},$$

$$\hat{\mathcal{F}}_t^0 := \sigma(X_s|s \geq t) \quad \text{("future")}.$$ 

Let $M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in S})$ be a Markov process. Then $M$ is called the canonical model.

**Definition 3.1.5.** Define the shift operator $\vartheta_t : \Omega \to \Omega$ for $t \geq 0$ by

$$\vartheta_t(\omega)(s) := \omega(s + t),$$

i.e.

$$\vartheta_t(\omega) = \omega(\cdot + t).$$

It is obvious that $\vartheta : \Omega \to \Omega$ is $\mathcal{F}_t^0/\mathcal{F}$-measurable and moreover, (exercise)

$$\vartheta_t^{-1}(\mathcal{F}) = \hat{\mathcal{F}}_t^0 \ \forall t \geq 0.$$

**Lemma 3.1.6.**

i. $\psi$ is $\hat{\mathcal{F}}_t^0$-measurable if and only if there exists an $\mathcal{F}$-measurable $\varphi$ such that

$$\psi = \varphi \circ \vartheta_t.$$

ii. Suppose $P_x, x \in S$, are given and

$$M := (\Omega, \mathcal{F}, (\mathcal{F}_t^0)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in S})$$

defined such that (M1) is fulfilled. Then (M2) holds if and only if

$$E_x[\varphi \circ \vartheta_t|\mathcal{F}_t^0] = E_{X_t}[\varphi] \quad P\text{-a.s.} \forall x \in S, \forall \varphi \text{ bounded and } \mathcal{F}\text{-measurable.} \quad (3.1.4)$$

**Proof.**

i. Exercise (Factorization lemma).

ii. “$\Rightarrow$” is clear. (Consider $\varphi = 1_{B(x)}, B \in S$.)

“$\Leftarrow$”: By 3.1.3 (i)(b) we know that (3.1.1) holds. By use of (3.1.1) we have shown (3.1.3).

Now Lemma 3.1.6 follows by “monotone classes”.

It is clear that (3.1.4) implies (elementary Markov processes)

$$E_x[\varphi \circ \vartheta_t|\mathcal{F}_t^0] = E_x[\varphi \circ \vartheta_t|\sigma(X_t)] \quad \forall t \geq 0. \quad (3.1.5)$$

**Interpretation of (3.1.5):** The conditional expectation of a future observable, given the past, only depends on the present at time $t$. The equivalent formulations of (3.1.5) in the following lemma have corresponding interpretations:

**Lemma 3.1.7.** Fix $x \in S$. Then the following statements are equivalent to (3.1.5):

i. 

$$E_x[\varphi_t^0|\hat{\mathcal{F}}_t^0] = E_x[\varphi_t^0|\sigma(X_t)] \quad P\text{-a.s.}, \forall t \geq 0, \forall \varphi_t^0 \text{ bounded and } \hat{\mathcal{F}}_t^0\text{-measurable.}$$
3. Markov Processes

\[ ii. \quad E_x[\varphi_0^0 \varphi_0^0 | \sigma(X_t)] = E_x[\varphi_0^0 | \sigma(X_t)]E_x[\varphi_0^0 | \sigma(X_t)] \quad P_x \text{-a.s.,} \]

for all \( t \geq 0 \) and for all \( \varphi_0^0, \varphi_0^0 \) bounded, \( \varphi_0^0 \mathcal{F}_t^0 \)-measurable and \( \varphi_0^0 \mathcal{F}_t^0 \)-measurable.

Proof. (3.1.5) \( \Rightarrow \) (i):

\[
E_x[\varphi_0^0 \varphi_0^0] = E_x[\varphi_0^0 | \sigma(X_t)]E_x[\varphi_0^0 | \sigma(X_t)] = E_x[\varphi_0^0 E_x[\varphi_0^0 | \sigma(X_t)]] = E_x[\varphi_0^0 | \sigma(X_t)].
\]

Therefore, \( E_x[\varphi_0^0 | \sigma(X_t)] \) is a \( P_x \)-version of \( E_x[\varphi_0^0 | \mathcal{F}_t^0] \). Hence, (i) holds.

(i) \( \Rightarrow \) (ii):

Let \( f \) be bounded and \( S \)-measurable and \( \varphi_0^0 \) be \( \mathcal{F}_t^0 \)-measurable. Then

\[
E_x[\varphi_0^0 \varphi_0^0 f(X_t)] = E_x[E_x[\varphi_0^0 | \mathcal{F}_t^0] \varphi_0^0 f(X_t)] \overset{(i)}{=} E_x[E_x[\varphi_0^0 | \sigma(X_t)] \varphi_0^0 f(X_t)] = E_x[E_x[\varphi_0^0 | \sigma(X_t)] E_x[\varphi_0^0 | \sigma(X_t)] f(X_t)].
\]

(ii) \( \Rightarrow \) (3.1.6):

\[
E_x[\varphi_0^0] = E_x[E_x[\varphi_0^0 | \sigma(X_t)] E[\varphi_0^0 | \sigma(X_t)]] = E_x[\varphi_0^0 E_x[\varphi_0^0 | \sigma(X_t)]].
\]

Hence,

\[
E_x[\varphi_0^0 | \sigma(X_t)] = E_x[\varphi_0^0 | \mathcal{F}_t^0].
\]

\[ \square \]

Remark 3.1.8. Consider the situation of 3.1.3(ii). Then \( P_x(X_0 = 0) = 1 \) if and only if \( p_0(x, \cdot) = \sigma_x \) (Dirac-measure in \( x \)). If this holds for every \( x \in S \), then \( M \) is called normal. In this case, \( p_t(x, A) = P_x(X_t \in A) \), which is the probability to be at time \( t \) in \( A \) starting in \( x \) (transition probability).

3.2. The Strong Markov Property

Consider the “canonical model”.

Recall: Let \( \mathcal{F}_t \) be some filtration and \( T \) an \( (\mathcal{F}_t) \)-stopping time. Define the \( \sigma \)-field of the \( T \)-past by

\[
\mathcal{F}_T := \{ A \in \mathcal{F} | A \cap \{ T \leq t \} \in \mathcal{F}_t \ \forall t \geq 0 \}.
\]

Definition 3.2.1. Let \( M = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in S}, (\vartheta_t)_{t \geq 0}) \) be a canonical Markov process. Then \( M \) satisfies the strong Markov property (SMP), if there exists a right-continuous filtration \( (\mathcal{F}_t)_{t \geq 0} \) on \( (\Omega, \mathcal{F}) \) (i.e. \( \mathcal{F}_t = \bigcap_{t>0} \mathcal{F}_{t+t} \)) such that for all \( (\mathcal{F}_t) \)-stopping times \( T \) we have (SMP)

\[
E_x[1_{\{T < \infty\}} \varphi \circ \vartheta_T | \mathcal{F}_T] = 1_{\{T < \infty\}} E_{X_T}[\varphi] \quad P\text{-a.s.} \quad \forall \varphi \text{ bounded, } \mathcal{F}\text{-measurable and } \forall x \in S.
\]

Remark 3.2.2. \( \quad i. \) It is clear that (SMP) implies the Markov property with respect to \( (\mathcal{F}_t) \) and therefore, with respect to \( \mathcal{F}_T^0 \) by (3.1.6). In general, the converse is false.

\( \quad ii. \) (SMP) holds if and only if for all \( (\mathcal{F}_t) \)-stopping times \( T \) we have

\[
E_x[\varphi \circ \vartheta_t; T < \infty] = E_x[E_x(\varphi); T < \infty] \quad P_x\text{-a.s.} \tag{3.2.6}
\]

for all \( \varphi \) bounded and \( \mathcal{F} \)-measurable and for all \( x \in S \).
3.2. The Strong Markov Property

Proof. “⇒” is clear, since \( \{ T < \infty \} \in \mathcal{F}_T \).

“⇐”: Let \( A \subset \mathcal{F}_T \) and \( \tilde{T} := T \cdot 1_A + \infty \cdot 1_{A^c} \). Then \( \tilde{T} \) is a \((\mathcal{F}_t)\)-stopping time because \( \{ \tilde{T} \leq t \} = \{ A \cap \{ T \leq t \} \} \in \mathcal{F}_t \).

Hence 
\[ E_x[\varphi \circ \vartheta_T; A \cap \{ T < \infty \}] = E_x[\varphi \circ \vartheta_{\tilde{T}}; \tilde{T} < \infty] \]

(3.2.6) 
\[ = E_x[E_{X_T}(\varphi); A \cap \{ T < \infty \}] \]

Proposition 3.2.3. Suppose \( \varphi \in \mathcal{S} = \sigma (\mathcal{C}_b(S)) \). Let \( M = (\Omega, \mathcal{F}, ((X_t)_{t \geq 0})_{t \geq 0}, (P_x)_{x \in S}, (\vartheta_t)) \) be a (canonical) Markov process with

i. right-continuous paths,

ii. \( p_t(\mathcal{C}_b(S)) \subset \mathcal{C}_b(S) \) (“Feller property”),

where \( p_t \) is the corresponding semigroup, then \( M \) fulfills the (SMP) with respect to \( \mathcal{F}_t := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \) (right-continuous).

Proof. Let \( T \) be an \((\mathcal{F}_t)\)-stopping time. Define 
\[ T_n := \sum_{k=1}^{\infty} \frac{1}{2^n} 1_{\{ \frac{k-1}{2^n} \leq T < \frac{k}{2^n} \}} + \infty \cdot 1_{\{ T = \infty \}}. \]

Then \( T_n \searrow T \). We have to show that (3.2.6) holds:
Without loss of generality 
\[ \varphi = f_0(X_{t_0})f_1(X_{t_1}) \ldots f_n(X_{t_n}), \quad f_i \in \mathcal{C}_b(S), \quad t_0 \leq t_1 \leq \ldots \leq t_n. \]

Then 
\[ E_x[\varphi \circ \vartheta_T; T < \infty] \overset{(i)}{=} \lim_{n \to \infty} E_x[\varphi \circ \vartheta_{T_n}, T < \infty] \]
\[ = \lim_{n \to \infty} \sum_{k=1}^{\infty} E_x \left[ \varphi \circ \vartheta_T \cdot \frac{k-1}{2^n} \leq T \leq \frac{k}{2^n} \right] \]
\[ = \lim_{n \to \infty} \sum_{k=1}^{\infty} E_x \left[ E_{X_{T_{2^{-n}}}^{T_{2^{-n}}}}[\varphi]; (k-1)2^{-n} \leq T < \frac{k}{2^n} \right] \]
\[ \overset{M.P.}{=} \lim_{n \to \infty} E_x[E_{X_{T_{2^{-n}}}^{T_{2^{-n}}}}[\varphi], T < \infty] \]
\[ = E_x[E_{X_T}[\varphi]; T < \infty] \quad \text{(Lebesgue)}, \]

since \( X_{T_n} \to X_T \) on \( [ T < \infty ] \) and 
\[ x \mapsto E_x[\varphi] = E_x[f_0(X_{t_0}), \ldots, f_n(X_{t_n})] = p_{t_0}(f_0p_{t_1-t_0}f_1p_{t_2-t_1}f_2 \ldots p_{t_n-t_{n-1}}f_n)(x) \]
are continuous on \( S \).

Remark 3.2.4 (Blumenthal’s 0–1 law). Let \( M \) be a canonical normal Markov process with respect to \( \mathcal{F}_t := \bigcap_{s \geq t} \mathcal{F}_s^0 \), \( t \geq 0 \) (on measurable state space \((S, \mathcal{S})) \). Then \( P_x = 0 \) or \( P_x = 1 \) on \( \mathcal{F}_0 \).

Proof. Let \( \varphi \) be \( \mathcal{F}_0 \)-measurable and bounded. Then for all \( x \in S \) we have 
\[ \varphi = E_x[\varphi|\mathcal{F}_0] = E_x[\varphi \circ \vartheta_0|\mathcal{F}_0] \overset{M.P.}{=} E_{X_0}[\varphi] = E_x[\varphi] \quad P_x \text{-a.s.}. \]

Hence, \( \varphi \) is constant \( P_x \)-a.e.
3. Markov Processes

3.3. Application to Brownian Motion

Let $S = \mathbb{R}^d$, $S = \mathcal{B}(\mathbb{R}^d)$ and

$$p_t(x, A) = \int_A \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y-x|^2}{2t}} \, dy, \quad x \in \mathbb{R}^d, \ t > 0, \ A \in \mathcal{B}(\mathbb{R}^d),$$

$p_0(x, \cdot) = \delta_x$.

**Lemma 3.3.1.** For all $t, x \geq 0$, we have

$$p_t p_s = p_{t+s}.$$

**Proof.** Exercise (by Fourier transform). \qed

Let $\Omega := \mathcal{C}([0, \infty[, \mathbb{R}^d)$, $X_t(\omega) := \omega(t)$, $P_0 := P$ be the Wiener measure on $\Omega$, $\mathcal{F} := \sigma(X_t| t \geq 0)$ and $P_x$ the image of $P_0$ under $\omega \mapsto \omega + x$. Then we have (cf. [Röc06])

$$E_x[f(X_{t_0}, \ldots, X_{t_p})] = \int p_0(x, dx_0)p_{t_1-t_0}p_{t_2-t_1-\ldots-t_{n-1}}(x_{n-1}, dx_n)f(x_0, \ldots, x_n),$$

for all $x \in \mathbb{R}^d$, for all $\mathcal{B}(\mathbb{R}^d)\text{-}\text{measurable, bounded functions } f \text{ and for } 0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$.

Hence,

$$\mathcal{M} := (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$$

is by Proposition 3.1.3(i)(a) a Markov Process on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. $\mathcal{M}$ has continuous sample paths and is normal. In particular, Remark 3.2.4 is fulfilled. But also, by Proposition 3.2.3 $\mathcal{M}$ has SMP with respect to

$$\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s, \ t \geq 0,$$

because of the following Proposition.

**Proposition 3.3.2.** $(p_t)_{t \geq 0}$ is strong Feller, that is,

$$p_t f \in \mathcal{C}_b(\mathbb{R}^d) \ \forall f \in \mathcal{B}_b(\mathbb{R}^d).$$

(Moreover, we have $p_t f \in \mathcal{C}^\infty$ for all $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d).$)

**Proof.** Let $x \in \mathbb{R}^d$ and $x_n \to x$. We have to show that $p_t f(x_n) \to p_t f(x)$. Let $\varepsilon > 0$ and $n_0$ such that $x_n \in B_1(x)$ for all $n \geq n_0$, there exists $h \in \mathcal{L}^1(\mathbb{R})$ such that $\frac{e^{-(y-x')^2}}{2t} \leq h(y)$ for all $y, x' \in \mathbb{R}$ such that $|x'-x| \leq 1$.

Define $g(x_1, \ldots, x_d) = h(x_1) \cdots h(x_d)$, then

$$\frac{e^{-(y-x')^2}}{2t} \leq g(y) \ \forall x' \in B_1(x) \ \forall y \in \mathbb{R}^d.$$

Hence

$$\lim_{n \to \infty} p_t f(x_n) = \frac{1}{\sqrt{(2\pi t)^d}} \lim_{n \to \infty} \int f(y) e^{-\frac{|y-x_n|^2}{2t}} \, dy = \frac{1}{\sqrt{(2\pi t)^d}} \int f(y) e^{-\frac{|y-x|^2}{2t}} \, dy = p_t f(x)$$

by Lebesgue’s dominated convergence theorem since $f$ is bounded and $\frac{e^{-(y-x')^2}}{2t} \leq g(y) \in \mathcal{L}^1$ for all $n \geq n_0$. \qed
3.3. Application to Brownian Motion

Corollary 3.3.3. i. Let \( M \) be a normal (canonical) Markov-Process with respect to \( \mathcal{F}_t := \bigcap_{s \geq t} \mathcal{F}_s \), \( t \geq 0 \). Then, by Blumenthal for \( t \) not fixed,
\[
P_x[\limsup_{t \to 0} 1_{\{X_t = x\}} = 1] \in \{0, 1\}.
\]
For Brownian motion this probability is equal to 1 because of the law of iterated logarithm.

ii. Let \( X \) be Brownian motion starting at \( x = 0 \) and let
\[
N(\omega) := \{0 \leq s \leq 1 | X_s(\omega) = 0\}.
\]
Then for \( P_0 \)-a.e. \( \omega \) we have that
a) \( N(\omega) \) is closed,
b) \( N(\omega) \) has Lebesgue-measure zero,
c) \( N(\omega) \) has no isolated points,
that is \( N(\omega) \) is a Cantor set for \( P_0 \)-a.e. \( \omega \), i.e. every point of \( N(\omega) \) is a cluster point of \( N(\omega) \).

iii. Let \( U \subset \mathbb{R}^d \), \( U \) open and bounded and \( X \) Brownian motion on \( \mathbb{R}^d \). Define the first exit time of \( U \)
\[
T := \inf\{t > 0 | X \not\in U\} = \sigma_U^c.
\]
\( T \) is an (\( \mathcal{F}_t \))-stopping time. Then there exists \( \varepsilon > 0 \) such that
\[
\sup_{x \in U} E_x[e^{\varepsilon T}] < \infty.
\]

Proof. i. Clear.

ii. a) \( X \) is \( P \)-a.s. continuous.
b) By Fubini, we get
\[
E \left[ \int_0^1 1_{N(\omega)}(s) \, ds \right] = \int_0^1 E_0[1_{N(\omega)}(s)] \, ds
= \int_0^1 P_0[X_s = 0] \, ds
= \int_0^1 p_s(0, \{0\}) \, ds = 0.
\]
c) According to our intuition, by law of (local) iterated logarithm \( t = 0 \) is not an isolated point of \( N(\omega) \). By (SMP) this is true for any \( s \in \mathbb{N}(\omega) \). Define
\[
J := \{\omega | N(\omega) \text{ has isolated points}\} = \bigcup_{r, s \in \mathbb{Q}, r < s} \{\omega | N(\omega) \cap [r, s] \text{ contains exactly one (isolated) point} \}.
\]
Let \( T_{\{0\}} := \inf\{t > 0 | X_t = 0\} \). Then \( T_{\{0\}} \) is an (\( \mathcal{F}_t \)) (\( = \bigcap_{s > t} \mathcal{F}_s \))-stopping time (exercise). Therefore, for \( T := T_{\{0\}} \circ \vartheta_r + r \) (\( \mathcal{F}_t \))-stopping time
\[
P_0[A, r, s] \leq P_0[r + T_{\{0\}} \circ \vartheta_r < s, T_{\{0\}} \circ \vartheta_r + T_{\{0\}} \circ \vartheta_r > 0]
= E_0[1_{\{T_{\{0\}} > 0\}} \circ \vartheta_T; T < s]
= E_0[E_{X_T}[1_{\{T_{\{0\}} > 0\}} \circ \vartheta_T | \mathcal{F}_T]; T < s]
= E_0[E_{X_T}[1_{\{T_{\{0\}} > 0\}}]; T < s]
= E_0[P_0[T_{\{0\}} > 0]; T < s] = 0,
\]
3. Markov Processes

by (i).

iii. Let \( R > 0 \) such that \( U \subset B_R \) (closed ball with \( R \) around 0). Define

\[
T_R := \inf \{ t > 0 \mid |X_t| > R \} = \sigma_{B_R^c}.
\]

By Doob-Meyer, \( |X_t|^2 - d \cdot t \) is a martingale under \( P_x \). Hence,

\[
0 \leq |x|^2 = E_x[|X_0|^2 - d \cdot 0] = E_x[|X_{T_R \wedge n}|^2 - d \cdot T_R \wedge n]
\leq E_x[|X_{T_R \wedge n}|^2] - d \cdot E_x[T_R \wedge n] \leq R^2 - d \cdot E_x[T_R \wedge n], \quad \forall x \in U,
\]

Therefore, taking \( n \) to \( \infty \), we get

\[
E_x[T] \leq E_x[T_R] \leq \frac{R^2}{d} < \infty, \quad \forall x \in U.
\]

Hence,

a) \[ P_x[T < \infty] = 1, \quad \forall x \in U, \]

b) for large \( t_0 \)

\[ \sup_{x \in U} P_x[T > t_0] \leq \sup_{x \in U} \frac{1}{t_0} E_x[T] < 1. \]

Define

\[
\varphi(t) := \sup_{x \in U} P_x[T > t].
\]

Then, we have

\[
\varphi(t + s) = \sup_{x \in U} E_x[1_{\{T > t+s\}}] = \sup_{x \in U} E_x[1_{\{T > t\} \cdot 1_{\{T > s\}}} \circ \vartheta_s] = \sup_{x \in U} E_x[E_x[1_{\{T > t\} \circ \vartheta_s} \mid \mathcal{F}_s]; T > s] \quad \text{(since \{T > s\} \in \mathcal{F}_s)}
\]

\[ \leq \sup_{x \in U} E_x[P_X[T > t]; T > s] \leq \sup_{x \in U} P_x[T > t] \cdot \sup_{x \in U} P_x[T > s] = \varphi(t)\varphi(s). \]

Claim: These two conditions, i.e.

a) there exists a \( t_0 > 0 \) such that \( \varphi(t_0) < 1 \),

b) \[ \varphi(t + s) \leq \varphi(t)\varphi(s), \quad (\star) \]

imply that \( \varphi \) is subexponential, i.e. there exist \( K > 0 \) and \( \lambda > 0 \) such that

\[ \varphi(t) \leq Ke^{-\lambda t}, \quad \forall t \geq t_0. \]

Proof. We have

\[ \varphi(s) = \varphi \left( \frac{s}{t_0} \cdot t_0 \right) \leq \varphi \left( \frac{t_0}{t_0} \cdot t_0 \right) \]
Hence, equation (\(\star\)) implies
\[
\varphi(s) \leq \varphi(t_0 + \ldots + t_0) \leq \varphi(t_0) \cdot \ldots \cdot \varphi(t_0) = \varphi(t_0)\left\lfloor \frac{s}{t_0} \right\rfloor. \tag{3.4.1}
\]

From now on, without loss of generality we consider \(\varphi(t_0) > 0\). Then, since \(\left\lfloor \frac{s}{t_0} \right\rfloor \leq \frac{s}{t_0} - 1\), it follows that
\[
\varphi(t_0)\left\lfloor \frac{s}{t_0} \right\rfloor = e^{\left\lfloor \frac{s}{t_0} \right\rfloor \ln \varphi(t_0)} \leq \exp \left(-s \frac{\ln \varphi(t_0)}{t_0} \right) \exp(\ln \varphi(t_0)) = Ke^{-s\lambda},
\]
where
\[
K := \frac{\varphi(t_0) - 1}{\varphi(t_0)} < 1 = \exp(\ln \varphi(t_0)) \quad \text{and} \quad \lambda := \frac{\ln \varphi(t_0)}{t_0}.
\]

Then for all \(x \in U\)
\[
E_x[e^{\varepsilon T}] = E_x \left[ \int_0^{e^{\varepsilon T}} 1 \, ds \right] = E_x \left[ \int_0^\infty 1_{[0,e^{\varepsilon T}]}(s) \, ds \right] \\
\text{Fubini} = \int_0^\infty P_x[e^{\varepsilon T} > s] \, ds \\
= \int_{e^{\varepsilon t_0}}^\infty P_x[e^{\varepsilon T} > s] \, ds + \int_{e^{\varepsilon t_0}}^\infty P_x[e^{\varepsilon T} > s] \, ds.
\]

For the first term we have easily
\[
\int_{e^{\varepsilon t_0}}^\infty P_x[e^{\varepsilon T} > s] \, ds \leq e^{\varepsilon t_0} < \infty.
\]

But, we can compute the second one:
\[
\int_{e^{\varepsilon t_0}}^\infty P_x[e^{\varepsilon T} > s] \, ds \leq \int_{e^{\varepsilon t_0}}^\infty \sup_{x \in U} P_x[e^{\varepsilon T} > s] \, ds \\
= \int_{t_0}^\infty e^{\varepsilon u} \sup_{x \in U} P_x[e^{\varepsilon T} > e^{\varepsilon u}] \, du \\
= \varepsilon \int_{t_0}^\infty e^{\varepsilon u} \varphi(u) \, du \\
= \varepsilon \cdot K \int_{t_0}^\infty e^{u(e-\lambda)} \, du < \infty,
\]
if \(\varepsilon < \lambda\).

\(\square\)

### 3.4. Sojourn Time

Let \(S\) be a polish space, \(\mathcal{S}\) the Borel-\(\sigma\)-algebra and \(\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in S})\) normal Markov process with respect to a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) with state space \(S\) and continuous paths.
3. Markov Processes

**Definition 3.4.1.** The time
\[ \tau_x := \inf \{ t > 0 | X_t \neq x \} = \sigma_{\{x\}^c} \]
is called sojourn time.

**Remark 3.4.2.** \( \tau_x \) is an \((F_t)_{t \geq 0}\)-stopping time because for all \( t \geq 0 \)
\[ \{ \tau_x < t \} = \{ X_s = x | 0 \leq s \leq t, s \in \mathbb{Q} \} \in F_t, \]
hence, \( \{ \tau_x < t \} \in F_t \). Since \( F_t = F_{t+} \), also
\[ \{ \tau_x < t \} = \bigcap_{k \in \mathbb{N}} \{ \tau_x < t + \frac{1}{k} \} \in F_{t+} = F_t. \]

**Proposition 3.4.3.**
\[ P_x[\tau_x > t] = \exp(-c_x \cdot t), \quad t \geq 0, \]
where \( c_x := -\ln P_x[\tau_x > 1] \in [0, \infty] \),
i.e. \( \tau_x \) is exponential distributed with constant \( c_x \) (cf. [KL99]).

**Proof.** (By Markov property.) Define for \( t \geq 0 \)
\[ f(t) := P_x[\tau_x > t]. \]

**Claim:** \( f(t+s) = f(t)f(s) \) for all \( t, s \geq 0 \).
\[ E_x[1_{\{\tau_x > t+s\}} | F_s] = E_x[1_{\{\tau_x > t\}} \cdot 1_{\{\tau_x > s\}}] \]
\[ = E_x[1_{\{\tau_x > t\}}]1_{\{\tau_x > s\}} = P_x[\tau_x > t] \cdot 1_{\{\tau_x > s\}}. \]

Hence,
\[ f(t+s) = P_x[\tau_x > t+s] = E_x[1_{\{\tau_x > t+s\}} | F_s] \]
\[ = E_x[1_{\{\tau_x > t+s\}}] \]
\[ = P_x[\tau_x > t]P_x[\tau_x > s] = f(t)f(s). \]

In particular,
\[ f(1) = f \left( 2^n \frac{1}{2^n} \right) = f \left( \frac{1}{2^n} \right)^{2^n} \]
and then
\[ f \left( \frac{k}{2^n} \right) = f \left( \frac{1}{2^n} \right)^k = f(1)^{kn}. \]

Hence, by the right-continuity of \( f \) we have \( f(t) = f(1)^t = e^{t\ln f(1)} \).

**Corollary 3.4.4.** Let \( T_x = \inf \{ t > 0 | X_t = x \} = \sigma_{\{x\}^c} \). Let \( M \) be as in Proposition 3.4.3, but assume in addition, that \( M \) has (SMP) with respect to \((F_t)_{t \geq 0}\) (right-continuity!). Let \( y \in S \) such that
\[ P_x[T_x < \infty] > 0. \]

Then for all \( u \geq 0 \), we have
\[ P_x[X_s = x, \forall s \in [T_x, T_x + u); T_x < \infty] = e^{-c_x u}, \]
where \( c_x \) is as in Proposition 3.4.3.
3.4. Sojourn Time

Proof. Let \( \varphi := 1_{\{X_s = x, \ \forall s \in [0, u]\}} \). Then

\[
P_x[X_s = x, \ \forall s \in [T_x, T_x + u); T_x < \infty] = E_x[\varphi \circ \vartheta_{T_x}; T_x < \infty]
\]

\[
= E_x[E_x[\varphi \circ \vartheta_{T_x}; \mathcal{F}_{T_x}]; T_x < \infty]
\]

\[
= E_x[E_{X_{T_x}}[\varphi]; T_x < \infty]
\]

\[
= E_x[\varphi] \cdot P_x[T_x < \infty]
\]

\[
= P_x[X_s = x, \ \forall s \in [0, u)] \cdot P_y[T_x < \infty]
\]

\[
= e^{-c_x u} P_x[T_x < \infty].
\]
3. Markov Processes
4. Girsanov Transformation

4.1. Problem Outline \((d = 1)\)

We want to construct a process such that it solves (in a “weak” sense) the following equation (“law of motion for the stochastic dynamics \((X_t)_{t \geq 0}\)”):

\[
\begin{align*}
    &dX_t = b(X_t, t) \, dt + dW_t, \\
    &X_0 = x_0 \in \mathbb{R}^d,
\end{align*}
\]

that is,

\[
X_t(\omega) = x_0 + \int_0^t b(X_s(\omega), s) \, ds + W_t(\omega).
\]

Here, \(X_t = x_0 + \int_0^t b(X_s, s) \, ds\) denotes the deterministic part and \(W_t\) the stochastic perturbation, i.e. \(W_t\) is a Wiener process.

One possible strategy of solving this equation is to find the strong solution, that is, for a given Wiener process \((W_t)_{t \geq 0}\) on a given probability space \((\Omega, \mathcal{F}, P)\) construct the paths \((X_t)_{t \geq 0}\) of the solution by classical methods (e.g. Picard-Lindelöf or Euler scheme).

**Example:** The Ornstein-Uhlenbeck process has the “law of motion”

\[
\begin{align*}
    &dX_t = -\alpha X_t \, dt + dW_t, \quad \alpha > 0, \\
    &X_0 = x_0 \in \mathbb{R}.
\end{align*}
\]

**Claim:** This problem has a strong solution

\[
X_t := e^{-\alpha t}x_0 + \int_0^t e^{-\alpha (t-s)} \, dW_s = F(x_0, (W_s)_{s \leq t})(t),
\]

hence, \(X_t\) is adapted to the Wiener filtration.

**Proof.** We apply Itô’s product rule to \(X_t = e^{-\alpha t} \cdot \left(x_0 + \int_0^t e^{\alpha s} \, dW_s\right)\) to get

\[
X_t = x_0 + \int_0^t e^{-\alpha s} \, d \left(x_0 + \int_0^s e^{\alpha u} \, dW_u\right) + \int_0^t \left(x_0 + \int_0^s e^{\alpha u} \, dW_u\right) (-\alpha) e^{-\alpha s} \, ds
\]

\[
= x_0 - \alpha \int_0^t X_s \, ds + \int_0^t 1 \, dW_s = X_0 - \alpha \int_0^t X_s \, ds + W_t.
\]

Instead of strong solutions one can construct “(probabilistically) weak solutions”. We want to construct a Brownian motion \((W_t)_{t \geq 0}\) and a process \((X_t)_{t \geq 0}\) on some probability space \((\Omega, \mathcal{F}, P)\) such that

\[
X_t = x_0 + \int_0^t b(X_s) \, ds + W_t
\]
4. Girsanov Transformation

holds, that is, construct \((X_t)_{t \geq 0}\) on a suitable probability space \((\Omega, \mathcal{F}, P)\) such that \((X_0 = 0)\)

\[
W_t := X_t - \int_0^t b(X_s, s) \, ds
\]

is a Brownian motion, e.g. take \((\Omega, \mathcal{F}), (X_t)_{t \geq 0}\) canonical, i.e.,

\[
\Omega := C([0, 1]),
X_t(\omega) := \omega(t),
\mathcal{F} := \sigma(X_t|t \geq 0),
\]

such that

\[
W_t(\omega) := X_t(\omega) - \int_0^t b(X_s(\omega), s) \, ds (= G(X_s(\omega)))
\] (4.1.1)

is a Brownian motion under \(P\).

But we have to identify \(P\)!

One technique to find \(P\) is using the Girsanov transformation. This approach has the following advantages:

- One can do this even if dependence on the past is very complicated.
- One can do this for very irregular \(b_s\).

**Method:** Let \(\Omega = C([0, 1])\), \((X_t)_{t \geq 0}\) be a coordinate process, i.e. \(X_t(\omega) = \omega(t)\) and \(P_0\) be the Wiener measure on \(C([0, 1]) = \Omega\). Then define

\[
P := \exp \left( \int_0^1 b_t \, dX_t - \frac{1}{2} \int_0^1 b_t^2 \, dt \right) P_0.
\]

We can check that

\[
W_t := X_t - \int_0^t b_s \, ds
\]

is a Brownian motion under \(P\), where \(b_s\) denotes the drift. \((X_t)_{t \geq 0}\) is a Brownian motion under \(P_0\), hence, a martingale under \(P_0\), but *not* a martingale under \(P\)!

**Catch:** We have to check, that \(P\) is a probability measure, i.e. we have to check, that

\[
\int e^{\int_0^1 b_t \, dX_t - \frac{1}{2} \int_0^1 b_t^2 \, dt} \, dP_0 = 1.
\]

This is the hard work in applications.

**Relation with Transformation Rule for Lebesgue Measure**

Define \(T : C([0, 1]) \to C([0, 1])\) by

\[
T(\omega) := X(\omega) - \int_0^\cdot b(X_s(\omega), s) \, ds.
\]

Then by Girsanov (under certain conditions)

\[
P \circ T^{-1} = \left( e^{\int_0^1 b_s(X_s, s) \, dX_s - \frac{1}{2} \int_0^1 b_s(X_s, s)^2 \, ds} \, dP_0 \right) \circ T^{-1} = P_0.
\]

\("\equiv \det DT"\)
4.2. The General Girsanov Transformation

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((\mathcal{F}_t)_{t \geq 0}\) be a right-continuous filtration (not necessarily “completed”). Then let \(\tilde{P}\) be another probability measure such that

\[ \tilde{P} \, \text{loc.} \ll P \quad \text{(i.e.} \, \tilde{P}|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t} \, \forall \, t \geq 0) \].

Then the Radon-Nikodym densities

\[ Z_t := \frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_t} := \frac{d\tilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} \]

exist and \((Z_t)_{t \geq 0}\) is a martingale (since for all \(t > s\) and \(F_s \in \mathcal{F}_s\)

\[ \int_{F_s} Z_t \, dP \big|_{\mathcal{F}_s \subset \mathcal{F}_t} = \int_{F_s} \tilde{Z}_s \, d\tilde{P} = \int_{F_s} Z_s \, dP, \]

i.e.

\[ E[\tilde{Z}_t|\mathcal{F}_s] = Z_s. \]

**Assumption** (from now on in force): \(Z\) has continuous sample paths \(P\)-a.s. (This is enough in most applications. Note that by Ito’s representation theorem any martingale with respect to a filtration “generated” by a Brownian motion has this property!)

**Lemma 4.2.1.** For every \((\mathcal{F}_t)\)-stopping time \(T\) we have

\[ \tilde{P} = Z_T P \quad \text{on} \quad \mathcal{F}_T \cap \{ T < \infty \} \quad \text{(trace \sigma-field)} \]

In particular, \(\tilde{P} \ll P\) on \(\mathcal{F}_T\) if \(T\) is finite.

**Proof.** Let \(A \in \mathcal{F}_t\). Then, by a lemma in [Röc06] for all \(t \geq 0\)

\[ A \cap \{ T \leq t \} \in \mathcal{F}_T \cap \{ T < \infty \} \subset \mathcal{F}_t. \]

Hence,

\[ \tilde{P}[A \cap \{ T \leq t \}] = \int_{A \cap \{ T \leq t \}} Z_t \, dP \bigg|_{A \cap \{ T \leq t \} \in \mathcal{F}_T \cap \{ T < \infty \}} = \int_{A \cap \{ T \leq t \}} Z_T \, dP. \]

Letting \(t \to \infty\) and applying monotone convergence the assertion follows. \(\square\)

The following lemma describes how martingales “behave” under change from \(P\) to \(\tilde{P}\). Define

\[ \xi(\omega) := \inf\{ t \geq 0 | Z_t(\omega) = 0 \}. \]

\(\xi(\omega)\) is a stopping time. Recall that then \(Z_t(\omega) = 0\) for all \(t \in [\xi(\omega), \infty[\), since \((Z_t)_{t \geq 0}\) is a positive (super)martingale (cf. [Röc06]).

**Lemma 4.2.2.**

i. \(\xi = \infty\) \(\tilde{P}\)-a.s. (not necessarily \(P\)-a.s.).

ii. For all \(s \leq t\), \(\varphi_t \mathcal{F}_t\)-measurable and positive we have

\[ E_{\tilde{P}}[\varphi_t|\mathcal{F}_s] = 1_{\{Z_s \neq 0\}} Z_s^{-1} E_{\tilde{P}}[\varphi_t Z_t|\mathcal{F}_s] \quad \tilde{P} - a.s.. \]

iii. Let \(\tilde{M} := (\tilde{M}_t)_{t \geq 0}\) be an adapted continuous process. Then \(\tilde{M}\) is a local \(\tilde{P}\)-martingale (up to \(\infty\)), if \(\tilde{M} \cdot Z\) is a local \(P\)-martingale (up to \(\xi\)).

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Proof. i. We have

\[ \tilde{P}[\xi < t] = \begin{cases} E_P[Z_{\xi\wedge t}, \xi < t] = E_P[Z_{\xi}, \xi < t] = 0. \end{cases} \]

Letting \( t \to \infty \), the assertion follows.

ii. For all \( \varphi, F_s \)-measurable and positive,

\[ E_P[\varphi] = E_P[\varphi 1_{\{Z_{\xi}\neq 0\}}] = E_P[\varphi 1_{\{Z_{\xi}\neq 0\}} E_P[Z_{\xi} | F_s]] = E_P[\varphi 1_{\{Z_{\xi}\neq 0\}}] E_P[Z_s^{-1} E_P[\varphi_t Z_t | F_s]] \]

iii. Let \( T_1 \leq T_2 \leq \ldots < \xi \) (on \( \{\xi > 0\} \)) be a localizing sequence for the local \( P \)-martingale \( M \cdot Z \) (hence in particular \( \sup_n T_n = \xi \)). Then for all bounded stopping times \( T \)

\[ E_P[\tilde{M}_{T \wedge T_n}] = E_P[M_{T \wedge T_n} Z_{T \wedge T_n}] = E_P[M_0 Z_0] = E_P[M_0]. \]

Hence, \( (\tilde{M}_{T_n \wedge T})_{t \geq 0} \) is a \( \tilde{P} \)-martingale and the assertion follows, since \( \xi = \infty \) \( \tilde{P} \)-a.s. by (i).

Proposition 4.2.3 (General Girsanov transform). Let \( M \) be a continuous local \( P \)-martingale (up to \( \infty \)). Then

\[ \tilde{M} := M - \int_0^\infty \frac{1}{Z_s} \, d(M, Z)_s \]

is a continuous local \( \tilde{P} \)-martingale (up to \( \infty \)).

Proof. By Lemma 4.2.2(iii) we have to show that \( \tilde{M} \) is a local \( P \)-martingale up to \( \xi \).

We have that \( M \cdot Z \) is \( P \)-a.s. continuous. Let \( \xi_n := \inf\{t \geq 0 | Z_t \leq \frac{1}{n}\} \) and \( T_n, n \in \mathbb{N} \), be stopping times, such that \( M_{T_n} \) is a localizing sequence for the following three local \( P \)-martingales \( M, M \cdot Z - \langle M, Z \rangle \) and

\[ \int_0^r \left( \int_0^s \frac{1}{Z_s} \, d(M, Z)_s \right) \, dZ_r. \]

Note that \( \int_0^r \frac{1}{Z_s} \, d(M, Z)_s \) is predictable, since it is \( P \)-a.s. continuous in \( r \) and adapted. We have that \( 0 \leq T_1 \leq T_2 \leq \ldots \leq T_n < \xi \) on \( \{\xi > 0\} \), since \( \xi_n < \xi \) on \( \{\xi > 0\} \) because \( Z \) is \( P \)-a.s. continuous. Furthermore, \( \xi_n' \), because \( Z \) is \( P \)-a.s. continuous. Then by Itô’s product rule for all \( t \geq 0 \)

\[ (M \cdot Z)_{t \wedge T_n} = (M \cdot Z)_{t \wedge T_n} - Z_{t \wedge T_n} \int_0^{t \wedge T_n} \frac{1}{Z_s} \, d\langle M, Z \rangle_s \]

\[ = (M \cdot Z)_{t \wedge T_n} - \int_0^{t \wedge T_n} Z_r \, \left( \int_0^r \frac{1}{Z_s} \, d\langle M, Z \rangle_s \right) - \int_0^{t \wedge T_n} \int_0^r \frac{1}{Z_s} \, d\langle M, Z \rangle_s \, dZ_r \]

\[ = (M \cdot Z)_{t \wedge T_n} - \int_0^{t \wedge T_n} Z_r \, \left( \int_0^r \frac{1}{Z_s} \, d\langle M, Z \rangle_s \right) - \int_0^r \frac{1}{Z_s} \, d\langle M, Z \rangle_s \, dZ_r \]

Here, \( \int_0^{t \wedge T_n} \frac{1}{Z_s} \, d\langle M, Z \rangle_s \, dZ_r \) is a \( P \)-martingale, and \( (M \cdot Z)_{t \wedge T_n} - \langle M, Z \rangle_{t \wedge T_n} \) is a \( P \)-martingale. Therefore, \( M \cdot Z \) is a local \( P \)-martingale up to \( \xi \).
4.3. Girsanov Transform with Brownian Motion

**Remark 4.2.4** ("Z as an exponential martingale"). We have by Itô formula for $t < \xi$ $P$-a.s.

$$\log Z_t = \log Z_0 + \int_0^t \frac{1}{Z_s} dZ_s - \int_0^t \frac{1}{Z_s^2} d\langle Z \rangle_s \quad (*)$$

$Y_t$ is a local $P$-martingale up to $\xi$. Since $Z$ is a $P$-martingale up to $\infty$ and has pathwise quadratic variation (cf. 1.4.2) we have

$$\langle Y \rangle_t = \int_0^t \frac{1}{Z_s} d\langle Z \rangle_s, \ t < \xi.$$ Exponentiating $(*)$ yields

$$Z_t = Z_0 \cdot e^{Y_t - \frac{1}{2} \langle Y \rangle_t} \quad \text{(exponential } P\text{-martingale)}.$$

Then $Z$ solves the following SDE for given $Y$ up to $\xi$:

$$dZ = Z \, dY \Leftrightarrow Z_t = Z_0 + \int_0^t Z_s \, dY_s, \quad (Z(0) = Z_0)$$

**Proof.** By the 2-dimensional Itô’s formula we have

$$Z_t = Z_0 + Z_0 \int_0^t e^{Y_s - \frac{1}{2} \langle Y \rangle_s} \, dY_s + \frac{1}{2} Z_0 \int_0^t e^{Y_s - \frac{1}{2} \langle Y \rangle_s} \, d\langle Y \rangle_s - \frac{1}{2} Z_0 \int_0^t e^{Y_s - \frac{1}{2} \langle Y \rangle_s} \, d\langle Y \rangle_s$$

$$= Z_0 + \int_0^t Z_s \, dY_s.$$ Note that the last terms in Itô’s formula don’t occur, since $\langle Y \rangle$ is of bounded variation. \hfill \Box

**Corollary 4.2.5.** For $\tilde{M}$ as in Proposition 4.2.3 we have $\tilde{M} = M - \langle M, Y \rangle$ $P$-a.s. up to $\xi$, hence $\tilde{P}$-a.s. up to $\infty$.

**Proof.**

$$\int_0^t \frac{1}{Z_s} d\langle M, Z \rangle_s = \langle N, H \cdot M \rangle = H \langle N, M \rangle \left( M, \int_0^t \frac{1}{Z_s} dZ_s \right) = \langle M, Y \rangle.$$ Hence, the assertion follows by Proposition 4.2.3. \hfill \Box

### 4.3. Girsanov Transform with Brownian Motion

Let $(X_t)_{t \in [0,1]}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a right-continuous filtration $(\mathcal{F}_t)_{t \in [0,1]}$. (Take for example $(\Omega, \mathcal{F}, P)$ to be the canonical Wiener space $(C[0,1], \mathcal{F}, P)$ with classical Wiener measure $P$.)

**Heuristics:** Let $Y$ be a local continuous $P$-martingale and $\tilde{P} \ll P$ with density

$$Z_t := e^{Y_t - \frac{1}{2} \langle Y \rangle_t} \text{ on } \mathcal{F}_t.$$ Then we know that $\tilde{M} = X - \langle X, Y \rangle$ is a local continuous $\tilde{P}$-martingale up to $\infty$, where $X$ is a martingale with respect to $P$ (as $M$ above). Since $\langle \tilde{M} \rangle_t = \langle X \rangle_t = t$, it follows by Levy’s characterization theorem of Brownian motion that $\tilde{M}$ is a Brownian motion under $\tilde{P}$. We want to get that

$$d\langle X, Y \rangle = B_t \, dt.$$ We succeed by the following

**Ansatz:**

$$Y_t := \int_0^t b_s \, dX_s.$$
Remark 4.3.1. In order to make this work, we need that
i. \((b_t)_{t \in [0,1]}\) is progressively measurable,

\[ P \left[ \int_0^1 b_s^2 \, ds < \infty \right] = 1 \]

(Then, \(Y\) is a continuous local \(P\)-martingale!) AND!

iii. \(\tilde{P} = e^{\int_0^1 b_s \, dX_s - \frac{1}{2} \int_0^1 b_s^2 \, ds} \cdot P\)
is a probability measure, i.e.

\[ E_P \left[ e^{\int_0^1 b_s \, dX_s - \frac{1}{2} \int_0^1 b_s^2 \, ds} \right] = 1 \quad (4.3.2) \]

Theorem 4.3.2 (Girsanov transform for \(M\) as a Brownian motion): Assume (i) - (iii) from Remark 4.3.1. Let

\[ Z_t := \exp \left[ \int_0^t b_s \, dX_s - \frac{1}{2} \int_0^t b_s^2 \, ds \right] , \ t \geq 0 , \]

and

\[ \tilde{P} := Z_1 \cdot P . \]

Then

\[ W_t := X_t - \int_0^t b_s \, ds , \ t \geq 0 , \]
is a Brownian motion under \(\tilde{P}\).

Proof. Claim: \((Z_t)_{t \geq 0}\) is a \(P\)-martingale up to \(\infty\).

Step 1: \((Z_t)_{t \geq 0}\) is a (global) \(P\)-supermartingale.

Proof. It is clear, that \((Z_t)\) is a local continuous \(P\)-martingale. Let \(0 \leq T_1 \leq \ldots \leq T_n \ldots < \infty\) on \(\{ \xi > 0 \}\) be a localizing sequence of stopping times for \((Z_t)_{t \geq 0}\). Then for \(0 \leq s < t\)

\[ E_P[Z_t|\mathcal{F}_s] = E_P[\lim_{n \to \infty} Z_{t \wedge T_n}|\mathcal{F}_s] \]

\[ \leq \liminf_{n \to \infty} E_P[Z_{t \wedge T_n}|\mathcal{F}_s] \]

\[ = \liminf_{n \to \infty} Z_{s \wedge T_n} = Z_s . \]

\[ \square \]

Step 2: \((Z_t)_{t \geq 0}\) is a \(P\)-martingale.

Proof. By (iii) for all \(s \in [0,1]\)

\[ 1 = E[Z_1] \leq E[Z_s] \leq E[Z_0] = 1 . \]

In addition,

\[ 0 \leq Z_s - E_P[Z_1|\mathcal{F}_s] \]

and

\[ \int (Z_s - E_P[Z_1|\mathcal{F}_s]) \, dP = 0 . \]

So, \(Z_s = E_P[Z_1|\mathcal{F}_s]\) \(P\)-a.s.

\[ \square \]
We have
\[ \langle X, \int_0^t b_s \, dX_s \rangle_t = \int_0^t b_s \, ds. \]
Hence, by Corollary 4.2.5 it follows that \( W \) is a continuous local \( \tilde{P} \)-martingale. But \( \langle W \rangle_t = \langle X \rangle_t = t \) \( P \)-a.s., hence \( P \)-a.s.. So, by Lévy (Proposition 1.5.1) the assertion follows.

\[ \square \]

**Remark 4.3.3.**

i. **Special case:**

\[ b_t(\omega) := b(X_t(\omega), t), \]

i.e. depending only on “present” time, where \( b : \mathbb{R} \times [0, 1] \to \mathbb{R} \) is \( \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0, 1]) \) measurable. Then 4.3.1(i) is fulfilled.

4.3.1(ii) is fulfilled, if e.g.

\[
\begin{align*}
E \left[ \int_0^1 b_t^2 \, dt \right] &= \int_0^1 E[b_t^2(X_t, t)] \, dt \\
&\equiv \int_0^1 p_t f(0) \, dt \\
p_t(0, dx) &= N(0, t) \int_0^1 b^2(x, t) N(0, t)(dx) \, dt \\
&= \int_0^1 \int b^2(x, t) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, dx \, dt < \infty.
\end{align*}
\]

In particular, it is not necessary that \( b \) is bounded.

To satisfy 4.3.1(iii) we have to work harder (see below)!

ii. In Theorem 4.3.2 \( (X_t)_{t \geq 0} \) solves the following SDE under \( \tilde{P} \)

\[ dX_t = dW_t + b_t \, dt. \]

Here, \( W_t \) is only a Brownian motion under \( \tilde{P} \).

**Example 4.3.4.** Consider \( b_t = \alpha \in \mathbb{R} \) fixed for all \( t \). Then, clearly, 4.3.1(i),(ii) hold, but also:

**Claim:**  (iii) holds.

**Proof.** Since \( P \circ X_1 \) is normal distributed, we have

\[
\begin{align*}
E \left[ e^{\alpha X_1 - \frac{1}{2} \alpha^2} \right] &= e^{-\frac{1}{2} \alpha^2} \int e^{\alpha x} N(0, 1)(dx) \\
&= e^{-\frac{1}{2} \alpha^2} \int e^{\alpha x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\
&= e^{-\frac{1}{2} \alpha^2} \cdot e^{\frac{1}{2} \alpha^2} = 1.
\end{align*}
\]

Here, we have used that for the Laplace-transform \( \mathcal{L} \) we have

\[
\mathcal{L}(N(0, \sigma^2))(\xi) = \int e^{\xi x} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} x^2} \, dx = e^{\frac{1}{2} \xi^2 \sigma^2}.
\]

\[ \square \]
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Hence, by Theorem 4.3.2 under $\tilde{P} := Z_1 \cdot P$ the process $W_t = X_t - \alpha t, t \geq 0$, is a Brownian motion.

Now we consider the canonical model for Brownian motion, i.e. $\Omega = C([0,1]), P$ as a Wiener measure and $X_t(\omega) := \omega(t), t \geq 0$. Define for $\alpha \in \mathbb{R}$

$$\Lambda_\alpha : C([0,1]) \to C([0,1])$$

by

$$\Lambda_\alpha(\omega)(t) = \omega + \alpha \cdot t, \quad t \in [0,1].$$

Note that by Theorem 4.3.2 we know $(Z_1 \cdot P) \circ W^{-1}$ is a Wiener measure! In particular,

$$\tilde{P} \circ \Lambda^{-1}_{-\alpha} = P.$$

**Proposition 4.3.5.** Let $P^\alpha := P \circ \Lambda^{-1}_{-\alpha}$. Then

$$P^{-\alpha} = \exp(\alpha X_1 - \frac{1}{2} \alpha^2) \cdot P.$$

**Proof.** Since by Theorem 4.3.2

$$\tilde{P} \circ \Lambda^{-1}_{-\alpha} = P$$

it follows that

$$P \circ \Lambda^{-1}_{-\alpha} = \tilde{P} = \exp(\alpha X_1 - \frac{1}{2} \alpha^2) \cdot P_{h(s)} = \exp \left( \int_0^1 \dot{h}(s) dX_s - \frac{1}{2} \int_0^1 (\dot{h}(s))^2 ds \right)$$

But we have even more:

**Theorem 4.3.6 (Cameron-Martin).** Let

$$h \in E := C([0,1])_0 := \{ h \in C([0,1]) | h(0) = 0 \}$$

and

$$X_h : \Omega \to C([0,1])$$

defined by

$$X_h(\omega) = X(\omega) + h.$$

Let $P_0$ be the law of $X$ on $C([0,1])$ (i.e. the classical Wiener measure) and $P_h := P_0 \circ X_h^{-1}$ be the law of $X_h$ on $C([0,1])$. Then the following assertions are equivalent:

i. $P_h \approx P_0$ with

$$\frac{dP_h}{dP_0} = \exp \left( \int_0^1 \dot{h}(s) dX(s) - \frac{1}{2} \int_0^1 (\dot{h}(s))^2 ds \right)$$

ii. $h \in H := \left\{ h \in C([0,1])_0 | h \text{ is absolutely continuous and } \dot{h} \in L^2([0,1], ds) \right\}$.

$H$ is called **Cameron-Martin space**. $H$ is a Hilbert space with inner product

$$\langle h, \tilde{h} \rangle_H := \int_0^1 \dot{h}(s) \dot{\tilde{h}}(s) ds, \quad h, \tilde{h} \in H.$$
Note that $\|h\|_H^2 = \langle h, h \rangle_H = 0$ implies $\dot{h}(s) = 0$ $\text{d}s$-a.s.. Hence,

$$h(t) = h(0) + \int_0^t \dot{h}(s) \, ds = h(0) = 0 \quad \forall t \geq 0.$$ 

So, $\|\cdot\|_H$ is not only a semi norm, but a norm. We have that $H$ is dense in $E := C([0, 1])_0$ with respect to $\|\cdot\|_H$. Then identifying $H$ with its dual $H'$ by Riesz map $R$ we get

$$E' \hookrightarrow H' \xrightarrow{R} H \subset E.$$ 

Before we prove Theorem 4.3.6, we want to characterize $E'$.

**Lemma 4.3.7.** For $M_0 := \{ \mu | \mu \text{ is a signed measure of finite total variation on } [0, 1], \text{ such that } \mu(1) := \int_1^1 d\mu = 0 \}$, we have

$$E' = M_0.$$ 

**Remark 4.3.8.** A signed measure $\mu$ can be written as $\mu = \mu_1 - \mu_2$ with positive measures $\mu_1$ and $\mu_2$ and the total variation

$$\sup_{A \in \mathcal{B}([0, 1])} \mu(A) = \|\mu\|_{\text{var}} < \infty.$$ 

In addition, for a signed measure $\mu$, there exist positive measures $\mu^+, \mu^-$ and $S \in \mathcal{B}([0, 1])$ such that $\mu = \mu^+ - \mu^-$ and

$$\mu^+(S^c) = 0 \quad \text{and} \quad \mu^-(S) = 0.$$ 

Then

$$\|\mu\|_{\text{var}} = \mu^+([0, 1]) + \mu^-([0, 1]).$$

**Proof of 4.3.7.** "$\supset$": Defining $f \mapsto \mu(f) := \int f \, d\mu$ for $\mu \in M_0$ we see that any $\mu$ defines an element in $E'$. Furthermore, if $\mu, \nu \in M_0$ such that $\mu(f) = \nu(f)$ $\forall f \in E$, then for all $f \in C([0, 1])$

$$\int f \, d\mu = \int f \, d\mu - f(0) \int 1 \, d\mu = \int_{E} (f - f(0)) \, d\mu$$

$$= \int (f - f(0)) \, d\nu = \int f \, d\nu - \int f(0) \, d\nu = \int f \, d\nu.$$ 

Hence, (since $\mu(1) = \nu(1) = 0$) $\mu$ and $\nu$ define the same element in $E'$. (See also [Cho69a] and [Cho69b].)

"$\subset$": Let $l \in E'$. Then by the Hahn-Banach theorem there exists an $\bar{l} \in (C[0, 1])'$ such that $\bar{l} = l$ on $E' \subset C([0, 1])$. Hence, by Riesz-Markov (cf. [Röc04]) there exists a signed measure $\nu$ on $[0, 1]$ of finite total variation such that

$$\bar{l}(f) = \int f \, d\nu \quad \forall f \in C([0, 1]).$$ 

Set $\mu := \nu - \nu(1)\delta_0$, where $\delta_0$ is the Dirac measure at $0 \in [0, 1]$. Then $\mu(1) = 0$, so $\mu \in M_0$ and for all $f \in E$

$$\int f \, d\mu = \int f \, d\nu - \nu(1) f(0) \int = 0 = \int f \, d\nu = \bar{l}(f) = l(f).$$
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**Proof of Theorem 4.3.6.** Without loss of generality consider \((\Omega, \mathcal{F}, P), \mathcal{F}_t\) and \((X_t)_{t \geq 0}\) as the canonical model, hence \(P = P_0\).

(ii) \(\Rightarrow\) (i): (An alternative proof using Fourier transform can be found in [MR92, Chapter II].) We first check the conditions of Remark 4.3.1 (i)-(iii):

(i) and (ii) are obviously satisfied for 
\[
  b_s(\omega) := \dot{h}(s) \quad \forall \omega \in \Omega, s \in [0, 1].
\]

(iii): Define 
\[
  Y_t := \int_0^1 \dot{h}(s) \, dX_s \quad \text{(Itô-Integral)}.
\]

Then \(Y_1\) is centered (i.e. \(E[Y_1] = 0\) for all \(t\) (martingale)) and normally distributed as an \(L^2\)-limit of centered, normally distributed random variables (cf. [Röc06]). Furthermore,

\[
  E[Y_1^2] = E[(Y_1)_1] = E\left(\int_0^1 (\dot{h}(s))^2 \, ds\right) = \|h\|_H^2,
\]

i.e.
\[
  Y_1 \sim N(0, \|h\|_H^2).
\]

Hence,
\[
  E\left[e^{Y_1 - \frac{1}{2}\|h\|_H^2}\right] = e^{-\frac{1}{2}\|h\|_H^2} E[e^{Y_1}] = 1
\]

So, 4.3.1(iii) holds. Therefore, applying Girsanov (Theorem 4.3.2) we obtain that under
\[
  \tilde{P} := Z_1 \cdot P := \exp\left[\int_0^1 \dot{h}(t) \, dX_t - \frac{1}{2} \int_0^1 \dot{h}(t)^2 \, dt\right] \cdot P
\]

\(W := X - h\) is a Brownian motion, i.e. \(\tilde{P} \circ W^{-1} = P_0\), i.e. \(\tilde{P} = P_0 \circ (W^{-1})^{-1} = P_0 \circ X_h^{-1} = P_h\).

But, since \(E[Y_1^2] < \infty\), we have \(Z_1 > 0\) \(P\)-a.s., therefore, \(P_h \approx P\).

(i) \(\Rightarrow\) (ii): We know \(P_0(E) = 1\) with \(E := C([0, 1])_0\). Clearly, we have that \(H \subset E\) dense with respect to the norms \(\|\cdot\|_H\) and \(\|\cdot\|_\infty\) (cf. functional analysis).

**Continuity:** Let \(h \in H\). Then (by definition of absolute continuity)
\[
  \dot{h}(t) = \dot{h}(0) + \int_0^t \dot{h}(s) \, ds \quad \forall t \in [0, 1],
\]

thus,
\[
  |h(t)| \leq \int_0^t |\dot{h}(s)| \, ds \leq \int_0^1 |\dot{h}(s)| \, ds \leq \left(\int_0^1 |\dot{h}(s)|^2 \, ds\right)^{\frac{1}{2}} = \|h\|_H, \quad \forall t \in [0, 1],
\]

hence,
\[
  \|h\|_\infty \leq \|h\|_H.
\]

Let \(R : H \to H\) the Riesz isomorphism. Then
\[
  E' \subset H' \xrightarrow{R} H \subset E \quad \text{(continuously)}.
\]

(Cf. Lemma 4.5.1 below: \(R(\mu) = -\int_0^t \mu([0, s]) \, ds, \quad \mu \in E'\).)
Claim 1: Let $\mu \in E' (= M_0)$. The map $C([0, 1]) \ni f \mapsto \mu(f) \in \mathbb{R}$ is Gaussian distributed under $P_0$, more precisely

$$P_0 \circ \mu^{-1} = N(0, E[\mu^2]).$$

Proof. Since $\mu = \lim_{n \to \infty} \sum_{i=1}^{N_n} \alpha_{i}^{(n)} \delta_{t_i^{(n)}}$, $\alpha_{i}^{(n)} \in \mathbb{R}$, $t_i^{(n)} \in [0, 1]$ weakly (cf. Bauer), we realize that $\mu_n : E \to \mathbb{R}, \ n \in \mathbb{N}$, are jointly Gaussian because

$$\sum_{i=1}^{N} (\gamma \mu_{n_i})(f) = \sum_{i=1}^{N} \gamma_{i} \sum_{i=1}^{N_{n_i}} \alpha_{i}^{(n)} \delta_{t_i^{(n)}}(f) = \sum_{i=1}^{N} \gamma_{i} \sum_{i=1}^{N_{n_i}} \alpha_{i}^{(n)} f(t_i^{(n)}) \sim X_{t_i^{(n)}}(f)$$

i.e.

$$\sum_{i=1}^{N} \gamma_{i} \mu_{n_i} = \sum_{i=1}^{N} \gamma_{i} \sum_{i=1}^{N_{n_i}} \alpha_{i}^{(n)} X_{t_i^{(n)}} \sim N(0, E(\sum_{i=1}^{N} \gamma_{i} \mu_{n_i})^2)$$

Hence, since $\mu_n \xrightarrow{n \to \infty} \mu$ weakly, $\hat{\mu}_n \xrightarrow{n \to \infty} \hat{\mu}$ and $\mu$ is centered Gaussian.

Claim 2: Let $\mu \in M_0$ of the form $\mu = \varrho \cdot dt$ and $\varrho$ bounded and

$$\mu(1) = \int_{0}^{1} \varrho \ dt = 0.$$  \hfill(*)

Then

$$E[\mu^2] = \int_{0}^{1} \left( \int_{0}^{t} \varrho(s) \ ds \right)^2 \ dt = \|R(\mu)\|_H^2.$$  \hfill(4.3.3)
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Proof.

\[ E[\mu^2] X_t(\omega) = E \left( \int_0^1 X_t \phi(t) \, dt \int_0^1 X_t' \phi(t') \, dt' \right) \]

Using Fubini = \int_0^1 \int_0^1 \phi(t) \phi(t') E[X_t X_t'] \, dt \, dt' = \int_0^1 \int_0^1 \phi(t) \phi(t') (t \land t') \, dt' \, dt

\[ = \int_0^1 \int_0^t \phi(t) \phi(t') \, dt' \, dt + \int_0^1 \phi(t) \int_t^1 \phi(t') \, dt' \, dt \]

= \int \phi(t) \left[ \int_0^t \phi(t') \, dt' \cdot t - \int_0^t \int_t^1 \phi(s) \, ds \cdot 1 \, dt' \right] \, dt \, \int \phi(t) + \int \phi(t') \, dt' \, dt

\[ = \int \phi(t) t \left( \int_0^t \phi(t') \, dt' + \int_0^1 \phi(t') \, dt' \right) \, dt - \int \phi(t) \int_0^t \phi(s) \, ds \, dt' \, dt \]

\[ = \int \left[ \int_0^t \phi(t') \, dt' \cdot t - \int_0^t \int_t^1 \phi(s) \, ds \cdot 1 \, dt' \right] \, dt \, t \int \phi(t) + \int \phi(t') \, dt' \, dt \]

\[ = \int \phi(t) t \left( \int_0^t \phi(t') \, dt' + \int_0^1 \phi(t') \, dt' \right) \, dt - \int \phi(t) \int_0^t \phi(s) \, ds \, dt' \, dt \]

\[ \overset{I.b.p.}{=} - \int_0^1 \phi(t) \, dt \cdot \int_0^1 \int_t^1 \phi(s) \, ds \, dt' + \int_0^0 \phi(t) \, dt \cdot \int_0^1 \int_t^1 \phi(s) \, ds \, dt' + \int_0^1 \left( \int_0^t \phi(s) \, ds \right)^2 \, dt \]

\[ = \int_0^1 \left( \int_0^t -\phi(s) \, ds \right)^2 \, dt \]

\[ = \int_0^1 \left( \frac{d}{dt} \int_0^t \left( \int_0^t -\phi(s) \, ds \right) \, dt' \right)^2 \, dt = \int_0^1 \left( \frac{d}{dt} \tilde{R}(\mu)(t) \right)^2 \, dt = \left\| \tilde{R}(\mu) \right\|_H^2. \]

(4.3.4)

For Claim 2 it remains to show that \( \tilde{R}(\mu) = R(\mu) \), i.e.

\[ R(\mu) = - \int \left( \int_0^t \phi(s) \, ds \right) \, dt'. \]

(4.3.5)

To this end let \( \tilde{h} \in H \). Then

\[ \mu(\tilde{h}) = \int_0^1 \tilde{h}(t) \phi(t) \, dt \overset{1.b.p.}{=} - \int \int \frac{d}{dt} \tilde{h}(t) \int_0^t \phi(t) \, ds \, dt = \left\langle \tilde{h}, - \int \left( \int_0^t \phi(s) \, ds \right) \, dt' \right\rangle \]

(4.3.6)

and Claim 2 is proved. \( \square \)

Since \( C_0^1([0, 1]) \) is dense in \( L^2([0, 1], dt) \) for all \( \tilde{h} \in H \), there exists a sequence \((v_n)_{n \in \mathbb{N}}\) in \( C_0^1([0, 1]) \) such that \( v_n \to \tilde{h} \) as \( n \to \infty \) in \( L^2([0, 1], dt) \). Hence

\[ u_n := \int_0^1 v_n \, dt \overset{n \to \infty}{\longrightarrow} \tilde{h} \quad \text{in } H. \]

Since by (4.3.5)

\[ u_n = R(\mu_n) \quad \text{for} \quad \mu_n := - \dot{v}_n \, dt \in E' \]

it follows that for

\[ \tilde{M}_0 := \left\{ \phi \, dt \left| \phi \in C([0, 1]), \int_0^1 \phi \, dt = 0 \right. \right\} \]

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that $R(\tilde{M}_0)$ is dense in $H$ with respect to $\|\cdot\|_H$. $R(\tilde{M}_0)$ is also a linear subspace of $H$.

Let $h \in E(= C([0,1])_0)$ such that $P_h \ll P$.

Claim: $\mu_n \in \tilde{M}_0$ such that $R(\mu_n) \xrightarrow{n \to \infty} 0$ in $H$. Then $\mu_n(h) \xrightarrow{n \to \infty} 0$.

Suppose the claim is true. Then $h \in H$.

**Proof.** By claim the map $R(\tilde{M}_0) \ni R(\mu) \mapsto R(\mu) \in \mathbb{R}$ (for fixed $h$) is a linear continuous functional on $(R(\tilde{M}_0), \|\cdot\|_H)$. Hence, by Riesz ($R(\tilde{M}_0)$ is dense in $H$, hence Riesz is applicable) there exists an unique $h_0 \in H$ such that

$$\mu(h) = \langle R(\mu), h_0 \rangle_H \quad \forall \mu \in \tilde{M}_0.$$  

But by (4.3.6) we also have $\langle R(\mu), h_0 \rangle_H = \mu(h_0)$. Therefore,

$$\mu(h) = \mu(h_0). \quad \text{(4.3.7)}$$

Hence, for all $q \in C([0,1])$ and for

$$\tilde{h} := h - \int_0^1 h(t) \, dt, \quad \tilde{h}_0 := h_0 - \int_0^1 h_0(t) \, dt$$

we have

$$\int q \tilde{h} \, dt = \int \left( q - \int_0^1 q \, ds \right) \tilde{h} \, dt = \int \left( q - \int_0^1 q \, ds \right) h \, dt$$

$$\overset{(4.3.7)}{=} \int \left( q - \int_0^1 q \, ds \right) h_0 \, dt = \int \left( q - \int_0^1 q \, ds \right) \tilde{h}_0 \, dt = \int q \tilde{h}_0 \, dt.$$  

Hence, $\tilde{h} = \tilde{h}_0$, therefore, $h = h_0$, because $h(0) = h_0(0) = 0$. Therefore, the assertion is true as long as claim holds. 

**Proof of Claim.** Since $R(\mu_n) \to 0$ in $H$, it follows by (4.3.3) that

$$E[\mu_n^2] = \|R(\mu_n)\|_H^2 \xrightarrow{n \to \infty} 0,$$

i.e. $\mu_n \to 0$ in $L^2(P)$, hence, also in $P$-measure, therefore, because of $P_h \ll P$, also in $P_h$-measure. Since $\{\mu_n, n \in \mathbb{N}\}$ is a Gaussian family under $P_h$ such that

$$E_{P_h}[\mu_n] = \int_{\mu_n(\omega) + h}^{\mu_n(\omega)} P(d\omega) = \int_{\mu_n(\omega) + h}^{\mu_n(\omega) + \mu_n(h)} P(d\omega) + \mu_n(h), \quad \text{(4.3.8)}$$

it follows by [Ròc06] that $\mu_n \xrightarrow{n \to \infty} 0$ in $L^p(P_h)$ for all $p \geq 1$, hence for $p = 1$

$$\limsup_{n \to \infty} |\mu_n(h)| \overset{(4.3.8)}{\leq} \limsup_{n \to \infty} E[|\mu_n|] = 0.$$

So, the claim is proved.
4. Girsanov Transformation

4.4. Novikov condition

Let \((\Omega, \mathcal{F}, P)\) be a probability space together with a right-continuous complete filtration \((\mathcal{F}_t)\) and let \((Y_t)_{t\geq 0}\) be a \(P\)-a.s. continuous local martingale. Then by 1.4.8

\[ Z_t := \exp(Y_t - \frac{1}{2} \langle Y \rangle_t), \quad t \geq 0, \]

is a \((P\)-a.s.) continuous local martingale. We want to develop a condition for (4.3.2).

Lemma 4.4.1. Let \(t \geq 0\) and \(\langle Y \rangle_t\) be bounded. Then \(Z_t \in \mathcal{L}^p\) for all \(p > 1\) and

\[ E(Z_t^p) \leq \exp\left(\frac{1}{2} p(p - 1) \|\langle Y \rangle_t\|_\infty\right). \]

Proof.

\[
E(Z_t^p) = E\left[\exp\left(pY_t - \frac{1}{2} p^2 \langle Y \rangle_t + \left(\frac{1}{2} p^2 - \frac{1}{2} p\right) \langle Y \rangle_t\right)\right]
\leq \exp\left(\frac{1}{2} p(p - 1) \|\langle Y \rangle_t\|_\infty, \Omega\right) E\left[\exp\left(pY_t - \frac{1}{2} p^2 \langle Y \rangle_t\right)\right].
\]

Set \(\tilde{Y}_t := pY_t\). Then \(\langle \tilde{Y} \rangle_t = p^2 \langle Y \rangle_t\) and, therefore,

\[
E(\exp(\tilde{Y}_t - \frac{1}{2} \langle \tilde{Y} \rangle_t)) \leq 1,
\]
since it is a supermartingale (by Fatou).

Now we come to the condition for (4.3.2).

Theorem 4.4.2 (Novikov). If \(E(\exp(\frac{1}{2} \langle Y \rangle_t)) < \infty\) for all \(t\), then \(E(Z_t) = 1\) for all \(t\).

Remark 4.4.3. i. In the examples above (see 4.3.4 and 4.3.6) we had \(Y_t = \int_0^t b_s \, dX_s\) and

\[ E\left(\exp\left(\frac{1}{2} \langle Y \rangle_t\right)\right) = E\left(\exp\left(\frac{1}{2} \int_0^t b_s^2 \, ds\right)\right) = E(\exp(\frac{1}{2} \langle Y \rangle_t)) < \infty \]

was always satisfied.

ii. For the proof of Theorem 4.4.2 we need that by a time change one can construct a Wiener process \(W\), such that

\[ Y_t = W(\langle Y \rangle_t), \]

and (for every fixed \(t\)) \(\langle Y \rangle_t\) is a stopping time with respect to a suitable filtration for which \(W\) is adapted. This means that \(Y_t\) has the form \(W_T\), and it holds

\[ \langle Y \rangle_t = T. \]

The details are presented in appendix A.

Now Theorem 4.4.2 follows from

Theorem 4.4.4. Let \((W_t)_{t\geq 0}\) be a Wiener process on \((\Omega, \mathcal{F}, P)\) and \(T\) be a stopping time. If

\[ E(\exp(\frac{1}{2} T)) < \infty, \]

then the "Wald identity" holds:

\[ E(\exp(W_T - \frac{1}{2} T)) = 1. \]
Remark 4.4.5. Set $M_t := \exp(W_t - \frac{1}{2}t)$. Then $(M_t)$ is a continuous martingale, since it is a continuous positive supermartingale and $E[M_t] = 1$ (Exercise, cf. 4.3.4). By the optional sampling theorem for unbounded stopping times (cf. [Röc06]), we have

$$E(M_T) \leq E(M_0) = 1.$$  

Thus, it is clear that $E(\exp(W_t - \frac{1}{2}T)) \leq 1$ in 4.4.4. For the proof of ”$\geq$” in 4.4.4 we will need two Lemmas.

Lemma 4.4.6. Let $\tilde{P}$ be a probability measure on $(\Omega, \mathcal{F}, P)$ with

$$\tilde{P}|_{\mathcal{F}_t} = \exp(W_t - \frac{1}{2}t) \cdot P|_{\mathcal{F}_t}, \forall t \geq 0,$$

and $T$ be a stopping time with $P(T < \infty) = 1$. Then

$$E(\exp(W_t - \frac{1}{2}T)) = \tilde{P}(T < \infty).$$ (4.4.9)

In particular, the Wald identity (cf. 4.4.4) holds if and only if

$$\tilde{P}(T < \infty) = 1.$$  

Proof. Since $M_t := \exp(W_t - \frac{1}{2}t)$ is a martingale and $\{T \leq t\} \in \mathcal{F}_{t \land T}$ we have

$$\tilde{P}[T \leq t] = E[1_{\{T \leq t\}} M_t] = E[1_{\{T \leq t\}} M_{t \land T}] = E[1_{\{T \leq t\}} M_T].$$

Letting $t \to \infty$ we get (since $P[T < \infty] = 1$)

$$\tilde{P}[T < \infty] = E[M_T].$$  

Lemma 4.4.7. Let $c > 0$ and define the “passage time of $(W_t - t)$” by

$$T_c := \inf \{t > 0 | W_t = t - c\}.$$  

Then

$$\tilde{P}(T_c < \infty) = 1$$

and, thus, the Wald identity holds for $T_c$. Furthermore,

$$E \left( \exp \left( \frac{1}{2} T_c \right) \right) = e^c.$$  

Proof. By the law of iterated logarithm we have

$$P(T_c < \infty) = 1.$$  

$\tilde{W}_t = W_t - t$ is a Brownian motion with respect to $\tilde{P}$ (cf. Example 4.3.4 with $\alpha = 1$), which means that $T_c$ is a passage time of $\tilde{W}_t$ with respect to $\tilde{P}$. Therefore, again by the law of iterated logarithm

$$\tilde{P}(T_c < \infty) = 1.$$  

Thus, by 4.4.6

$$1 = E \left( \exp \left( W_{T_c} - \frac{1}{2} T_c \right) \right) = e^{-c} E \left( \exp \left( \frac{1}{2} T_c \right) \right).$$  

\[\square\]
4. Girsanov Transformation

Proof of Theorem 4.4.4. It remains to show that "$\geq$" holds:
By 4.4.5 $M_t := \exp(W_t - \frac{1}{2}t), t \geq 0$, is a positive continuous supermartingale. Hence,

$$1 \geq E(M_{T \wedge T}) \geq E(M_T)^{\text{4.4.7}} = 1.$$ 

But then for all $c > 0$

$$1 = E(M_{T \wedge T}) \overset{\text{Wien}}{=} E\left(\exp\left(\frac{1}{2}c\right)\exp(-c), T_c \leq T\right) + E\left(\exp\left(W_T - \frac{1}{2}T\right), T_c \geq T\right)$$

$$\leq e^{-c} E(e^{\frac{1}{2}T}) + E\left(\exp\left(W_t - \frac{1}{2}T\right)\right)$$

$$\iff E\left(\exp\left(W_t - \frac{1}{2}T\right)\right)$$

$\square$
4.5. Integration by Parts on Wiener Space: A First Introduction to the Malliavin Calculus Following J.M. Bismut

Fix the following framework:
Let $P$ be the Wiener measure on $\Omega = C([0, 1])_0 (= E)$, $(X_t)$ the coordinate process,
$$H := \left\{ h \in C([0, 1])_0 \mid h \text{ is absolutely continuous and } \int_0^1 \dot{h}(s)^2 \, ds < \infty \right\}$$
the Cameron-Martin space. $H$ is a Hilbert space with inner product
$$\langle h, g \rangle_H = \int_0^1 \dot{h}(s) \dot{g}(s) \, ds = \langle \dot{h}, \dot{g} \rangle_{L^2([0, 1], dt)},$$
and $F : \Omega \to \mathbb{R}$ be the Wiener functional. We already know
$$E' = \{ \mu | \mu \text{ is a signed measure on } [0, 1] \text{ of bounded variation such that } \mu(1) = 0 \}$$
and
$$E' \subset H' \xrightarrow{\text{R}} H \subset E$$
continuous and densely, where $R$ denotes the Riesz map.

Lemma 4.5.1.
$$R(\mu) = \int_0^1 \mu([s, 1]) \, ds \left( = - \int_0^1 ([0, s]) \, ds \right).$$
(Cf. (4.3.5) as a special case.)

Proof. Let $h \in H$. Then
$$\mu(h) = \int_0^1 \int_0^t \dot{h}(s) \, ds \, d\mu(dt) = \int_0^1 \int_0^1 \dot{h}(s) \, ds \, d\mu(dt) = \int_0^1 \int_0^1 \dot{1}_{[s, 1]}(t) \dot{h}(s) \, ds \, d\mu(dt)$$
$$= \int_0^1 \left( \int_0^1 \mu([r, 1]) \, dr \right) \dot{h}(s) \, ds = \left\langle \int_0^1 \mu([s, 1]) \, ds, h \right\rangle_H$$
Hence $R(\mu) = \int_0^1 \mu([s, 1]) \, ds$ by 4.3.2.

We recall

Definition 4.5.2. $F : C([0, 1])_0 \to \mathbb{R}$ is called Fréchet-differentiable in $\omega \in C([0, 1])_0$, if there exists $F'(\omega) \in E'$ such that
$$F(\omega + \eta) = F(\omega) + F'(\omega)(\eta) + o(\|\eta\|), \quad \forall \eta \in C([0, 1])_0.$$
In this case
$$\nabla F(\omega) := R(F'(\omega)) \in H$$
is called gradient of $F$ at $\omega$. Note that $\nabla F(\omega)$ is in the "tangent space" of $H$ in $\omega$. 

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Remark 4.5.3. Because of Lemma 4.5.1 we have for the measure

\[ F'(\omega)(dt) = F'(\omega, dt) \]

that

\[ \nabla F(\omega)(\cdot) = R(F'(\omega, dt)) = \int_0^1 F'(\omega, s, 1) \, ds. \tag{4.5.10} \]

Then by definition of the derivative

\[ \lim_{\lambda \to 0} \frac{F(\omega + \lambda \eta) - F(\omega)}{\lambda} = \langle \nabla F(\omega), \eta \rangle_H \quad \forall \eta \in H (\subset E := C([0, 1]))_0. \]

Definition 4.5.4. \( F \in L^2(P) \) is called \( H \)-differentiable, if for all \( (\mathcal{F}_t) \)-adapted real processes \( (u_s)_{s \in [0,1]} \), product-measurable, bounded and

\[ U_t(\omega) := \int_0^t u_s(\omega) \, ds, \quad t \in [0,1] \quad (H - \text{vector field on } E) \]

(i.e. \( U(\omega) \in H \)) there exists a \( \mathcal{F}/\mathcal{B}(H) \)-measurable map \( \nabla F : E \to H \) such that

\[ E(\|\nabla F\|_H^2) < \infty \]

and

\[ \frac{F(\omega + \lambda U) + F(\omega)}{\lambda} \to \langle \nabla F(\omega), U(\omega) \rangle_H \quad \text{in } L^2(P) \]

or equivalently

\[ \nabla_U F := \lim_{\lambda \to 0} \frac{F(X + \lambda U) + F(X)}{\lambda} = \langle \nabla F, U \rangle_H \quad \text{in } L^2(P). \]

Here, \( \nabla F \) is called the Malliavin gradient (cf. [Wat84]). Define the Malliavin derivative

\[ D_t F := (\nabla F)(\omega)(t). \]

In particular, \( (D_t F)_{0 \leq t \leq 1} \) is a process. (Fact is that this process always has a version that is product-measurable in \((\omega, t)\)!

Geometric interpretation of \( \nabla F : H \text{-vector field on } E \).

Geometric interpretation of \( \nabla F : L^2([0, 1], dt) \text{-vector field on } E \).

Remark 4.5.5. Let \( u \) and \( U \) as in Definition 4.5.4.

i. We have

\[ \langle \nabla F(\omega), U(\omega) \rangle_H = \langle D F(\omega), u \rangle_{L^2([0,1], dt)} \]

ii. Let

\[ Z_\lambda^t := \exp \left( \lambda \int_0^t u_s \, dX_s - \frac{1}{2} \lambda^2 \int_0^t u_s^2 \, ds \right). \]

Then Novikov’s condition is fulfilled, since \( u \) is bounded. So, Girsanov’s theorem implies that \( X_\lambda := X - \lambda U \) is a Wiener process under \( P^\lambda := Z_1^\lambda P \). Hence,

\[ E_P[F(X_\lambda)] = \int F(X_\lambda) \, dP^\lambda = \int F(X) \, dP = E_P[F(X)] \tag{4.5.11} \]

Lemma 4.5.6.

\[ \lim_{\lambda \to 0} \frac{Z_\lambda^1 - 1}{\lambda} = \int_0^1 u_s \, dX_s \in L^2(P). \]
In fact this limit exists even in $\mathcal{L}^p(P)$ for all $p \geq 1$ (exercise).

**Proof.** By Itô
\[ Z_t^\lambda = 1 + \lambda \int_0^t Z_s^\lambda u_s \, dX_s. \] (*)&

Hence, for $t = 1$
\[ \frac{Z^\lambda_1 - 1}{\lambda} - \int_0^1 u_s \, dX_s = \int_0^1 (Z^\lambda_s - 1)u_s \, dX_s. \]

But by Ito-isometry
\[ E \left[ \left( \int_0^1 (Z^\lambda_s - 1)u_s \, dX_s \right)^2 \right] = E \left[ \int_0^1 (Z^\lambda_s - 1)^2 u_s^2 \, ds \right] \leq \|u\|_\infty^2 E \left[ \int_0^1 (Z^\lambda_s - 1)^2 \, ds \right]. \]

$Z^\lambda_s - 1$ is a martingale, since $Z^\lambda_s$ is in $\mathcal{L}^2(P)$ (see below), thus, by $(\ast)$ $(Z^\lambda_s - 1)^2$ is a submartingale. Therefore,
\[ E \left[ \left( \int_0^1 (Z^\lambda_s - 1)u_s \, dX_s \right)^2 \right] = \|u\|_\infty^2 E[(Z^\lambda_1 - 1)^2]. \]

It remains to prove that $(Z^\lambda_1 - 1)^2 \to 0 \in \mathcal{L}^2(P).$ Clearly, $(Z^\lambda_1 - 1)^2 \to 0$ P-a.s., since for (fixed) $\omega \in E$ it is differentiable in $\lambda = 0,$ in particular continuous. But the set $\{(Z^\lambda_1 - 1)^2 | 0 \leq \lambda \leq 1\}$ is uniformly $P$-integrable because $\{(Z^\lambda_1)^p | 0 \leq \lambda \leq 1\}$ is $\mathcal{L}^1$ bounded for all $p \geq 2$:
\[ E[(Z^\lambda)^p] \leq \exp \left[ \frac{1}{2} \lambda^2 p(p - 1) \left\| \int_0^1 u_s^p \, ds \right\|_{\infty} \right] \leq \exp \left[ \frac{1}{2} p(p - 1) \lambda^2 \|u\|_{\infty} \right]. \]

Hence, the assertion follows by Lebesgue’s dominated convergence theorem. \(\square\)

**Proposition 4.5.7** (Bismut’s integration by Parts formula on Wiener space). Let $u, U$ be as in Proposition 4.5.4 and let $F : E \to \mathbb{R}$ be $H$-differentiable. Then
\[ \left( E \left[ (D,F,U)_{L^2([0,1],dt)} \right] \right) = E[(\nabla F,U)] = E \left[ F \int_0^1 u_s \, dX_s \right]. \]

**Remark 4.5.8.** Proposition 4.5.7 identifies duality between $D$ and $\int \cdot \, dX$ (i.e. between the Malliavin derivative and the Itô-integral). This is the starting point for defining an extension of the Itô-integral, namely the Skorohod integral.

The extension is simply defined as the adjoint $D^*$ of $D.$ Note that $\text{dom } D^*$ contains also non-$(F_t)$-adapted processes (cf. Lectures on Malliavin calculus)! $\text{dom } D$ is the set of all $H$-differentiable functions $F : E \to \mathbb{R} \subset L^2(E, P_0),$
\[ D : \text{dom } D \subset L^2(E, P) \to L^2(E \to L2([0,1], dt), P_0). \]

Hence,
\[ D^* : \text{dom } D^* \subset L^2(E \to L^2([0,1], dt), P_0) \to L^2(E, P_0). \]

**Proof of 4.5.7.** (4.5.11) implies
\[ E_{(P_0)} \left[ \frac{F(X^\lambda) - F(X)}{\lambda} \right] = E_{(P_0)} \left[ \frac{F(X)}{\lambda} \right]. \]

Therefore,
\[ E \left[ \frac{F(X^\lambda) - F(X)}{\lambda} Z_1^\lambda \right] = E \left[ -F(X) \frac{Z_1^\lambda - 1}{\lambda} \right] \xrightarrow{\lambda \to 0} E \left[ -F \int_0^1 u_s \, dX_s \right] \]
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by Lemma 4.5.6. But the left hand side is equal to
\[ E \left[ \frac{1}{\lambda} \left( (F(X^\lambda) - F(X)) + \langle \nabla F, U \rangle_H \right) Z_1^\lambda \right] - E \left[ \langle \nabla F, U \rangle_H Z_1^\lambda \right]. \]

Since \( \langle \nabla F, U \rangle_H \in L^2(E, P_0) \) and \( Z_1^\lambda \to 1 \) in \( L^2(P_0) \) (see Proof of 4.5.6), the second summand converges to \(-E[\langle \nabla F, U \rangle_H] \) as \( \lambda \to 0 \). The first summand converges to 0 by Cauchy-Schwarz and Lemma 4.5.6 as \( \lambda \to 0 \).

First application:
Identification of the integrand in Itô’s representation theorem.

Corollary 4.5.9 (Clark-Formula). Let \( F \in L^2(P_0) \) \( H \)-differentiable. Then
\[ F = E[F] + \int_0^1 E[D_tF|\mathcal{F}_t] \, dX_t \quad P\text{-a.s..} \]

Exercise: Show that \((t, \omega) \mapsto E[D_tF|\mathcal{F}_t](\omega)\) is \( \bar{P}_X \)-a.e. equal to a \( B([0,1]) \otimes \mathcal{F}\)-measurable function. Hint: First consider \( N_t(\omega) = 1_{[a,b]}(t)1_A(\omega) \) for \( A \in \mathcal{F} \) instead of \( D_tF \) (Here, \( \bar{P}_M(dt, d\omega) = d\langle M \rangle_t(\omega)P_0(d\omega)0dtP_0(d\omega) \)).

Proof of 4.5.9. Without loss of generality \( E[F] = 0 \). Let \( G \in L^2(P_0) \). Then
\[ G = E[G] + \int_0^1 u_t \, dX_t, \]
where \( u \in L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}_x) \) and \( u \) adapted. Then by Bismut’s Integration by Parts-formula we have for
\[ u^{(n)} := (u \wedge n) \vee (-n), \quad n \in \mathbb{N}, \]
bounded, that
\[ E[FG] = \lim_{n \to \infty} E \left[ F \int_0^1 u_t^{(n)} \, dX_t \right] \overset{4.5.7}{=} \lim_{n \to \infty} E \left[ \int_0^1 D_tFu_t^{(n)} \, dt \right] \overset{\text{Fubini}}{=} \lim_{n \to \infty} \int_0^1 E[D_tFu_t^{(n)}] \, dt = \int_0^1 E[D_tFu_t] \, dt \overset{\text{u adapted}}{=} \int_0^1 E[D_tF|\mathcal{F}_t]u_t \, dt \overset{2.4.1}{=} E \left[ \left( \int_0^1 E[D_tF|\mathcal{F}_t] \, dX_t \right) G \right]. \]

Example 4.5.10. i. Define
\[ F := \int_0^1 X_t \, dt. \]
Note that \( E \ni \omega \mapsto F(\omega) = \int_0^1 X_t \, dt \) is linear and continuous in \( E \), hence \( F \in E' \). Then \( E' \ni F'(\omega, dt) = dt - \delta_0 \), hence
\[ D_tF(\omega) = F'(\omega, [t, 1]) = 1 - t. \]
So, by 4.5.9
\[ \int_0^1 X_t \, dt = F = \int_0^1 (1 - t) \, dX_t. \]
Note that this can be also (in fact much more easily) proved by Itô’s product rule (see 2.5.6).
ii. \( F := f(X_1) \), where \( f \in \mathcal{C}^2(\mathbb{R}) \). Then by the chain rule (which also holds for Frechet-differentiable functions on Banach spaces) \( F \) is Frechet-differentiable and

\[
F'(\omega, dt) = f'(X_1(\omega)) \cdot \delta_1(-f(X_1(\omega))) \delta_0 \quad (\in E').
\]

Thus,

\[
D_t F(\omega) = F'(\omega, |t, 1]) = f'(X_1(\omega)).
\]

Hence, by Corollary 4.5.9

\[
f(X_1) = F = E[f(X_1)] + \int_0^1 E[f'(X_1)|\mathcal{F}_t] \, dX_t. \tag{*}
\]

\( F \) depends only on time \( t = 1 \), but is represented as an integral along paths. The corresponding integrands can be interpreted in terms of solutions \( h(X_t(\omega), t) \) to the following “final value problem”

\[
h(\cdot, 1) = f(\cdot), \\
\frac{1}{2} h_{xx} + h_t = 0.
\]

The solution has the form \( h(x, t) = p_{1-t} f(x) \) (Fact by elementary calculation, cf. Chapter III). Hence by Itô’s formula (cf. 1.3.1 (iii))

\[
h(X_1, 1) = h(0, 0) + \int_0^1 h_x(X_s, s) \, dX_s, \tag{**}
\]

where \( h(X_1, 1) = f(X_1) \) and \( h(0, 0) = p_{1} f(0) = E[f(X_1)] \). By (*) and (**) and uniqueness in Itô’s representation theorem we can conclude that

\[
h_x(X_s, s) = E[f'(X_1)|\mathcal{F}_s] = (\partial_s(E[fX_{1-s} + x]))_{x=X_s}.
\]

To show that \( F \in \mathcal{L}^2(P) \) is \( H \)-differentiable is rather difficult in general. The following sufficient condition might be useful to check \( H \)-differentiability.

**Proposition 4.5.11.** The following is a sufficient condition for the \( H \)-differentiability of \( F \) in \( \mathcal{L}^2(P) \):

There exists a kernel \( F'(\omega, dt) \) from \( \Omega \) to \( E([0, 1]) \) such that for all \( U \) as in 4.5.4

i. \[
\frac{F(X + \lambda U)}{\lambda} - F(X) \rightarrow_{\lambda \rightarrow 0} 1 \int_0^1 F'(\omega, dt) U_t(\omega) \quad \text{for } P\text{-a.e. } \omega \in E.
\]

ii. For all \( c > 0 \)

\[
|F(X + U) - F(X)| \leq c \| U \|_{\infty} \quad \text{P-a.s.}
\]

In this case

\[
\langle H \rangle \nabla F(\omega) = R(F'(\omega, dt) - F'(\omega, [0, 1]) \cdot \delta_0) = \int \mathcal{F}_0 F'(\omega, [s, 1]) \, ds.
\]

**Proof.** By 4.5.1 we have

\[
\int F'(\omega, dt) U_t(\omega) \bigg|_{U_0=0} = \int \mu_\omega(dt) U_t(\omega) = \langle R(\mu_\omega), U(\omega) \rangle_H = \langle \nabla F(\omega), U(\omega) \rangle_H.
\]

Hence, the assertion follows by Lebesgue’s dominated convergence theorem. \( \square \)
4. Girsanov Transformation

**Example 4.5.12.** $F(\omega) := \max_{0 \leq t \leq 1} \omega$ is not Frechet-differentiable, but define

$$F'(\omega, dt) := \delta_{T(\omega)}(dt),$$

where

$$T := \inf \{ t > 0 | X_T = F \}.$$

**Exercise:** Show that $T = \sup_n T_n$, where $T_n := \inf \{ t > 0 | X_t > -F - \frac{1}{n}, t \in \mathbb{Q} \}$ and that $T_n$ are stopping times for all $n$, hence, $T$ is a stopping time.

Then it follows by Proposition 4.5.11 that

$$D_tF(\omega) = F'(\omega, [t, 1]) = \delta_{T(\omega)}([t, 1]) = 1_{\{T > t\}}(\omega).$$

**Next step:** Identify $E[D_tF|\mathcal{F}_t]$ in order to use Clark formula.

Define

$$M_t := \max_{0 \leq s \leq t} X_s.$$ 

Then we have for $P$-a.e. $\omega \in E$

$$E[D_tF|\mathcal{F}_t](\omega) = P[T > t|\mathcal{F}_t](\omega)$$

$$= P\left( \max_{t \leq s \leq 1} X_s > M_t|\mathcal{F}_t \right)(\omega)$$

$$= P\left( \max_{0 \leq s \leq 1-t} X_{s+t} > M_t|\mathcal{F}_t \right)(\omega).$$

Now, we use the superstrong Markov property, i.e. for any $\mathcal{F}$-measurable functions $\vartheta_t$ and any $\mathcal{F}_t$-measurable function $\varphi$ we have

$$E_x[H(\vartheta_t, \varphi_t)|\mathcal{F}_t](\omega) = E_{X_t}(\omega)[H(\cdot, \omega)].$$

Hence,

$$E[D_tF|\mathcal{F}_t](\omega) = P_{X_t}(\omega)\left( \max_{0 \leq s \leq 1-t} X_s(\omega) > M_t(\omega) \right)$$

$$= P_{X_t}(\omega)\left( \max_{0 \leq s \leq 1-t} X_s(\omega) - X_t(\omega) > M_t(\omega) - X_t(\omega) \right).$$

By the reflection principle, this is equal to

$$2 \cdot P_{X_t}(X_{1-t}(\omega) - X_t(\omega) > M_t(\omega) - X_t(\omega))$$

$$= 2 \cdot P_0(X_{1-t}(\omega) > M_t(\omega) - X_t(\omega))$$

$$= 2 \cdot N(0, (1-t))(\{M_t(\omega) - X_t(\omega), \infty\})$$

$$= 2 \cdot N(0, 1)\left( \frac{M_t(\omega) - X_t(\omega)}{\sqrt{1-t}}, \infty\right)$$

$$= 2 \int_{\frac{M_t(\omega) - X_t(\omega)}{\sqrt{1-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \left( 1 - \Theta \left( \frac{M_t(\omega) - X_t(\omega)}{\sqrt{1-t}} \right) \right).$$
A. Time Transformation

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \((\mathcal{F}_t)_{t \geq 0}\) a right-continuous, complete filtration and \(Y\) a \(P\)-a.s. continuous locale martingale up to \(\infty\) (with respect to \((\mathcal{F}_t)\)) such that \(Y_0 \equiv 0\). For simplicity let
\[
\langle Y \rangle_\infty : = \lim_{t \to \infty} \langle Y \rangle_t = \infty \quad P\text{-a.s.}
\]
Define the "inverse" of \(\langle Y \rangle_t\) by
\[
C_t := \inf \{ s > 0 \mid \langle Y \rangle_s > t \}.
\]

**Theorem A.0.1.** Let \(\langle Y \rangle_\infty = \infty\) \(P\)-a.s. and \(W_t := Y_{C_t}, t \geq 0\). Then, \(W\) is a \((\mathcal{G}_t)\) \(t \geq 0 = \mathcal{F}_{C_t}\) Brownian motion (cf. 1.5.1) and \(Y_t = W_{\langle Y \rangle_t}\) for all \(t \geq 0\).

For the proof we need the following two lemmas.

**Lemma A.0.2.**

\(i.\) The map \(t \mapsto C_t\) is increasing and right-continuous.

\(ii.\) \(C_t\) is an \((\mathcal{F}_s)\)-stopping time for all \(t\).

\(iii.\)
\[
\langle Y \rangle_{C_t} = t \quad P\text{-a.s. } \forall t.
\]

\(iv.\) \(t \leq C_{\langle Y \rangle_t}\). (Note that in general case \(t \neq C_{\langle Y \rangle_t}\) !)

**Proof.** (iii): Without loss of generality \(\langle Y \rangle\) is continuous everywhere. Then \(\langle Y \rangle_{C_t} \geq t\), since \(t \mapsto \langle Y \rangle_t\) is (right-)continuous.

Assumption: \(\langle Y \rangle_{C_t} \geq t + \epsilon\) for \(\epsilon > 0\).

Then, since \(t \mapsto \langle Y \rangle_t\) is continuous, there exists a \(\delta > 0\) such that \(\langle Y \rangle_{C_t + \delta} > t + \frac{\epsilon}{2} > t\). Hence, \(C_t \leq C_t - \delta\). Since this is impossible it follows that \(\langle Y \rangle_{C_t} < t + \epsilon\) for all \(\epsilon > 0\).

(i): Obviously \(t \mapsto C_t\) is increasing. It remains to show that
\[
\lim_{u \searrow t} C_u \leq C_t.
\]
("\(\geq\)" is clear since \(C_t\) is increasing.)

Let \(\epsilon > 0\). Then \(\langle Y \rangle_{C_t + \epsilon} > t\). Hence, there exists a \(\delta > 0\) such that
\[
\langle Y \rangle_{C_t + \epsilon} > u \quad \forall u \in [t, t + \delta].
\]
Thus,
\[
C_u \leq C_t + \epsilon \quad \forall u \in [t, t + \delta].
\]

(ii): We have
\[
\{ C_t < u \} = \{ \langle Y \rangle_u > t \} \quad \forall u, t, \quad (A.0.1)
\]
because "\(\subset\)" is clear and, if \(\langle Y \rangle_u > t\), then there exists an \(\epsilon > 0\) such that \(\langle Y \rangle_u - \epsilon > t\) and, therefore, \(C_t \leq u - \epsilon < u\). But
\[
\{ \langle Y \rangle_u > t \} \in \mathcal{F}_u.
\]
Hence, by \((A.0.1)\)
\[
\{ C_t \leq u \} \in \mathcal{F}_{u+} = \mathcal{F}_u.
\]

(iv): If \(C_{\langle Y \rangle_t} < t\), then by \((A.0.1)\) we would get \(\langle Y \rangle_t > \langle Y \rangle_t\). Therefore, \(C_{\langle Y \rangle_t} \geq t\). \(\square\)
A. Time Transformation

Note that $\mathcal{G}_t := \mathcal{F}_{C_t}$, $t \geq 0$ (filtration, since $C_t$ is increasing!).

Lemma A.0.3.  

i. $(\mathcal{G}_t)_{t \geq 0}$ is right-continuous and complete.

ii. $(Y)_t$ is a stopping time with respect to $(\mathcal{G}_s)_{s \geq 0}$ for all $t$.

Proof. (ii) is clear by (A.0.1).

(i): Right-continuity: Let $A \in \bigcap_{s>0} \mathcal{F}_{C_{t+s}}$ (in particular, $A \in \mathcal{F}_{C_{t+\frac{1}{n}}}$). Hence, for all $s$

\[
A \cap \{C_{t+\frac{1}{n}} < s\} \in \mathcal{F}_s \quad \forall n, s,
\]

therefore,

\[
A \cap \bigcup_{n=1}^\infty \{C_{t+\frac{1}{n}} < s\} \in \mathcal{F}_s \quad \forall s,
\]

thus $A \cap \{C_t \leq s\} \in \mathcal{F}_{s^+} = \mathcal{F}_s \forall s$, and $A \in \mathcal{F}_{C_t}$.

Completeness: Let $A_0 \in \mathcal{F}_{C_t}$ with $P[A_0] = 0$ and $A \subset A_0$. Then for all $s$

\[
A \cap \{C_t \leq s\} \in \mathcal{F}_s
\]
as a subset of an $\mathcal{F}_s$-measurable $P$-zero set $A_0 \cap \{C_t \leq s\}$, since $\mathcal{F}_s$ is complete. Hence, $A \in \mathcal{F}_{C_t}$.

Proof of A.0.1. Since $Y$ is $P$-a.s. continuous and $(C_t)$ is right-continuous, $(W_t)$ is $P$-a.s. right-continuous.

**Step 1.** $t \mapsto W_t = Y_{C_t}$ is $P$-a.s. continuous:

It suffice to show that (cf. Exercises)

\[
P[Y_u = Y_t \quad \text{for } t \leq u \leq \sigma_t \quad \forall t \geq 0] = 1, \tag{A.0.2}
\]

where

\[
\sigma_t := \inf\{s > t|\langle Y\rangle_s > \langle Y\rangle_t\},
\]

hence,

\[
\langle Y \rangle_u = \langle Y \rangle_t \quad \text{for } u \in [t, \sigma_t].
\]

(A.0.2) is sufficient, since then $Y$ is constant on the interval, where $\langle Y \rangle$ is constant and by definition this is the case on $[C_{t-}, C_t]$. Therefore, $Y_{C_t} = Y_{C_{t-}}$. For (A.0.2) it remains to show that for all $r \in \mathbb{Q}^+$

\[
P[Y_u = Y_r \quad \text{for } r \leq u \leq \sigma_r \quad \forall r \in \mathbb{Q}^+] = 1, \tag{A.0.3}
\]

because, if $t \geq 0$ with $t < \sigma_t$, then for all $r \in [t, \sigma_t) \cap \mathbb{Q}^+ \sigma_r = \sigma_t$ and (A.0.3) implies (A.0.2), since $Y$ is $P$-a.s. (right-)continuous.

Let $(T'_n)_{n \in \mathbb{N}}$ be a localizing sequence for $Y$. Then

\[
T_n := \inf\{t > 0|Y_t > n\} \land T'_n, \quad n \in \mathbb{N},
\]
is again a localizing sequence. Fix $r \in \mathbb{Q}^+$. For $n \in \mathbb{N}$ set

\[
N^{(n)}_t := Y_{(r+t)\land \sigma_r \land T_n} - Y_{r \land T_n}, \quad t \geq 0,
\]

\[
\bar{F}_t := \mathcal{F}_{t+r}, \quad t \geq 0.
\]

Then $N^{(n)}$ is a continuous bounded martingale with respect to $(\bar{F}_t)$, since by the stopping theorem $\forall s \leq t$

\[
E[N^{(n)}_t | \bar{F}_s] = E[Y_{(r+t)\land \sigma_r \land T_n} - Y_{r \land T_n} | \bar{F}_{r+s}] = Y_{(r+s)\land \sigma_r \land T_n} - Y_{r \land T_n} = N^{(n)}_s,
\]

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since $\sigma_r$ is a stopping time with respect to $(\mathcal{F}_t)$. Additionally,
\[
(N^{(n)})_t = \langle Y \rangle_{(r+t)\wedge \sigma_r \wedge T_n} - \langle Y \rangle_{r\wedge T_n} = 0 \quad \forall t \geq 0.
\]
Hence,
\[
E[(N^{(n)}_t)^2] = E[(N^{(n)}_t)] = 0 \quad \forall t \geq 0,
\]
thus
\[
N^{(n)}_t = 0 \quad \text{P-a.s.}
\]
Letting $n \to \infty$ implies
\[
Y_{(r+t)\wedge \sigma_r} - Y_r = 0 \quad \text{P-a.s.} \quad \forall t \geq 0,
\]
which yields (A.0.3).

**Step 2.** $W$ is a local martingale (up to $\infty$) with respect to $(\mathcal{G}_t)$:

By A.0.2(ii) $C_t$ is an $(\mathcal{F}_u)$-stopping time. For $t \geq 0$
\[
E[(Y^{C_t})_u] = E[(Y^{C_t})] \leq E[(Y)_C] = t.
\]
Therefore, by Corollary 1.4.7 $Y^{C_t}$ is a martingale and
\[
E[(Y^{C_t}_s)^2] \leq \liminf_n E[(Y^{C_t}_s)^2] = \liminf_n E[(Y)_s | C_t \wedge T_n] \leq t.
\]
Hence, $(Y^{C_t}_s)_{s \geq 0}$ and $(Y^{C_t}_s)^2)_{s \geq 0}$ are uniformly integrable.

Moreover, $C_{(Y)_T_n}$ is a $(\mathcal{F}_t)$-stopping time, because:
\[
\{C_{(Y)_T_n} < u\} \overset{(A.0.1)}{=} \{\langle Y \rangle_u > \langle Y \rangle_{T_n}\}
\]
\[= \bigcup_{r \in \mathbb{Q}} \{\langle Y \rangle_u > r\} \cap \{r > \langle Y \rangle_{T_n}\}\]
\[= \bigcup_{r \in \mathbb{Q}} \{C_r < u\} \cap \{C_r > T_n\}\]
\[= \bigcup_{r \in \mathbb{Q}} \{C_r < u\} \cap \{C_r \wedge u > T_n\}\]

Thus,
\[
\{C_{(Y)_T_n} \leq u\} \in \mathcal{F}_{u+} = \mathcal{F}_u.
\]
Furthermore, $(Y)_T_n$ is a stopping time with respect to $(\mathcal{G}_t)$, since
\[
\{\langle Y \rangle_{T_n} \leq u\} \overset{(A.0.1)}{=} \{C_u \geq T_n\} \in \mathcal{F}_{T_n \wedge Cu} \subset \mathcal{F}_{Cu} = \mathcal{G}_k.
\]
Hence, for all $t > s$ (since $C_{t \wedge (Y)_T_n} = C_t \wedge C_{(Y)_T_n}$, because $(C_t)$ is increasing)
\[
E[W_{t \wedge (Y)_T_n} | G_s] = E[W_{t \wedge (Y)_T_n} | G_s] = E[Y_{C_{t \wedge (Y)_T_n}} | G_s]
\]
\[= E[Y_{C_{t \wedge C_{(Y)_T_n}} \wedge T_n} | \mathcal{F}_u] = E[Y^{C_{t \wedge C_{(Y)_T_n}}} | \mathcal{F}_u]
\]

Since $(Y^{C_t})_{s \geq 0} \wedge C_{(Y)_T_n}$ is an uniformly integrable martingale (note that $C_s$ is not necessarily bounded), it follows that
\[
= Y^{C_t}_{s \geq t} \wedge C_{(Y)_T_n} = W_{s \wedge (Y)_T_n}.
\]
A. Time Transformation

Hence, $W$ is a local martingale with localising sequence $(Y_T)_n \in \mathbb{N}$ (up to $\infty$).

**Step 3.** $(W)_t = t \, \forall t$. In particular, $W$ is a Brownian motion:

Since by $L^2$-martingale convergence theorem $(Y^C_s)_{s \geq 0}$ is uniformly integrable and $(Y^C_s)_{s \geq 0} \leq (Y^C_s)_{s \geq 0}$, it follows that $(W^C_s)^2 - (Y^C_s)$ is an uniformly integrable martingale. Hence, (though $C_t, s$ are not bounded) by the stopping theorem we get that for all $t > s$

$$E[W_t^2 - t|G_s] = E[Y^C_t - (Y^C_s)_t|G_s]$$

$$= E[(Y_t^C)^2 - (Y^C_s)_t|G_s] = (Y_t^C)^2 - (Y_t^C)_s$$

$$= Y_s^2 - (Y^C_s)_s = W_s^2 - s.$$ Therefore, $(W^2_t - t)_{t \geq 0}$ is a continuous martingale with respect to $(G_t)_{t \geq 0}$. By the Doob-Meyer decomposition it follows that $(W)_t = t$ for all $t$ and by 1.4.7 that $(W_t)_{t \geq 0}$ is a martingale. By Lévy's characterization theorem 1.5.1 $W_t$ is a Brownian motion.

**Step 4.** $Y_t = W(Y)_t$ for all $t \geq 0$:

We have

$$W(Y)_t = Y C(Y)_t$$

and since $s \mapsto \langle Y \rangle_s$ is increasing and continuous

$$C(Y)_t = \inf \{ s > 0 | \langle Y \rangle_s > \langle Y \rangle_t \} = \inf \{ s > t | \langle Y \rangle_s > \langle Y \rangle_t \} = \sigma_t.$$ Therefore, $(Y)$ is constant on $[t, C(Y)_t]$. By (A.0.2) we get that $Y$ is constant on $[t, C(Y)_t]$, thus, $W(Y)_t = Y C(Y)_t = Y_t$.

**Remark A.0.4.** We have supposed that $P[\langle Y \rangle_\infty = \infty] = 1$. (This has been necessary as the counter example

$$\Omega := \{ \omega \}, \quad Y \equiv 0$$

shows.) Basically, the theorem also holds for the case, where $P[\langle Y \rangle_\infty < \infty] > 0$. But, one possibly has to enlarge $\Omega$.

**Construction of the enlarged Wiener space:**

Let $(W_t)_{t \geq 0}$ be a Wiener process on $(\Omega, F, P)$ with respect to $(F_t)$. Set

$$\hat{\Omega} := \Omega \times \Omega', \quad \hat{F} := F \otimes F', \quad \hat{P} := P \times P',$$

$$C_t := \left\{ \begin{array}{ll} \inf \{ s | \langle Y \rangle_s > t \} & \text{on } \{ \langle Y \rangle_\infty > t \}, \\ \infty & \text{on } \{ \langle Y \rangle_\infty \leq t \}. \end{array} \right.$$ Let

$$\hat{F}_t := \sigma(F_{C_t \wedge s} | s \geq 0), \quad \hat{G}_t := \hat{F}_t \times F'_t,$$

$$W_t := \left\{ \begin{array}{ll} Y_{C_t} & \text{on } \{ \langle Y \rangle_\infty > t \}, \\ W'_t - W(Y)_\infty + \langle Y \rangle_\infty & \text{on } \{ \langle Y \rangle_\infty \leq t \}. \end{array} \right.$$ Then $W$ is a Wiener process on $(\hat{\Omega}, \hat{F}, \hat{P})$ with respect to $(\hat{G}_t)_{t \geq 0}$, $Y_t$ is a $(\hat{G}_t)$-stopping time and we have

$$Y_t = W(Y)_t.$$ For the proof cf. [IW89, Chapter II, Theorem 7.21].
Literaturverzeichnis


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