Noname manuscript No.
(will be inserted by the editor)

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Modules over Motivic Cohomology

Oblatum date & date

Abstract For fields of characteristic zero, we show the homotopy category of modules over the motivic ring spectrum representing motivic cohomology is equivalent to Voevodsky's big category of motives. The proof makes use of some highly structured models for motivic stable homotopy theory, motivic Spanier-Whitehead duality, the homotopy theories of motivic functors and of motivic spaces with transfers as introduced from ground up in this paper. Combining the above with Morita theory identifies the category of mixed Tate motives over any noetherian and separated schemes of finite Krull dimension with the homotopy category of modules over a symmetric endomorphism ring spectrum. Working with rational coefficients, we extend the equivalence for fields of characteristic zero to all perfect fields by employing the techniques of alterations and homotopy purity in motivic homotopy theory.

Mathematics Subject Classification (2000) 14F42, 55U35

1 Introduction

In homotopy theory it is convenient not to be tied down to any particular choice of a model for a homotopy category. For example, simplicial sets and topological spaces have equivalent homotopy categories. One of the lessons of homotopical algebra is that model structures contain more information than homotopy categories. The homotopical approach naturally leads to homotopy theories for algebras and modules. An important precursor to this paper is that modules over the integral Eilenberg-MacLane ring spectrum provide a homotopy theoretic model for the derived category of abelian groups turning homological algebra into a matter of stable homotopy theory [36].

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The motivic Eilenberg-MacLane spectrum $\mathsf{M}\mathbb{Z}$ is of fundamental interest in motivic homotopy theory [32]. As a cohomology theory, $\mathsf{M}\mathbb{Z}$ represents motivic cohomology groups. The slice filtration gives ubiquitous examples of motivic spectra equipped with $\mathsf{M}\mathbb{Z}$-module structures [21], [35]. In this paper we study $\mathsf{M}\mathbb{Z}$ as a ring in the highly structured models for motivic stable homotopy theory [7], [18]. To this end we employ the model structure on its module category $\mathsf{M}\mathbb{Z} - \mathbf{mod}$. This allows us to make precise the idea that modules over motivic cohomology provide a non-linear setting for motives:

Theorem 1.1 *If k is a field of characteristic zero, the homotopy category of* $\mathsf{M}\mathbb{Z} - \mathbf{mod}$ *is equivalent to Voevodsky's big category of motives* $\mathbf{DM}_k$ *of k. The equivalence preserves the monoidal and triangulated structures.*

The equivalence follows from a zig-zag of Quillen equivalences of models. Using models allow us to control all higher order structure such as mapping spaces, suspension and loop functors, cofiber and fiber sequences and also the monoidal smash products in both categories. Put bluntly, any homotopy theoretic result in one model translates immediately into a similar result in the other model. For example, the blow-up distinguished triangle for motives of not necessarily smooth schemes from [33] follows essentially since any free $\mathsf{M}\mathbb{Z}$-module maps to its corresponding motive under the Quillen equivalence. An outline of the proof of this equivalence has been published in [26], and applications have been considered in [10] and [21].

In fact Theorem 1.1 holds for any commutative unital coefficient ring and noetherian and separated base schemes of finite Krull dimension for which the motivic stable homotopy category is generated by dualizable objects. Using de Jong's work on alterations [19] we extend the equivalence between modules over $\mathsf{M}\mathbb{Z}$ and motives to all perfect fields and coefficients in algebras over the rational numbers. Voevodsky discusses several facets of the case of rational coefficients in the concluding remarks of his ICM address [32].

For general base schemes as above we show the category of mixed Tate motives is equivalent to the homotopy category of spherical modules over $\mathsf{M}\mathbb{Z}$. Moreover, we employ spherical modules over motivic cohomology, Morita theory in the context of stable model structures, and motivic spaces with transfers to identify mixed Tate motives with the modules over a symmetric endomorphism ring spectrum. We refer to Section 6 for precise statements.

Theorem 1.2 *The category of mixed Tate motives over any noetherian and separated scheme of finite Krull dimension is equivalent to the associated homotopy category of the module category of a symmetric endomorphism ring spectrum with many objects.*

Another application emerging from Theorem 1.1 is that the homotopy category of $\mathsf{M}\mathbb{Z} - \mathbf{mod}$ satisfies the conditions in Voevodsky's cross functor theorem [3], and thus it should open for a wholesale use of Grothendieck's six functor formalism for motives. This approach toward functoriality for motives is related to Voevodsky's conjecture stating that a base scheme map f of finite type induces a natural isomorphism $f^*\mathsf{M}\mathbb{Z} \longrightarrow \mathsf{M}\mathbb{Z}$ [35, Conjecture 17]. In loc.cit. it is noted that the conjecture holds for any morphism of regular base schemes over a field.

2 Homotopy theory of presheaves with transfers

Let S be a noetherian and separated scheme of finite Krull dimension. Denote by Sm_S the site of smooth S-schemes of finite type in the Nisnevich topology. For legibility, we shall suppress S in the notation. The category $\mathbf{M}$ of motivic spaces consists of contravariant functors from Sm to pointed simplicial sets. A scheme U in Sm defines a representable motivic space U_+. We consider $\mathbf{M}$ in its motivic model structure [7, Theorem 2.12].

Let Cor denote the Suslin-Voevodsky category of finite correspondences in Sm [30], [31]. The category $\mathbf{M}^{\mathrm{tr}}$ of motivic spaces with transfers consists of all contravariant $\mathbb{Z}$-linear functors from Cor to simplicial abelian groups, i.e. simplicial objects in the category $\mathbf{Pre}^{\mathrm{tr}}$ of presheaves with transfers of S. A scheme U in Sm defines a representable motivic space with transfers U^{tr}.

Let $\mathscr{U}: \mathbf{M}^{\mathrm{tr}} \longrightarrow \mathbf{M}$ denote the evident forgetful functor induced by the graph $\mathrm{Sm} \longrightarrow \mathrm{Cor}$. Its left adjoint transfer functor $\mathbb{Z}^{\mathrm{tr}}: \mathbf{M} \longrightarrow \mathbf{M}^{\mathrm{tr}}$ is the left Kan extension determined by

$$\mathbb{Z}^{\mathrm{tr}}(U \times \Delta^n)_+ := U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\Delta^n].$$

This definition makes implicit use of the tensor product $\otimes^{\mathrm{tr}}$ for motivic spaces with transfers, which exists by general results in [2]. If A is a motivic space, let A^{tr} be short for $\mathbb{Z}^{\mathrm{tr}}(A)$. Let $\mathbf{M}(A,B)$ denote the usual $\mathbf{M}$-object of maps, or internal hom, and $\mathbf{sSet}_{\mathbf{M}}(A,B)$ the function complex of simplicial sets between motivic spaces A and B. We employ similar notations for the exact same type of constructs in $\mathbf{M}^{\mathrm{tr}}$ which is enriched in simplicial abelian groups, and hence in simplicial sets via the forgetful functor [2].

The proof of the next lemma relating $\mathscr{U}$ and $\mathbb{Z}^{\mathrm{tr}}$ to the closed symmetric monoidal structures in $\mathbf{M}$ and $\mathbf{M}^{\mathrm{tr}}$ is a routine check.

Lemma 2.1 *The transfer functor $\mathbb{Z}^{\mathrm{tr}}: \mathbf{M} \longrightarrow \mathbf{M}^{\mathrm{tr}}$ is strict symmetric monoidal and its right adjoint $\mathscr{U}$ is lax symmetric monoidal. In particular, there are natural isomorphisms*

$$\mathbf{M}(A, \mathscr{U}(B)) \xrightarrow{\cong} \mathscr{U}\mathbf{M}^{\mathrm{tr}}(A^{\mathrm{tr}}, B).$$

The above extends verbatim to coefficients in any commutative ring with unit. In Section 7, we consider algebras over the rational numbers.

2.1 Unstable theory

In this section we introduce in broad strokes an unstable homotopy theory of motivic spaces with transfers akin to the work of Morel-Voevodsky [23] on motivic unstable homotopy theory, see also [7, Section 2.1].

Definition 2.2 A map $A \longrightarrow B$ in $\mathbf{M}^{\mathrm{tr}}$ is a schemewise weak equivalence if for all $U \in \mathrm{Sm}$, $A(U) \longrightarrow B(U)$ is a weak equivalence of underlying simplicial sets. The schemewise fibrations and schemewise cofibrations are defined similarly. A projective cofibration is a map having the left lifting property with respect to all schemewise acyclic fibrations. Note that every motivic space with transfers is schemewise fibrant since simplicial abelian groups are fibrant simplicial sets.

It is straightforward to prove the following result, cf. [6, Theorem 4.4].

Theorem 2.3 *The schemewise weak equivalences, schemewise fibrations and projective cofibrations equip* $\mathbf{M}^{\mathrm{tr}}$ *with the structure of a proper, monoidal, simplicial and combinatorial model category. The sets*

$$\{U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n]\}_{U\in \mathrm{Sm}}^{n\geq 0}$$

and

$$\{U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\Lambda^n_i \hookrightarrow \Delta^n]\}_{U\in \mathrm{Sm}}^{0<i\leq n}$$

generate the projective cofibrations and acyclic projective cofibrations.

The schemewise model is a step toward constructing the motivic model which reflects properties of the Nisnevich topology. We wish to emphasize the importance of working with a sufficiently fine topology; for example, what follows does not hold in the Zariski topology. Recall the Nisnevich topology is generated by elementary distinguished squares, i.e. pullback squares

$$Q \quad = \quad \begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow{\psi} & X \end{array}$$

where ϕ is étale, ψ is an open embedding and $\phi^{-1}(X \smallsetminus U) \longrightarrow (X \smallsetminus U)$ is an isomorphism of schemes (with the reduced structure) [23, Definition 3.1.3]. Let $\mathcal{Q}$ denote the set of elementary distinguished squares in Sm.

Definition 2.4 A motivic space with transfers Z is motivically fibrant if

– $\mathscr{U} Z(Q)$ is a homotopy pullback square of simplicial sets for all $Q \in \mathcal{Q}$.
– $\mathscr{U} Z(U) \longrightarrow \mathscr{U} Z(U \times_S \mathbb{A}^1)$ is a weak equivalence for all $U \in \mathrm{Sm}$.

A motivic space with transfers is flasque if it satisfies the first condition in Definition 2.4. The Nisnevich local model arise by localizing the schemewise model with respect to the flasque motivic spaces with transfers. The map in the second condition is induced by the canonical projection $U \times_S \mathbb{A}^1 \longrightarrow U$. Let $(-)^c \longrightarrow \mathrm{id}_{\mathbf{M}^{\mathrm{tr}}}$ denote a schemewise cofibrant replacement functor.

Definition 2.5 A map $f : A \longrightarrow B$ of motivic spaces with transfers is a motivic weak equivalence if for every motivically fibrant Z, the map

$$\mathbf{sSet}_{\mathbf{M}^{\mathrm{tr}}}(f^c, Z) : \mathbf{sSet}_{\mathbf{M}^{\mathrm{tr}}}(B^c, Z) \longrightarrow \mathbf{sSet}_{\mathbf{M}^{\mathrm{tr}}}(A^c, Z)$$

is a weak equivalence of pointed simplicial sets.

By using Theorem 2.3 and the localization theory for combinatorial model categories, we obtain the motivic model for motivic spaces with transfers; it is the starting point for the stable theory introduced in the next section.

Theorem 2.6 *The motivic weak equivalences and the projective cofibrations form a left proper, simplicial, and combinatorial model structure on* $\mathbf{M}^{\mathrm{tr}}$.

In the following we show that there exists a set J' of acyclic cofibrations with finitely presentable domains and codomains such that any map $f\colon A \longrightarrow B$ is a motivic fibration with motivically fibrant codomain if and only if it has the right lifting property with respect to J'. Note that if B is motivically fibrant, then f is a motivic fibration if and only if it is a schemewise fibration and A is motivically fibrant [12, Proposition 3.3.16].

Using the simplicial mapping cylinder we factorize the map $(\mathbb{A}^1 \longrightarrow S)^{\mathrm{tr}}$ via a projective cofibration $c^{\mathrm{tr}}_{\mathbb{A}^1}\colon (\mathbb{A}^1 \rightarrowtail C_{\mathbb{A}^1})^{\mathrm{tr}}$ and a simplicial homotopy equivalence. Then $Z(\mathbb{A}^1 \times -)$ and Z are schemewise weakly equivalent if and only if there is a schemewise weak equivalence between internal hom objects

$$\mathbf{M}^{\mathrm{tr}}(c^{\mathrm{tr}}_{\mathbb{A}^1}, Z)\colon \mathbf{M}^{\mathrm{tr}}(C^{\mathrm{tr}}_{\mathbb{A}^1}, Z) \longrightarrow \mathbf{M}^{\mathrm{tr}}\big((\mathbb{A}^1)^{\mathrm{tr}}, Z\big).$$

Now the schemewise projective model is monoidal, which implies $\mathbf{M}^{\mathrm{tr}}(c^{\mathrm{tr}}_{\mathbb{A}^1}, Z)$ is a schemewise fibration. Hence Z is motivically fibrant if and only if Z is flasque and the map $Z \twoheadrightarrow *$ has the right lifting property with respect to the set of pushout product maps

$$\{(\mathbb{A}^1_U)^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\Delta^n] \cup_{(\mathbb{A}^1_U)^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n]} C^{\mathrm{tr}}_{\mathbb{A}^1_U} \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n]\}^{n\geq 0}_{U\in \mathrm{Sm}}. \quad (1)$$

For an elementary distinguished square Q the simplicial mapping cylinder yields a projective cofibration $P^{\mathrm{tr}} \rightarrowtail Cyl$ where Cyl is simplicial homotopy equivalent to Y^{tr}. Similarly, the map from $A^{\mathrm{tr}}_Q := U^{\mathrm{tr}} \cup_{P^{\mathrm{tr}}} Cyl$ to X^{tr} factors through a projective cofibration $A^{\mathrm{tr}}_Q \rightarrowtail B^{\mathrm{tr}}_Q$ of finitely presentable motivic spaces with transfers, where B^{tr}_Q is simplicial homotopy equivalent to X^{tr}. It follows that $Z(Q)$ is a homotopy pullback square if and only if the fibration

$$\mathbf{sSet}_{\mathbf{M}^{\mathrm{tr}}}(B^{\mathrm{tr}}_Q, Z) \twoheadrightarrow \mathbf{sSet}_{\mathbf{M}^{\mathrm{tr}}}(A^{\mathrm{tr}}_Q, Z)$$

is a weak equivalence. Let J' denote the union of the generating schemewise acyclic cofibrations in Theorem 2.3, the maps displayed in (1) and

$$\{A^{\mathrm{tr}}_Q \otimes^{\mathrm{tr}} \mathbb{Z}[\Delta^n] \cup_{A^{\mathrm{tr}}_Q \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n]} B^{\mathrm{tr}}_Q \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n]\}^{n\geq 0}_{Q\in\mathcal{Q}}. \quad (2)$$

Then J' consists of acyclic cofibrations and satisfies the desired properties. In what follows we use J' to show the motivic model arise from Kan's recognition principle for the pair $(\mathbb{Z}^{\mathrm{tr}}, u)$ [13, Theorem 2.1.19], cf. [12, Theorem 11.3.2].

The transfer functor $\mathbb{Z}^{\mathrm{tr}}$ furnishes a bijection between the respective sets of generating (schemewise acyclic) projective cofibrations. It also maps the set detecting motivic fibrations of motivic spaces with motivically fibrant codomains in [7, Theorem 2.7, Definition 2.14] bijectively to the set J'. Thus a sufficient condition to apply Kan's recognition principle is that the forgetful functor $\mathscr{U}$ maps acyclic cofibrations to weak equivalences of motivic spaces. In fact, since $\mathscr{U}$ preserves schemewise equivalences by definition, it suffices to check that $\mathscr{U}$ maps every map in J'-cell to a motivic weak equivalence. Here J'-cell denotes the class of maps of sequential compositions of cobase changes of coproducts of maps in J'. Since $\mathscr{U}$ preserves filtered colimits and weak equivalences of motivic spaces are closed under filtered colimits, it suffices to prove that $\mathscr{U}$ sends the cobase change of a map in J' to a weak equivalence.

First we consider maps in the set (1). As in classical algebraic topology, an inclusion $f\colon A \hookrightarrow B$ of motivic spaces is an $\mathbb{A}^1$-deformation retract if there exist a map $r\colon B \longrightarrow A$ such that $r \circ f = \mathrm{id}_A$ and an $\mathbb{A}^1$-homotopy $H\colon B \times \mathbb{A}^1 \longrightarrow B$ between $f \circ r$ and id_B which is constant on A. To define the same notion for motivic spaces with transfers one adds transfers to the affine line. Then $\mathbb{A}^1$-deformation retracts are motivic weak equivalences, and closed under cobase change, and smash or tensor products. Note that $\mathbb{Z}^{\mathrm{tr}}$ and $\mathscr{U}$ preserves $\mathbb{A}^1$-deformation retracts, since both functors are lax monoidal. The zero section of the affine line induces for all U in Sm an $\mathbb{A}^1$-deformation retract $(U \longrightarrow \mathbb{A}^1_U)^{\mathrm{tr}}$, so it remains to consider maps in (2) because the case of generating schemewise acyclic cofibrations is trivial.

Recall that an elementary distinguished square Q furnishes a projective cofibration $A^{\mathrm{tr}}_Q \rightarrowtail B^{\mathrm{tr}}_Q$. To show $\mathscr{U}(A^{\mathrm{tr}}_Q \longrightarrow B^{\mathrm{tr}}_Q)$ is a weak equivalence it suffices to check that it induces isomorphisms on all the Nisnevich sheaves of homotopy groups. By construction, the latter coincides with the map of homology sheaves induced by the map

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & P^{\mathrm{tr}} \arrow[r] \arrow[d] & U^{\mathrm{tr}} \oplus Y^{\mathrm{tr}} \arrow[r] \arrow[d, "\phi^{\mathrm{tr}} - \psi^{\mathrm{tr}}"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X^{\mathrm{tr}} \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

of chain complexes of Nisnevich sheaves. This lets us conclude since

$$0 \longrightarrow P^{\mathrm{tr}} \longrightarrow U^{\mathrm{tr}} \oplus Y^{\mathrm{tr}} \xrightarrow{\phi^{\mathrm{tr}} - \psi^{\mathrm{tr}}} X^{\mathrm{tr}} \longrightarrow 0 \tag{3}$$

is an exact sequence of Nisnevich sheaves by [30, Proposition 4.3.9]. We note that (3) is not exact as a sequence of Zariski sheaves. It is straightforward to extend this to cobase changes of maps of the form in (2).

The above lets us conclude there is a model structure on $\mathbf{M}^{\mathrm{tr}}$ such that $\mathscr{U}$ detects weak equivalences and fibrations. It has the same cofibrations and fibrant objects as the motivic model in Theorem 2.6. Using a straightforward argument involving framings – see [13, Chapter 5] – this implies that the models coincide. In particular, the motivic model is right proper and in fact we have proved the following lemma.

Lemma 2.7 *A map between motivic spaces with transfers is a motivic weak equivalence or a motivic fibration if and only if it is so when considered as a map between ordinary motivic spaces.*

The next lemma is included here mainly for reference to Hovey's work on symmetric spectra in [15].

Lemma 2.8 *The motivic model is a symmetric monoidal model structure.*

Proof The proof consists of first showing that tensoring with any cofibrant object preserves motivic weak equivalences and second combine this fact with left properness of the motivic model, cf. the proof of [7, Corollary 2.19]. □

Consider now the full embedding $\mathbf{M}^{\mathrm{tr}\sim} \hookrightarrow \mathbf{M}^{\mathrm{tr}}$ for motivic spaces with transfers whose underlying motivic spaces are simplicial Nisnevich sheaves. There is a proper, simplicial and monoidal model structure on $\mathbf{M}^{\mathrm{tr}\sim}$ and the inclusion is a right Quillen equivalence which detects and preserves both weak equivalences and fibrations. The left adjoint Nisnevich sheafification functor is strict symmetric monoidal, which implies the homotopy categories are equivalent as closed symmetric monoidal categories [13, Theorem 4.3.3].

2.2 Stable theory

Let $S \hookrightarrow \mathbb{A}^1 \smallsetminus \{0\}$ be the closed embedding obtained from the unit section. The simplicial mapping cylinder yields a factorization of the induced map $S_+ \longrightarrow (\mathbb{A}^1 \smallsetminus \{0\})_+$ into a cofibration and a simplicial homotopy equivalence

$$S_+ \rightarrowtail Cyl \xrightarrow{\simeq} (\mathbb{A}^1 \smallsetminus \{0\})_+.$$

Let $\mathbb{G}$ denote the cofibrant and finitely presentable motivic space Cyl/S_+. It is weakly equivalent to the Tate circle $S^1_t := (\mathbb{A}^1 \smallsetminus \{0\}, 1)$. Set $T := S^1_s \wedge \mathbb{G}$. Then T is weakly equivalent to the pointed projective line $(\mathbb{P}^1, 1)$.

For a general discussion of spectra and symmetric spectra, we refer to [15]. A motivic spectrum with transfers E consists of motivic spaces with transfers $(E_0, E_1, \ldots)$ and structure maps $T^{\mathrm{tr}} \otimes^{\mathrm{tr}} E_m \longrightarrow E_{m+1}$ for $m \geq 0$. Let $\mathbf{MS}^{\mathrm{tr}}$ denote the category of motivic spectra with transfers. The unit $T \longrightarrow \mathscr{U}(T^{\mathrm{tr}})$ induces an evident forgetful functor $\mathscr{U}_{\mathbb{N}} \colon \mathbf{MS}^{\mathrm{tr}} \longrightarrow \mathbf{MS}$ to the corresponding category of motivic spectra. Its left adjoint $\mathbb{Z}^{\mathrm{tr}}_{\mathbb{N}}$ is obtained by applying the strict symmetric monoidal transfer functor levelwise.

A motivic symmetric spectrum with transfers is a motivic spectrum with transfers E, together with an action of the symmetric group Σ_m on E_m for all integers $m \geq 0$. The group actions furnish motivic symmetric spectra with transfers $\mathbf{MSS}^{\mathrm{tr}}$ with the structure of a closed symmetric monoidal category. Let $\mathbf{MSS}$ denote Jardine's category of motivic symmetric spectra [18]. Since $\mathscr{U}_{\mathbb{N}}$ is lax symmetric monoidal, there is an induced lax symmetric monoidal functor $\mathscr{U}_{\Sigma} \colon \mathbf{MSS}^{\mathrm{tr}} \longrightarrow \mathbf{MSS}$. It acquires a strict symmetric monoidal left adjoint functor $\mathbb{Z}^{\mathrm{tr}}_{\Sigma}$ obtained by adding transfers levelwise to motivic spaces.

Theorem 2.9 *The main results in this section are as follows.*

1. *Motivic spectra with transfers acquires a stable model structure such that a map is a weak equivalence or fibration if and only if its underlying map of motivic spectra is so.*
2. *Motivic symmetric spectra with transfers acquires a monoidal stable model structure such that the right Quillen equivalence $\mathscr{U}_{\Sigma}$ detects and preserves weak equivalences and fibrations between stably fibrant objects.*
3. *There is a zig-zag of monoidal Quillen equivalences between $\mathbf{MSS}^{\mathrm{tr}}$ and symmetric $\mathbb{G}^{\mathrm{tr}}_m[1]$-spectra of non-connective (unbounded) chain complexes of presheaves with transfers.*
4. *If S is the Zariski spectrum of a perfect field k, the homotopy category of $\mathbf{MSS}^{\mathrm{tr}}$ is equivalent to Voevodsky's big category of motives $\mathbf{DM}_k$ over k. The equivalence respects the monoidal and triangulated structures.*

Let $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ denote symmetric spectra of connective (positively graded) chain complexes of presheaves with transfers with respect to the suspension coordinate $\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1,1)$. It turns out the Dold-Kan equivalence between $\mathbf{M}^{\mathrm{tr}}$ and connective chain complexes of presheaves with transfers $\mathbf{Ch}^{\mathrm{tr}}_+ := \mathbf{Ch}_+(\mathbf{Pre}^{\mathrm{tr}})$ extends to an equivalence between symmetric spectra:

$$\mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1} \rightleftarrows \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$$

Here $\mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1}$ denotes the category of symmetric $\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1,1)$-spectra of motivic spaces with transfers. Let $\mathbb{G}^{\mathrm{tr}}_m[1] := \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1 \smallsetminus \{0\},1) \longrightarrow 0$ be the shift of the chain complex $\mathbb{G}^{\mathrm{tr}}_m$ consisting of the motivic space with transfers $\mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1 \smallsetminus \{0\},1)$ concentrated in degree zero. Part (3) of Theorem 2.9 concerns the monoidal Quillen equivalences in the diagram:

$$\mathbf{MSS}^{\mathrm{tr}} \rightleftarrows \mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1} \leftrightarrows \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1} \rightleftarrows \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]} \rightleftarrows \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}^{\mathrm{tr}}_m[1]}$$

Remark 2.10 The proof of Theorem 2.9 occupies the rest of this section. In [25] we vary the site and show a partial generalization to other Grothendieck topologies of arithmetic interest, e.g. for etale and h-topologies.

2.2.1 Motivic spectra with transfers

The evaluation functor Ev_m sends motivic symmetric spectra with transfers to their m-th terms. Its left adjoint $\mathrm{Fr}_m\colon \mathbf{M}^{\mathrm{tr}} \longrightarrow \mathbf{MSS}^{\mathrm{tr}}$ takes values in shifted motivic symmetric suspension spectra with transfers [15, Section 1]. Notationwise we shall identify objects B of $\mathbf{M}^{\mathrm{tr}}$ and their suspension spectra $\mathrm{Fr}_0 B$, and similarly for motivic spaces. The motivic sphere spectrum with transfers $\mathbb{I}^{\mathrm{tr}} := S^{\mathrm{tr}}_+$ is the unit for the monoidal smash product in $\mathbf{MSS}^{\mathrm{tr}}$.

Let $\mathscr{L}_T$ and $\mathscr{L}^{\mathrm{tr}}_T$ be the left adjoints of the respective forgetful functors between the (co)tensored and $\mathbf{M}^{\mathrm{tr}}$-enriched spectra categories in the diagram:

$$\begin{tikzcd} \mathbf{MS} \arrow[r, shift left, "\mathbb{Z}^{\mathrm{tr}}_{\mathbb{N}}"] \arrow[d, shift right, "\mathscr{L}_T"'] & \mathbf{MS}^{\mathrm{tr}} \arrow[l, shift left, "\mathscr{U}_{\mathbb{N}}"] \arrow[d, shift left, "\mathscr{L}^{\mathrm{tr}}_T"] \\ \mathbf{MSS} \arrow[r, shift left, "\mathbb{Z}^{\mathrm{tr}}_{\Sigma}"] \arrow[u, shift right, "\mathscr{U}_T"'] & \mathbf{MSS}^{\mathrm{tr}} \arrow[l, shift left, "\mathscr{U}_{\Sigma}"] \arrow[u, shift left, "\mathscr{U}^{\mathrm{tr}}_T"] \end{tikzcd} \tag{4}$$

The right adjoints on display in (4) commute, and therefore the left adjoints commute up to a unique natural isomorphism. We note the identity

$$\mathscr{U}^{\mathrm{tr}}_T \circ \mathbb{Z}^{\mathrm{tr}}_{\Sigma} = \mathbb{Z}^{\mathrm{tr}}_{\mathbb{N}} \circ \mathscr{U}_T.$$

Definition 2.11 Let $f\colon E \longrightarrow F$ be a map of motivic symmetric spectra. It is a levelwise weak equivalence (resp. levelwise fibration) if $\mathrm{Ev}_m(f)$ is a motivic equivalence (resp. motivic fibration) for all $m \geq 0$.

It is straightforward to check that these maps define a model structure on motivic symmetric spectra. Let $(-)^c$ denote a cofibrant replacement functor in the levelwise model and $\mathbf{sSet}_{\mathbf{MSS}}$ function complexes of motivic symmetric spectra given by $\mathbf{sSet}_{\mathbf{MSS}}(E,F)_n := \mathbf{MSS}(E \wedge \Delta^n_+, F)$ as in [18, Section 4].

Definition 2.12 A levelwise fibrant motivic symmetric spectrum G is stably fibrant if for all $n \geq 0$ the adjoint structure map $G_n \longrightarrow \mathbf{M}(T, G_{n+1})$ is a motivic equivalence. A map $f: E \longrightarrow F$ is a stable equivalence if

$$\mathbf{sSet_{MSS}}(f^c, G)\colon \mathbf{sSet_{MSS}}(F^c, G) \longrightarrow \mathbf{sSet_{MSS}}(E^c, G)$$

is a weak equivalence for every stably fibrant motivic symmetric spectrum G. It is a stable fibration if it has the right lifting property with respect to all maps which are simultaneously a cofibration and a stable equivalence.

The stable equivalences and stable fibrations in **MS** are defined similarly in terms of function complexes.

Theorem 2.13 *The classes of stable equivalences and stable fibrations form model structures on motivic symmetric spectra and motivic spectra such that $\mathscr{L}_T$ is a left Quillen equivalence and* **MSS** *is a symmetric monoidal model with the property that smashing with any cofibrant object preserves stable equivalences.*

Proof The existence of the model structures and the left Quillen equivalence follow from results in [7, Section 2.1] and [15, Sections 3, 8, 10]. We note the model structure is Quillen equivalent via the identity map to Jardine's model structure on **MSS** [18, Theorem 4.31] (see also [7, Section 3.7]). The last statement follows by combining [15] and [18, Proposition 4.41]. □

Let $\Theta\colon \mathbf{MS} \longrightarrow \mathbf{MS}$ be the functor which maps E with adjoint structure maps σ^E to the motivic spectrum whose n-th term $(\Theta E)_n$ is $\mathbf{M}(T, E_{n+1})$ and adjoint structure maps given by $\mathbf{M}(T, \sigma^E)$. There is a natural transformation $\mathrm{id}_{\mathbf{MS}} \longrightarrow \Theta$. Denote by $\Theta^\infty(E)$ the motivic spectrum whose n-th term is the colimit of the diagram

$$E_n \xrightarrow{\sigma_n^E} \mathbf{M}(T, E_{n+1}) \xrightarrow{\mathbf{M}(T,\sigma_{n+1}^E)} \mathbf{M}(T^{\wedge 2}, E_{n+2}) \xrightarrow{\mathbf{M}(T^{\wedge 2},\sigma_{n+2}^E)} \cdots .$$

Lemma 2.14 *A map f between levelwise fibrant motivic spectra is a stable equivalence if and only if $\Theta^\infty(f)$ is a levelwise weak equivalence.*

Proof This is a special case of [15, Theorem 4.12]. □

In what follows we aim to show that $\mathscr{U}_T$ detects stable equivalences. First, let $t_n\colon \mathrm{Fr}_{n+1}(T \wedge A) \longrightarrow \mathrm{Fr}_n(A)$ be the map of motivic symmetric spectra (resp. motivic spectra) adjoint to the inclusion $T \wedge A \hookrightarrow \Sigma_{n+1} \wedge T \wedge A$ of the identity in Σ_{n+1} (resp. adjoint to the identity on $T \wedge A$). The adjoint of the n-th structure map of E is then naturally isomorphic to

$$\mathbf{M}(t_n, E)\colon \mathbf{M}(\mathrm{Fr}_n(T^0), E) \longrightarrow \mathbf{M}(\mathrm{Fr}_{n+1}(T \wedge T^0), E).$$

The canonical map $\mathrm{Fr}_1 T \longrightarrow \mathrm{Fr}_0 T^0$ induces a natural transformation from $\mathrm{id}_{\mathbf{MSS}}$ to $\mathbb{T} := \mathbf{MSS}(\mathrm{Fr}_1 T, -)$. By iteration $\mathbb{T}^n(E) \cong \mathbf{MSS}(\mathrm{Fr}_n T^{\wedge n}, E)$, so that in particular $\mathrm{Ev}_m\big(\mathbb{T}^n(E)\big) \cong \mathbf{M}(T^{\wedge n}, E_{n+m})$, and there is an induced natural transformation

$$\mathrm{id}_{\mathbf{MSS}} \longrightarrow \mathbb{T}^\infty := \mathrm{colim}_{m \geq 0}\, \mathbb{T}^m.$$

Lemma 2.15 *If E is a motivic symmetric spectrum there is an isomorphism*

$$\mathrm{Ev}_m\big(\mathbb{T}^\infty(E)\big) \cong \mathrm{Ev}_m\big(\theta^\infty(\mathscr{U}_T E)\big)$$

for every $m \geq 0$ which is natural in E.

Proof Let $\Omega\colon \mathbf{MSS} \longrightarrow \mathbf{MSS}$ denote the extension $\mathbf{MSS}(\mathrm{Fr}_0 T, -)$ of the T-loops functor. Recall that $(\Omega E)_n = \mathbf{M}(T, E_n)$ and if τ interchanges the two copies of T, the n-th adjoint structure map of ΩE is the composition

$$\sigma_n^{\Omega E}\colon \mathbf{M}(T, E_n) \xrightarrow{\mathbf{M}(T,\sigma_n^E)} \mathbf{M}\big(T, \mathbf{M}(T, E_{n+1})\big) \xrightarrow{\tau} \mathbf{M}\big(T, \mathbf{M}(T, E_{n+1})\big). \quad (5)$$

According to the definitions, $\mathrm{Ev}_m\big(\mathbb{T}^\infty(E)\big)$ is the colimit of the diagram

$$E_m \xrightarrow{\sigma_m^E} \mathbf{M}(T, E_{m+1}) \xrightarrow{\sigma_{m+1}^{\Omega E}} \mathbf{M}(T^{\wedge 2}, E_{m+2}) \xrightarrow{\sigma_{m+2}^{\Omega^2 E}} \cdots,$$

and $\mathrm{Ev}_m\big(\theta^\infty(\mathscr{U}_T E)\big)$ is the colimit of the diagram

$$E_m \xrightarrow{\sigma_m^E} \mathbf{M}(T, E_{m+1}) \xrightarrow{\mathbf{M}(T,\sigma_{m+1}^E)} \mathbf{M}(T^{\wedge 2}, E_{m+2}) \xrightarrow{\mathbf{M}(T^{\wedge 2},\sigma_{m+2}^E)} \cdots.$$

We claim there exists permutations $\rho_n \in \Sigma_n$ and commutative diagrams

$$\begin{array}{ccc} \mathbf{M}(T^{\wedge n}, E_{m+n}) & \xrightarrow{\sigma_{m+n}^{\Omega^n E}} & \mathbf{M}(T^{\wedge n+1}, E_{m+n+1}) \\ {\scriptstyle\rho_n}\downarrow & & \downarrow{\scriptstyle\rho_{n+1}} \\ \mathbf{M}(T^{\wedge n}, E_{m+n}) & \xrightarrow{\mathbf{M}(T^{\wedge n},\sigma_{m+n}^E)} & \mathbf{M}(T^{\wedge n+1}, E_{m+n+1}) \end{array} \quad (6)$$

for $n \geq 0$. The maps in (5) show there exists a composition of n transpositions $\tau_n = (12) \circ (23) \circ \cdots \circ (n-1n)$ such that $\tau_n \circ \sigma_{m+n}^{\Omega^n E} = \mathbf{M}(T^{\wedge n}, \sigma_{m+n}^E)$. Thus the inductively defined sequence of isomorphisms $\rho_{n+1} := \rho_n \circ \tau_n$ renders (6) commutative and the claimed isomorphism follows taking colimits; naturality holds because the choices of permutations are independent of E. $\square$

Theorem 2.16 *Let $f\colon E \longrightarrow F$ be a map in* **MSS** *such that $\mathscr{U}_T(f)$ is a stable equivalence of motivic spectra. Then f is a stable equivalence.*

Proof Since $\mathscr{U}_T$ detects levelwise equivalences, we may assume that f is a map between cofibrant and levelwise fibrant motivic symmetric spectra. If $\mathscr{U}_T(f)$ is a stable equivalence, then $\mathrm{Ev}_m\big(\theta^\infty \mathscr{U}_T(f)\big)$ is a motivic equivalence for all $m \geq 0$. Hence $\mathrm{Ev}_m\big(\mathbb{T}^\infty(f)\big)$ is a motivic equivalence by Lemma 2.15. In particular, $\mathbb{T}^\infty(f)$ is a stable equivalence. We need to show $\mathbf{sSet}_{\mathbf{MSS}}(f, G)$ is a weak equivalence for G any stably fibrant motivic symmetric spectrum.

There is a simplicial model structure on motivic symmetric spectra with levelwise equivalences as weak equivalences and levelwise monomorphisms as cofibrations [18, Lemma 2.1.2]. The fibrations in this model structure are called injective fibrations. Let $G \longrightarrow G^{if}$ denote an injective fibrant replacement; it is a levelwise equivalence of levelwise fibrant spectra. It follows that $\mathbf{sSet}_{\mathbf{MSS}}(f, G)$ is a weak equivalence if and only if $\mathbf{sSet}_{\mathbf{MSS}}(f, G^{if})$ is.

Since G is stably fibrant if and only if the adjoint structure maps σ_n^G are motivic equivalences [15, Theorem 8.8] we conclude that G^{if} is stably fibrant.

The assumption on G implies the map $G \longrightarrow \mathbb{T}(G)$ is a stable equivalence of stably fibrant motivic symmetric spectra. Hence so is $h_G\colon G \longrightarrow \mathbb{T}^\infty(G)$. In particular, h_G is a levelwise acyclic monomorphism, which implies G^{if} is a retract of $\mathbb{T}^\infty(G^{if})$ since there exists a filler $l\colon \mathbb{T}^\infty(G^{if}) \longrightarrow G^{if}$ in the diagram:

$$\begin{tikzcd} G^{if} \arrow[r, equal] \arrow[d, "h_{G^{if}}"'] & G^{if} \arrow[d] \\ \mathbb{T}^\infty(G^{if}) \arrow[r] & * \end{tikzcd}$$

Now since $\mathbb{T}^\infty$ is a simplicial functor there are induced maps

$$\begin{tikzcd} \mathbf{sSet}_{\mathbf{MSS}}(F, G^{if}) \arrow[r, "\mathbb{T}^\infty"] \arrow[d, "f^*"'] & \mathbf{sSet}_{\mathbf{MSS}}(\mathbb{T}^\infty(F), \mathbb{T}^\infty(G^{if})) \arrow[r, "h_F^* \circ l_*"] \arrow[d, "\mathbb{T}^\infty(f^*)"'] & \mathbf{sSet}_{\mathbf{MSS}}(F, G^{if}) \arrow[d, "f^*"'] \\ \mathbf{sSet}_{\mathbf{MSS}}(E, G^{if}) \arrow[r, "\mathbb{T}^\infty"] & \mathbf{sSet}_{\mathbf{MSS}}(\mathbb{T}^\infty(E), \mathbb{T}^\infty(G^{if})) \arrow[r, "h_E^* \circ l_*"] & \mathbf{sSet}_{\mathbf{MSS}}(E, G^{if}) \end{tikzcd}$$

such that the horizontal composite maps equal the respective identity maps. This shows $\mathbf{sSet}(f, G^{if})$ is a retract of $\mathbf{sSet}(\mathbb{T}^\infty(f), G^{if})$, which concludes the proof because, as noted above, $\mathbb{T}^\infty(f)$ is a stable equivalence. □

Remark 2.17 Theorem 2.16 is analogous to [16, Theorem 3.1.11] and also [18, Proposition 4.8].

Lemma 2.18 *A map $f\colon E \longrightarrow F$ of motivic spectra is a stable fibration if and only if it is a levelwise fibration and*

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "f_n"'] & \mathbf{M}(T, E_{n+1}) \arrow[d, "\mathbf{M}(T, f_{n+1})"] \\ F_n \arrow[r] & \mathbf{M}(T, F_{n+1}) \end{tikzcd} \tag{7}$$

is a homotopy pullback square of motivic spaces for $n \geq 0$.

Proof According to [15, Corollary 4.14] a levelwise fibration f of levelwise fibrant motivic spectra is a stable fibration if and only if

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "f_n"'] & (\Theta^\infty E)_n \arrow[d, "(\Theta^\infty f)_n"] \\ F_n \arrow[r] & (\Theta^\infty F)_n \end{tikzcd} \tag{8}$$

is a homotopy pullback square for all $n \geq 0$. We consider the factorization:

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "f_n"'] & \mathbf{M}(T, E_{n+1}) \arrow[r] \arrow[d, "\mathbf{M}(T, f_{n+1})"] & (\Theta^\infty E)_n \arrow[d, "(\Theta^\infty f)_n"] \\ F_n \arrow[r] & \mathbf{M}(T, F_{n+1}) \arrow[r] & (\Theta^\infty F)_n \end{tikzcd} \tag{9}$$

If f is a stable fibration of levelwise fibrant motivic spectra, then $\Theta(f)$ is a stable fibration by [15, Lemma 3.8]. Hence the component squares in (9) are homotopy pullback squares. It follows that (7) is a homotopy pullback square [6, Lemma 3.13]. Conversely, if f is a levelwise fibration of levelwise fibrant motivic spectra such that (7) is a homotopy pullback, then so is (8) because it is a filtered colimit of homotopy pullback squares, and weak equivalences as well as fibrations with fibrant codomains are preserved under filtered colimits. To conclude in general we may apply a levelwise fibrant replacement functor and refer to [6, Lemma 6.9]. $\square$

Next we discuss the model structures on $\mathbf{MS}^{\mathrm{tr}}$ and $\mathbf{MSS}^{\mathrm{tr}}$. The respective levelwise model structures exist, for example, by comparison with $\mathbf{MS}$ and $\mathbf{MSS}$ using [13, Theorem 2.1.19]. The classes of stable equivalences and stable fibrations for motivic (symmetric) spectra with transfers are defined using the exact same script as in Definition 2.12.

Next we characterize stable equivalences and stable fibrations of motivic spectra with transfers. The first part is a special case of [15, Theorem 4.12].

Lemma 2.19 *A map f of levelwise fibrant motivic spectra with transfers is a stable equivalence if and only if $\Theta^\infty(f)$ is a levelwise weak equivalence.*

Lemma 2.20 *A map $f\colon E \longrightarrow F$ of motivic spectra with transfers is a stable fibration if and only if it is a levelwise fibration and*

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "f_n"'] & \mathbf{M}^{\mathrm{tr}}(T^{\mathrm{tr}}, E_{n+1}) \arrow[d, "\mathbf{M}^{\mathrm{tr}}(T^{\mathrm{tr}}, f_{n+1})"] \\ F_n \arrow[r] & \mathbf{M}^{\mathrm{tr}}(T^{\mathrm{tr}}, F_{n+1}) \end{tikzcd} \tag{10}$$

is a homotopy pullback square of motivic spaces with transfers for $n \geq 0$.

Proof The proof of Lemma 2.18 applies verbatim. $\square$

Let $\Theta\colon \mathbf{MS}^{\mathrm{tr}} \longrightarrow \mathbf{MS}^{\mathrm{tr}}$ denote the transfer analog of $\Theta\colon \mathbf{MS} \longrightarrow \mathbf{MS}$, and similarly for Θ^∞. Since $\mathscr{U}$ commutes with $\mathbf{M}(T, -)$ we get the equality $\mathscr{U}_{\mathbb{N}} \circ \Theta = \Theta \circ \mathscr{U}_{\mathbb{N}}$.

Lemma 2.21 *The forgetful functor $\mathscr{U}_{\mathbb{N}}\colon \mathbf{MS}^{\mathrm{tr}} \longrightarrow \mathbf{MS}$ both detects and preserves stable fibrations and stable equivalences.*

Proof $\mathscr{U}\colon \mathbf{M}^{\mathrm{tr}} \longrightarrow \mathbf{M}$ detects and preserves motivic weak equivalences and motivic fibrations, see Lemma 2.7, so it detects and preserves homotopy pullback squares. Lemmas 2.1, 2.18 and 2.20 imply easily the statement about stable fibrations. To prove the claim concerning stable equivalences, we use that $\mathscr{U}_{\mathbb{N}}$ commutes with the stabilization functor Θ^∞ and combine this with Lemmas 2.14 and 2.19. $\square$

Theorem 2.22 *The stable equivalences and stable fibrations form Quillen equivalent model structures on motivic symmetric spectra with transfers and motivic spectra with transfers. Moreover, the stable model structure on $\mathbf{MSS}^{\mathrm{tr}}$ is symmetric monoidal and the lax symmetric monoidal functor $\mathscr{U}_{\Sigma}$ detects and preserves stable equivalences and stable fibrations between stably fibrant objects.*

Proof Combining Theorem 2.6 and results due to Hovey [15, Sections 3, 8, 10] shows the model structures exist and also that $\mathbf{MSS}^{\mathrm{tr}}$ is symmetric monoidal. Since T satisfies the cyclic permutation condition [15, Definition 10.2], the model structures are Quillen equivalent via a zig-zag as in [15, Section 10]. The claims concerning $\mathscr{U}_\Sigma$ follow since a stable equivalence between stably fibrant objects is a levelwise weak equivalence, and ditto for stable fibrations. □

The proof of the next result uses a rather ad hoc method which allows us to strengthen [15, Theorem 10.1] in the case of motivic spaces with transfers.

Theorem 2.23 *The adjoint* $\mathscr{L}_T^{\mathrm{tr}}\colon \mathbf{MS}^{\mathrm{tr}} \longrightarrow \mathbf{MSS}^{\mathrm{tr}}$ *of the forgetful functor* $\mathscr{U}_T^{\mathrm{tr}}$ *is a left Quillen equivalence.*

In effect, we consider spectra $\mathbf{ChS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ and symmetric spectra $\mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ of non-connective chain complexes of presheaves with transfers with respect to the suspension coordinate $\mathbb{G}_m^{\mathrm{tr}}$.

Proposition 2.24 *The functor* $\mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}\colon \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}} \longrightarrow \mathbf{ChS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ *detects stable equivalences.*

Proof In view of Lemma 2.25, the proof of Theorem 2.16 applies verbatim since $\mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}$ forgets the symmetric group actions and Lemma 2.15 concerning the functors θ and $\mathbb{T}$ holds for formal reasons. □

Lemma 2.25 *The category* $\mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ *acquires a model structure with weak equivalences the levelwise schemewise quasi-isomorphisms and cofibrations the monomorphisms.*

Proof A folk theorem attributed to Joyal says the category of non-connective chain complexes in any Grothendieck abelian category supports the so-called 'injective' model structure with weak equivalences the quasi-isomorphisms and cofibrations the monomorphisms [14]. Presheaves with transfers form a Grothendieck abelian category, so the model structure exists by comparison with the isomorphic chain complex category of the Grothendieck abelian category of symmetric $\mathbb{G}_m^{\mathrm{tr}}$-spectra of presheaves with transfers. □

Theorem 2.26 *The adjoint* $\mathscr{L}_{\mathbf{Ch}}^{\mathrm{tr}}\colon \mathbf{ChS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}} \longrightarrow \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ *of the forgetful functor* $\mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}$ *is a left Quillen equivalence.*

Proof The right adjoint $\mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}$ detects and preserves levelwise equivalences and fibrations, and also all stably fibrant objects. Hence it detects and preserves stable equivalences between stably fibrant objects, as well as stable fibrations with stably fibrant codomain. This implies $\mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}$ is a right Quillen functor by [5, Corollary A.2]. To conclude it is a Quillen equivalence, it suffices to check the derived unit

$$E \longrightarrow \mathscr{U}_{\mathbf{Ch}}^{\mathrm{tr}}(\mathscr{L}_{\mathbf{Ch}}^{\mathrm{tr}}E)^f$$

is a stable equivalence for all cofibrant spectra E. Since the model structure on $\mathbf{ChS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}}$ is stable and cofibrantly generated, we may assume $E = \mathrm{Fr}_n U^{\mathrm{tr}}$ for a finite type and smooth S-scheme U.

If $n = 0$, then the unit $\mathrm{Fr}_0 U^{\mathrm{tr}} \longrightarrow \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}} \mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} \mathrm{Fr}_0 U^{\mathrm{tr}}$ is the identity, so according to Proposition 2.24 it suffices to construct $r\colon \mathrm{Fr}_0 U^{\mathrm{tr}} \longrightarrow \mathrm{Fr}_0 U^{\mathrm{tr}}$ such that $\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}(r)$ is a stable equivalence and $\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}(\mathrm{Fr}_0 U^{\mathrm{tr}})$ is stably fibrant. In effect, suppose $R\colon \mathbf{Ch}^{\mathrm{tr}} \longrightarrow \mathbf{Ch}^{\mathrm{tr}}$ is a fibrant replacement functor which entails a natural map $\mathbb{G}^{\mathrm{tr}}_m \otimes^{\mathrm{tr}} R(C) \longrightarrow R(\mathbb{G}^{\mathrm{tr}}_m \otimes^{\mathrm{tr}} C)$ and a commutative diagram:

$$\begin{tikzcd} & \mathbb{G}^{\mathrm{tr}}_m \otimes C \arrow[dl] \arrow[dr] & \\ \mathbb{G}^{\mathrm{tr}}_m \otimes R(C) \arrow[rr] & & R(\mathbb{G}^{\mathrm{tr}}_m \otimes C) \end{tikzcd} \tag{11}$$

The diagram (11) implies that R extends to a functor of symmetric spectra. We may construct R by applying the small object argument with respect to the set of $\mathbb{G}^{\mathrm{tr}}_m$-suspensions of generating acyclic cofibrations for $\mathbf{Ch}^{\mathrm{tr}}$ following our construction in [6, Section 3.3.2]. Let $Q\colon \mathbf{Ch}^{\mathrm{tr}} \longrightarrow \mathbf{Ch}^{\mathrm{tr}}$ be the functor given by

$$C \longmapsto \operatorname{colim}_n \mathbf{Ch}^{\mathrm{tr}}\Big((\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n}, R\big((\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes C\big)\Big).$$

Using (11) and setting $\big(Q(F)\big)_n := Q(F_n)$ we find that Q extends to a functor $Q\colon \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}^{\mathrm{tr}}_m} \longrightarrow \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}^{\mathrm{tr}}_m}$. In the special case $F = \mathrm{Fr}_0 U^{\mathrm{tr}}$ and level n, let

$$r_n\colon (\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} \longrightarrow Q(\mathrm{Fr}_0 U^{\mathrm{tr}})_n$$

be the canonical composite map

$$(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} \longrightarrow R\big(\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}(\mathrm{Fr}_0 U^{\mathrm{tr}})_n\big) \longrightarrow \big(\Theta^\infty \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}} \circ R(\mathrm{Fr}_0 U^{\mathrm{tr}})\big)_n.$$

In particular, $\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}\big(r\colon \mathrm{Fr}_0 U^{\mathrm{tr}} \longrightarrow Q(\mathrm{Fr}_0 U^{\mathrm{tr}})\big)$ is a stable equivalence. Hence r is a stable equivalence of symmetric spectra by Proposition 2.24. Moreover, $Q(\mathrm{Fr}_0 U^{\mathrm{tr}})$ is stably fibrant since $Q(U^{\mathrm{tr}}) \longrightarrow \mathbf{Ch}^{\mathrm{tr}}\big(\mathbb{G}^{\mathrm{tr}}_m, Q(\mathbb{G}^{\mathrm{tr}}_m \otimes U^{\mathrm{tr}})\big)$ is a schemewise weak equivalence. This implies the derived unit

$$\mathrm{Fr}_0 U^{\mathrm{tr}} \longrightarrow \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}} Q(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} \mathrm{Fr}_0 U^{\mathrm{tr}})$$

is a stable equivalence.

If $n > 0$, let $s_n\colon \mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} \longrightarrow \mathrm{Fr}_0 U^{\mathrm{tr}}$ be the map of motivic spectra with transfers adjoint to the identity $(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} = (\mathrm{Fr}_0 U^{\mathrm{tr}})_n$. Since s_n is an isomorphism in all levels $\geq n$, it is a stable equivalence and so is $\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} s_n)^f$. In the commutative diagram

$$\begin{tikzcd} \mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} \arrow[r] \arrow[d, "s_n"'] & \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}\big(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} \mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}}\big)^f \arrow[d, "\sim"] \\ \mathrm{Fr}_0 U^{\mathrm{tr}} \arrow[r] & \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}\big(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} \mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}}\big)^f \end{tikzcd}$$

the upper horizontal map is a stable equivalence. We claim the derived unit $\mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}} \longrightarrow \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}\big(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}} \mathrm{Fr}_n(\mathbb{G}^{\mathrm{tr}}_m)^{\otimes n} \otimes U^{\mathrm{tr}}\big)^f$ is connected to the map $(\mathbb{G}_m^{\mathrm{tr}})^{\otimes n} \otimes \mathrm{Fr}_n U^{\mathrm{tr}} \longrightarrow (\mathbb{G}_m^{\mathrm{tr}})^{\otimes n} \otimes \mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}(\mathscr{L}^{\mathrm{tr}}_{\mathbf{Ch}}\mathrm{Fr}_n U^{\mathrm{tr}})^f$ via a zig-zag of stable equivalences. In effect, $\mathscr{U}^{\mathrm{tr}}_{\mathbf{Ch}}$ maps cofibrations to levelwise cofibrations and tensoring with the suspension coordinate $\mathbb{G}_m^{\mathrm{tr}}$ preserves levelwise weak equivalences of levelwise cofibrant objects. The result follows since tensoring with $(\mathbb{G}_m^{\mathrm{tr}})^{\otimes n}$ is a Quillen equivalence. □

Proof (of Theorem 2.23) The shift functor of $\mathbf{Ch}^{\mathrm{tr}}$ is a Quillen equivalence. Thus (symmetric) $\mathbb{G}_m^{\mathrm{tr}}$-spectra and $\mathbb{G}_m^{\mathrm{tr}}[1]$-spectra of non-connective chain complexes of presheaves with transfers are Quillen equivalent. The zig-zag of Quillen equivalences relating (symmetric) $\mathbb{G}_m^{\mathrm{tr}}[1]$-spectra of non-connective chain complexes with transfers and the motivic (symmetric) spectra with transfers is compatible with the functor forgetting symmetric group actions. Using Theorem 2.26 we conclude that $\mathscr{L}_T^{\mathrm{tr}}$ is a left Quillen equivalence. □

Recall that a set of objects generates a triangulated category $\mathcal{C}$ if $\mathcal{C}$ is the smallest localizing subcategory of $\mathcal{C}$ which contains the given set of objects. A full subcategory of a triangulated category is localizing if it is closed under arbitrary direct sums, retracts and cofiber sequences.

The following lemma shows a distinguished property of shifted motivic symmetric suspension spectra.

Lemma 2.27 *The triangulated homotopy category of* $\mathbf{MSS}^{\mathrm{tr}}$ *is generated by shifted motivic symmetric suspension spectra of representable motivic spaces with transfers. An analogous statement holds for* $\mathbf{MSS}$.

Proof The class of cofibrations in $\mathbf{MSS}^{\mathrm{tr}}$ is generated by the set

$$\{\mathrm{Fr}_m(U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n])\}^{m,n\geq 0}_{U\in \mathrm{Sm}}. \tag{12}$$

Likewise, the generating cofibrations in $\mathbf{MSS}$ are of the form

$$\mathrm{Fr}_m\big((U \times [\partial\Delta^n \hookrightarrow \Delta^n])_+\big).$$

Hence the result follows by riffs of arguments in [13, Section 7]. □

These generators are compact, i.e. in the homotopy category $\mathrm{Ho}(\mathbf{MSS}^{\mathrm{tr}})$ the corresponding representable functors preserve arbitrary coproducts. In (12), it suffices to consider only quasi-projective S-schemes, cp. the proof of Lemma 3.2.

2.2.2 The discrete model

The pointed projective line $(\mathbb{P}^1, 1)$ is the pushout of the diagram

$$* \longleftarrow S_+ \longrightarrow \mathbb{P}^1_+$$

where the right hand side map is given by the rational point $1\colon S \hookrightarrow \mathbb{P}^1$. It is not a cofibrant motivic space, but its image $(\mathbb{P}^1, 1)^{\mathrm{tr}}$ in $\mathbf{M}^{\mathrm{tr}}$ is cofibrant: Since the rational point in question is a section of the structure map of $\mathbb{P}^1$, the short exact sequence

$$0 \longrightarrow S^{\mathrm{tr}} \longrightarrow (\mathbb{P}^1)^{\mathrm{tr}} \longrightarrow (\mathbb{P}^1, 1)^{\mathrm{tr}} \longrightarrow 0$$

splits, so that $(\mathbb{P}^1, 1)^{\mathrm{tr}}$ is a retract of the cofibrant motivic space with transfers represented by the projective line.

Lemma 2.28 *Suppose $i\colon U \longrightarrow V$ is an open or closed embedding in* Sm *and consider the simplicial mapping cylinder factorization*

$$(U \rightarrowtail Cyl \xrightarrow{\cong} V)^{\mathrm{tr}}.$$

Then $(Cyl/U \longrightarrow V/U)^{\mathrm{tr}}$ is a schemewise weak equivalence.

Proof Using the Dold-Kan correspondence, it suffices to consider the map of presheaves of homology groups associated to the normalized chain complexes of $\cdots \longrightarrow 0 \longrightarrow U^{\mathrm{tr}} \longrightarrow V^{\mathrm{tr}}$ and $\cdots \longrightarrow 0 \longrightarrow V^{\mathrm{tr}}/U^{\mathrm{tr}}$. The result follows immediately since $(i\colon U \longrightarrow V)^{\mathrm{tr}}$ is a monomorphism. $\square$

There is a zig-zag of weak equivalences between $S^1_s \wedge \mathbb{G}$ with $(\mathbb{P}^1, 1)$ which consists of finitely presented cofibrant motivic spaces except at the endpoint $(\mathbb{P}^1, 1)$. It is obtained using the simplicial mapping cylinder [7, Section 4.2]. Lemma 2.28 shows $\mathbb{Z}^{\mathrm{tr}}$ turns this zig-zag into a zig-zag of weak equivalences between finitely presented cofibrant motivic spaces with transfers. Thus using [15, Theorem 9.4] – which requires cofibrant suspension coordinates – we get:

Proposition 2.29 *There is a zig-zag of strict monoidal Quillen equivalences between $\mathbf{MSS}^{\mathrm{tr}}$ and $\mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1}$.*

2.2.3 Spectra of chain complexes

Let $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ denote the category of symmetric spectra of connective chain complexes of presheaves with transfers, where the suspension coordinate is the normalized chain complex

$$\cdots \longrightarrow 0 \longrightarrow (\mathbb{P}^1, 1)^{\mathrm{tr}} \tag{13}$$

of $(\mathbb{P}^1, 1)^{\mathrm{tr}}$. Similarly, we let $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]}$ denote the category of symmetric spectra of connective chain complexes of presheaves with transfers, where the suspension coordinate is the normalized chain complex

$$\cdots \longrightarrow 0 \longrightarrow (\mathbb{A}^1 \smallsetminus \{0\}, 1)^{\mathrm{tr}} \longrightarrow 0 \tag{14}$$

of $\big(S^1_s \wedge (\mathbb{A}^1 \smallsetminus \{0\}, 1)\big)^{\mathrm{tr}}$. The zig-zag of weak equivalences between $S^1_s \wedge \mathbb{G}$ and $(\mathbb{P}^1, 1)$ induces a zig-zag of weak equivalences of finitely presented cofibrant connective chain complexes between (13) and (14). This implies the chain complex analog of Proposition 2.29:

Proposition 2.30 *There is a zig-zag of strict symmetric monoidal Quillen equivalences between $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ and $\mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]}$.*

Next we consider non-connective chain complexes $\mathbf{Ch}^{\mathrm{tr}} := \mathbf{Ch}(\mathbf{Pre}^{\mathrm{tr}})$. Inserting zero in negative degrees we get a strict symmetric monoidal and full embedding $i_0\colon \mathbf{Ch}^{\mathrm{tr}}_+ \hookrightarrow \mathbf{Ch}^{\mathrm{tr}}$ of connective chain complexes into non-connective chain complexes. Its right adjoint, the good truncation functor τ_0 sends $C = (C_n, d_n)_{n\in\mathbb{Z}}$ to the connective chain complex

$$\tau_0(C) = \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow \ker d_0 \longrightarrow 0.$$

There is a naturally induced strict symmetric monoidal and full embedding $i\colon \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]} \hookrightarrow \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}^{\mathrm{tr}}_m[1]}$. Its right adjoint τ sends $E = (E_0, E_1, \dots)$ to its connective cover $\big(\tau_0(E_0), \tau_0(E_1), \dots\big)$ equipped with the structure maps

$$\begin{aligned}\mathbb{G}^{\mathrm{tr}}_m[1] \otimes \tau_0(E_n) &\longrightarrow \tau_0\big(i_0(\mathbb{G}^{\mathrm{tr}}_m)\big) \otimes \tau_0(E_n)\\ &\longrightarrow \tau_0\big(i_0(\mathbb{G}^{\mathrm{tr}}_m) \otimes E_n\big)\\ &\longrightarrow \tau_0(E_{n+1}).\end{aligned}$$

Hence τ prolongs τ_0 and the unit $\mathrm{id} \longrightarrow \tau \circ i$ is the identity.

The schemewise model on $\mathbf{Ch}^{\mathrm{tr}}$ is obtained by applying [6, Theorem 4.4] to the standard projective model structure on non-connected chain complexes [13, Theorem 2.3.11]. Next the motivic model is defined by localizing the schemewise model using the exact same approach as for motivic spaces with transfers in Section 2.1. One checks it is monoidal by considering generating (acyclic) cofibrations, which have bounded below domains and codomains. Then (i_0, τ_0) is a Quillen adjoint pair because i_0 maps the set of generating (acyclic) cofibrations to a subset of the set of generating (acyclic) cofibrations. Note also that the model structures on symmetric spectra of (non-connective) chain complexes with transfers exist by work of Hovey [15].

Proposition 2.31 *There is a strict symmetric monoidal Quillen equivalence*

$$i\colon \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]} \rightleftarrows \mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}^{\mathrm{tr}}_m[1]} : \tau$$

Proof The adjunction is clearly a Quillen adjoint pair for the levelwise model structure on symmetric spectra. Second, the isomorphism between standard internal hom objects

$$\tau_0\mathbf{Ch}^{\mathrm{tr}}\big(i_0(C), D\big) \cong \mathbf{Ch}^{\mathrm{tr}}_+\big(C, \tau_0(D)\big)$$

implies that it is also a Quillen adjoint pair for the stable model structure. Since i_0 preserves fibrant chain complexes with transfer and also schemewise equivalences, one gets that i preserves fibrant symmetric spectra. The unit $\mathrm{id} \longrightarrow \tau \circ i$ is the identity, hence the derived unit

$$E \longrightarrow \tau\big(i(E)^f\big)$$

is a stable equivalence for all $E \in \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{G}^{\mathrm{tr}}_m[1]}$. To conclude (i, τ) is a Quillen equivalence, it remains to check that τ detects weak equivalences of fibrant symmetric spectra. Suppose $f\colon E \longrightarrow F$ is a map such that $\tau(f)$ is a weak equivalence. Now tensoring with $\mathbb{G}^{\mathrm{tr}}_m[1] = \mathbb{G}^{\mathrm{tr}}_m \otimes (\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0)$ is a Quillen equivalence by construction and thus ditto for the shift functor. Hence we may assume $F = 0$. The counit $i_0\tau_0 E_n \longrightarrow E_n$ induces the identity on homology groups H_m for $m \geq 0$, and thus $H_m E_n = 0$ for $m \geq 0$. Since $\mathbb{G}^{\mathrm{tr}}_m$ is discrete and cofibrant, applying $\mathbf{Ch}^{\mathrm{tr}}(\mathbb{G}^{\mathrm{tr}}_m, -)$ to the schemewise weak equivalence

$$\begin{tikzcd}
\cdots \arrow[r] & (E_n)_1 \arrow[r] \arrow[d] & (E_n)_0 \arrow[r] \arrow[d, "d_0"] & (E_n)_{-1} \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \operatorname{im} d_0 \arrow[r, hook] & (E_n)_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

of (schemewise fibrant) chain complexes with transfers induces a schemewise weak equivalence, i.e. $H_k\mathbf{Ch}^{\mathrm{tr}}(\mathbb{G}_m^{\mathrm{tr}}, E_n) = 0$ for $k \geq 0$. Thus the groups

$$\begin{aligned} H_{-k}E_n &\cong H_{-k}\mathbf{Ch}^{\mathrm{tr}}\big((\mathbb{G}_m^{\mathrm{tr}})^{\otimes k}[k], E_{n+k}\big) \\ &\cong H_{-k}\mathbf{Ch}^{\mathrm{tr}}\big((\mathbb{G}_m^{\mathrm{tr}})^{\otimes k}, E_{n+k}\big)[-k] \\ &\cong H_0\mathbf{Ch}^{\mathrm{tr}}\big((\mathbb{G}_m^{\mathrm{tr}})^{\otimes k}, E_{n+k}\big) \end{aligned}$$

are trivial for all integers $k \geq 0$ and n since $E_n \longrightarrow \mathbf{Ch}^{\mathrm{tr}}\big((\mathbb{G}_m^{\mathrm{tr}})^{\otimes k}[k], E_{n+k}\big)$ is a weak equivalence. This finishes the proof. □

2.2.4 The Dold-Kan equivalence

The Dold-Kan equivalence

$$\mathcal{D}^{\mathrm{tr}} \colon \mathbf{M}^{\mathrm{tr}} \rightleftarrows \mathbf{Ch}_+^{\mathrm{tr}} \colon \mathcal{K}^{\mathrm{tr}}$$

assigns to any motivic space with transfers A its normalized connective chain complex. The shuffle maps $\nabla^{\mathrm{tr}}_{A,B} \colon \mathcal{D}^{\mathrm{tr}}(A) \otimes \mathcal{D}^{\mathrm{tr}}(B) \longrightarrow \mathcal{D}^{\mathrm{tr}}(A \otimes^{\mathrm{tr}} B)$ which are introduced in the work of Eilenberg and MacLane [8, (5.3)] show that $\mathcal{D}^{\mathrm{tr}}$ is a lax symmetric monoidal functor. In [28], Schwede and Shipley discuss the Dold-Kan equivalence for simplicial abelian groups from a model categorical viewpoint. Quite amusingly, it turns out the Dold-Kan equivalence is easier to deal with on the level of symmetric spectra for motivic spaces with transfers than simplicial abelian groups. Here the trick is to suspend with respect to a discrete motivic space with transfers.

Lemma 2.32 *For cofibrant motivic spaces with transfers A and B the shuffle map $\nabla^{\mathrm{tr}}_{A,B}$ is a quasi-isomorphism.*

Proof If A and B are (co)domains of generating cofibrations this follows as in the classical Eilenberg-Zilber Theorem [9]. By definition $\mathcal{D}^{\mathrm{tr}}$ is a left Quillen functor for the schemewise model structures. Thus, if A' is obtained from A by attaching a generating cofibration and $\nabla^{\mathrm{tr}}_{A,B}$ is a quasi-isomorphism, then so is $\nabla^{\mathrm{tr}}_{A',B}$ by [13, Lemma 5.2.6]. Proceeding inductively by attaching cells, the result follows. □

If E is a motivic symmetric spectrum with transfers, then the sequence of connective chain complexes of presheaves with transfers $\mathcal{D}^{\mathrm{tr}}(E_0), \mathcal{D}^{\mathrm{tr}}(E_1), \ldots$ forms a symmetric spectrum on account of the canonical maps

$$\mathcal{D}^{\mathrm{tr}}(E_n) \otimes (\mathbb{P}^1, 1)^{\mathrm{tr}} \xrightarrow{\cong} \mathcal{D}^{\mathrm{tr}}\big(E_n \otimes^{\mathrm{tr}} (\mathbb{P}^1, 1)^{\mathrm{tr}}\big) \longrightarrow \mathcal{D}^{\mathrm{tr}}(E_{n+1}).$$

The functor $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ obtained by this construction manifestly prolongs $\mathcal{D}^{\mathrm{tr}}$ to the level of symmetric spectra.

Theorem 2.33 *The functor $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}} \colon \mathbf{MSS}^{\mathrm{tr}}_{\mathbb{P}^1} \longrightarrow \mathbf{ChSS}^{\mathrm{tr}}_{+,\mathbb{P}^1}$ is lax symmetric monoidal and a right Quillen equivalence which preserves the unit strictly. Its left adjoint $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ prolongs $\mathcal{K}^{\mathrm{tr}}$ and preserves the unit strictly, and moreover*

$$\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}(A \wedge B) \longrightarrow \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}(A) \wedge \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}(B)$$

is a weak equivalence provided A and B are cofibrant.

Proof The unit of the adjunction $(\mathcal{K}^{\mathrm{tr}}, \mathcal{D}^{\mathrm{tr}})$ is not monoidal [28, Remark 2.14]. However, on discrete objects it clearly coincides with the identity adjunction. This implies $(\mathcal{K}^{\mathrm{tr}}, \mathcal{D}^{\mathrm{tr}})$ is a $\mathbf{Pre}^{\mathrm{tr}}$-adjoint pair, and hence it extends using the proof of [15, Theorem 9.3] to an adjoint pair $(\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}, \mathcal{D}^{\mathrm{tr}}_{\mathrm{st}})$ of symmetric spectra for any suspension coordinate in $\mathbf{Pre}^{\mathrm{tr}}$ – for example $(\mathbb{P}^1, 1)^{\mathrm{tr}}$ – such that both the unit and counit is given levelwise. In particular, $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ and $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ are levelwise isomorphisms and $(\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}, \mathcal{D}^{\mathrm{tr}}_{\mathrm{st}})$ is an equivalence of categories. Since $\mathcal{K}^{\mathrm{tr}}$ and $\mathcal{D}^{\mathrm{tr}}$ detect and preserve motivic weak equivalences and also motivic fibrations, $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ and $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ preserve cofibrations of symmetric spectra and they are left adjoints in a Quillen equivalence between levelwise model structures.

The functors $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ and $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ commute with the left adjoint Fr_n of Ev_n, thus up to a natural isomorphism the map

$$\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}\Big(\mathrm{Fr}_{n+1}\big(A \wedge (\mathbb{P}^1, 1)^{\mathrm{tr}}\big) \longrightarrow \mathrm{Fr}_n A\Big)$$

coincides with

$$\mathrm{Fr}_{n+1}\big(\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}} A \wedge (\mathbb{P}^1, 1)^{\mathrm{tr}}\big) \longrightarrow \mathrm{Fr}_n \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}} A,$$

and similarly for $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$. In particular, $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ and $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ are left Quillen functors by [15, Theorem 2.2]. Hence $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ and $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ are right Quillen equivalences since any stably fibrant symmetric spectrum E of chain complexes is weakly equivalent to $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}\big(\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}(E)\big)$, where $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}(E)$ is stably fibrant as a symmetric spectrum of motivic spaces with transfers [15, Proposition 2.3].

The claim about the units is immediate. The functor $\mathcal{D}^{\mathrm{tr}}$ is lax symmetric monoidal. Since $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}} \circ \mathrm{Fr}_n \cong \mathrm{Fr}_n \circ \mathcal{D}^{\mathrm{tr}}$, writing symmetric spectra as colimits of free symmetric spectra of representable objects lets us conclude that $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ is lax symmetric monoidal. Moreover, for cofibrant symmetric spectra with transfers E and F, Lemma 2.32 implies the monoidality map

$$\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}(E) \wedge \mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}(F) \longrightarrow \mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}(E \wedge F)$$

is a levelwise schemewise quasi-isomorphism. Since $\mathcal{D}^{\mathrm{tr}}_{\mathrm{st}}$ is also lax symmetric monoidal, $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ acquires an induced op-lax monoidal structure.

If A and B are cofibrant connective chain complexes, then Lemma 2.32 implies there are isomorphisms and weak equivalences

$$\begin{aligned}
\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}\big(\mathrm{Fr}_m(A) \wedge \mathrm{Fr}_n(B)\big) &\cong \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}\big(\mathrm{Fr}_{m+n}(A \wedge B)\big) \\
&\cong \mathrm{Fr}_{m+n}\big(\mathcal{K}^{\mathrm{tr}}(A \wedge B)\big) \\
&\xrightarrow{\sim} \mathrm{Fr}_{m+n}\big(\mathcal{K}^{\mathrm{tr}}(A) \wedge \mathcal{K}^{\mathrm{tr}}(B)\big) \\
&\cong \mathrm{Fr}_m\big(\mathcal{K}^{\mathrm{tr}}(A)\big) \wedge \mathrm{Fr}_n\big(\mathcal{K}^{\mathrm{tr}}(B)\big) \\
&\cong \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}\big(\mathrm{Fr}_m(A)\big) \wedge \mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}\big(\mathrm{Fr}_n(B)\big).
\end{aligned}$$

One concludes, by attaching cells as in the proof of Lemma 2.32, the op-lax monoidal map for $\mathcal{K}^{\mathrm{tr}}_{\mathrm{st}}$ is a weak equivalence (in fact levelwise schemewise) for all cofibrant symmetric spectra of chain complexes. □

Remark 2.34 The adjunction in Theorem 2.33 is a weak monoidal Quillen equivalence in the sense of Schwede-Shipley [28, Definition 3.6].

2.3 Comparison with motives

Throughout this section the base scheme is a perfect field k. Sheafification yields left Quillen equivalences $\mathbf{Ch}^{\mathrm{tr}} \longrightarrow \mathbf{Ch}^{\mathrm{tr}\sim}$ between the Nisnevich local and motivic model structures, cp. Section 2.1. Its strict symmetric monoidal structure extends this to left Quillen functors $\mathbf{ChSS}^{\mathrm{tr}}_{\mathbb{G}_m^{\mathrm{tr}}[1]} \hookrightarrow \mathbf{ChSS}^{\mathrm{tr}\sim}_{\mathbb{G}_m^{\mathrm{tr}}[1]}$ of stable model structures which are Quillen equivalences [15, Theorem 9.3].

A chain complex of Nisnevich sheaves with transfers is called motivic if its homology sheaves are homotopy invariant. Voevodsky's category $\mathbf{DM}^{\mathrm{eff}}_{-}$ of effective motives is the full subcategory of the derived category of bounded below chain complexes of Nisnevich sheaves with transfers $\mathcal{D}^{-}(\mathrm{Shv}^{\mathrm{tr}\sim})$ which consists of motivic chain complexes [33, Section 3].

The homotopy category $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim})$ of the Nisnevich local model, which is defined as for motivic spaces with transfers in Section 2.1, is equivalent to the derived category of Nisnevich sheaves with transfers since Nisnevich local weak equivalences and quasi-isomorphisms coincide in $\mathbf{Ch}^{\mathrm{tr}\sim}$. We let $\mathbf{DM}^{\mathrm{eff}}$ be the full subcategory of $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim})$ consisting of motivic chain complexes. Note that it contains $\mathbf{DM}^{\mathrm{eff}}_{-}$ as a dense and full subcategory. If $C \in \mathbf{Ch}^{\mathrm{tr}\sim}$ is motivically fibrant, then the Nisnevich sheafification of the presheaf

$$U \longmapsto H_n\big(C(U)\big)$$

is $\mathbb{A}^1$-homotopy invariant [33, Theorem 3.1.12]. This shows that the motivic homotopy category $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim}_{\mathrm{mo}})$ is equivalent to the full subcategory of $\mathbf{DM}^{\mathrm{eff}}$ of fibrant chain complexes. By localization theory, $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim}_{\mathrm{mo}})$ gets identified with the localization of $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim})$ with respect to the localizing subcategory generated by chain complexes of the form

$$(U \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} U)^{\mathrm{tr}}.$$

Using the universal property of localizations we conclude there exists an equivalence between $\mathbf{DM}^{\mathrm{eff}}$ and $\mathrm{Ho}(\mathbf{Ch}^{\mathrm{tr}\sim}_{\mathrm{mo}})$.

The triangulated structures are preserved under all the equivalences and embeddings employed in the above: Recall the monoidal structure in $\mathbf{DM}^{\mathrm{eff}}_{-}$ is defined descending the monoidal structure in $\mathcal{D}^{-}(\mathrm{Shv}^{\mathrm{tr}\sim})$ [33, Section 3.2]. The latter is determined by the monoidal structure in the model category of (bounded below) chain complexes of Nisnevich sheaves with transfers, which implies the equivalences and embeddings preserve the monoidal structures.

In [33, Section 2.1] Voevodsky introduces the Spanier-Whitehead category $\mathbf{DM}_{\mathrm{gm}}$ of effective geometric motives $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ which has the effect of inverting the Tate object. As in stable homotopy theory there exists a monoidal and triangulated Spanier-Whitehead functor given by

$$\begin{aligned} SW\colon \mathbf{DM}_{\mathrm{gm}} &\longrightarrow \mathrm{Ho}(\mathbf{ChSS}^{\mathrm{tr}\sim}_{\mathbb{G}_m^{\mathrm{tr}}[1]}) \\ (C,n) &\longmapsto SW(C,n) := (\mathbb{G}_m^{\mathrm{tr}}[1])^{\otimes n} \otimes C. \end{aligned}$$

Here, if $n < 0$, $(\mathbb{G}_m^{\mathrm{tr}}[1])^{\otimes n}$ is interpreted as a shifted suspension spectrum.

Since $SW(C,n)$ is represented by a finite cofibrant symmetric spectrum of chain complexes, applying the functor SW and letting $p \geq -n, -n'$ we get that $\mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}}\big((C,n),(C',n')\big)$ is isomorphic to

$$\mathrm{colim}_p\,\mathrm{Hom}_{\mathbf{DM}_{-}^{\mathrm{eff}}}\big((\mathbb{G}_m^{\mathrm{tr}}[1])^{\otimes p+n} \otimes C, (\mathbb{G}_m^{\mathrm{tr}}[1])^{\otimes p+n'} \otimes C'\big),$$

and to

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{ChSS}^{\mathrm{tr}\sim}_{\mathbb{G}_m^{\mathrm{tr}}[1]})}\big(SW(C,n), SW(C',n')\big).$$

Hence the Spanier-Whitehead functor is a full embedding. The identification in [33, Theorem 3.2.6] of $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ as a full subcategory of $\mathbf{DM}^{\mathrm{eff}}_{-}$ implies the first isomorphism. The second isomorphism follows since the model structure on $\mathbf{ChSS}^{\mathrm{tr}\sim}_{\mathbb{G}_m^{\mathrm{tr}}[1]}$ has generating (acyclic) cofibrations with finitely presentable (co)domains. Finally, we note the vertical functors in the diagram

$$\begin{tikzcd}
\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r] \arrow[d] & \mathbf{DM}^{\mathrm{eff}} \arrow[d] \\
\mathbf{DM}_{\mathrm{gm}} \arrow[r] & \mathrm{Ho}(\mathbf{ChSS}^{\mathrm{tr}\sim}_{\mathbb{G}_m^{\mathrm{tr}}[1]})
\end{tikzcd}$$

are full embeddings according to Voevodsky's cancellation theorem [31]. The homotopy category is Voevodsky's big category of motives $\mathbf{DM}_k$ consisting of $\mathbb{G}_m^{\mathrm{tr}}$-spectra of non-connected chain complexes of Nisnevich sheaves with transfers having homotopy invariant homology sheaves.

2.4 Motivic cohomology

We define motivic cohomology to be the motivic symmetric spectrum

$$\mathrm{M}\mathbb{Z} := \big(\mathscr{U}(S^{\mathrm{tr}}_{+}),\, \mathscr{U}(T^{\mathrm{tr}}),\, \mathscr{U}(T^{\wedge 2})^{\mathrm{tr}},\, \ldots\big).$$

The structure maps of motivic cohomology are obtained by inserting $A = T$ and $B = T^{\wedge n}$ into the canonical maps

$$A \wedge \mathscr{U}(B^{\mathrm{tr}}) \longrightarrow \mathscr{U}(A^{\mathrm{tr}}) \wedge \mathscr{U}(B^{\mathrm{tr}}) \longrightarrow \mathscr{U}(A^{\mathrm{tr}} \otimes^{\mathrm{tr}} B^{\mathrm{tr}}) \xrightarrow{\cong} \mathscr{U}(A \wedge B)^{\mathrm{tr}}. \tag{15}$$

In [7, Section 4.2] it is shown that $\mathrm{M}\mathbb{Z}$ is weakly equivalent to Voevodsky's motivic Eilenberg-MacLane spectrum introduced in [32]. By [7, Example 3.4] $\mathrm{M}\mathbb{Z}$ is a commutative motivic symmetric ring spectrum. The multiplication and unit maps are determined by

$$\mathscr{U}(A^{\mathrm{tr}}) \wedge \mathscr{U}(B^{\mathrm{tr}}) \longrightarrow \mathscr{U}(A^{\mathrm{tr}} \otimes^{\mathrm{tr}} B^{\mathrm{tr}}) \xrightarrow{\cong} \mathscr{U}(A \wedge B)^{\mathrm{tr}},$$

and

$$A \longrightarrow \mathscr{U}(A^{\mathrm{tr}}).$$

Definition 2.35 An $\mathrm{M}\mathbb{Z}$-module is a motivic symmetric spectrum E with an action $\mathrm{M}\mathbb{Z} \wedge E \longrightarrow E$ such that the usual module conditions are satisfied. Let $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ denote the category of $\mathrm{M}\mathbb{Z}$-modules.

Standard constructions turn $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ into a closed symmetric monoidal category. We denote the smash product of $\mathrm{M}\mathbb{Z}$-modules E and F by $E \wedge_{\mathrm{M}\mathbb{Z}} F$. Note that $\mathrm{M}\mathbb{Z}$ is the unit. By neglect of structure, there exists a lax symmetric monoidal functor $\mathrm{M}\mathbb{Z} - \mathbf{mod} \longrightarrow \mathbf{MSS}$. Its strict symmetric monoidal left adjoint is the free module functor $\mathrm{M}\mathbb{Z} \wedge -\colon \mathbf{MSS} \longrightarrow \mathrm{M}\mathbb{Z} - \mathbf{mod}$. The $\mathrm{M}\mathbb{Z}$-module structure on $\mathrm{M}\mathbb{Z} \wedge E$ is given by the multiplication on $\mathrm{M}\mathbb{Z}$. It turns out there is a canonical model structure on $\mathrm{M}\mathbb{Z} - \mathbf{mod}$. A map of $\mathrm{M}\mathbb{Z}$-modules is a weak equivalence or a fibration if and only if the underlying map of motivic symmetric spectra is so.

Proposition 2.36 *The weak equivalences and the fibrations of* $\mathrm{M}\mathbb{Z}$*-modules form a proper monoidal model structure. The triangulated homotopy category* $\mathrm{Ho}(\mathrm{M}\mathbb{Z} - \mathbf{mod})$ *is generated by free* $\mathrm{M}\mathbb{Z}$*-modules of shifted motivic symmetric suspension spectra of representable motivic spaces.*

Proof The monoid axiom holds in $\mathbf{MSS}$ by inspecting [18, Proposision 4.19]. Hence $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ has a model structure according to [27, Theorem 4.1] and the second statement is now an easy consequence of Lemma 2.27. ☐

If E is a motivic symmetric spectrum with transfers, there is a canonical $\mathrm{M}\mathbb{Z}$-module structure on its underlying motivic symmetric spectrum $\mathscr{U}_\Sigma(E)$. The action $\mathrm{M}\mathbb{Z} \wedge \mathscr{U}_\Sigma(E) \longrightarrow \mathscr{U}_\Sigma(E)$ is defined by the maps

$$\mathscr{U}(T^{\wedge n})^{\mathrm{tr}} \wedge \mathscr{U}(E_m) \longrightarrow \mathscr{U}\big((T^{\mathrm{tr}})^{\otimes^{\mathrm{tr}} n} \otimes^{\mathrm{tr}} E_m\big) \longrightarrow \mathscr{U}(E_{m+n}).$$

These data determine a functor $\Psi\colon \mathbf{MSS}^{\mathrm{tr}} \longrightarrow \mathrm{M}\mathbb{Z} - \mathbf{mod}$. Since Ψ is a functor between locally presentable categories which preserves all limits and filtered colimits, it has a left adjoint Φ.

Lemma 2.37 *The functor* Ψ *detects and preserves weak equivalences and fibrations between stably fibrant objects and acyclic fibrations. In particular,* (Φ, Ψ) *is a Quillen adjoint pair.*

Proof The first statement follows from Theorem 2.22 and Proposition 2.36, and it implies the second by [5, Corollary A.2]. ☐

The construction of Ψ implies there is a commutative diagram:

$$\begin{tikzcd} \mathbf{MSS}^{\mathrm{tr}} \arrow[rr, "\Psi"] \arrow[dr, "\mathscr{U}_\Sigma"'] & & \mathrm{M}\mathbb{Z} - \mathbf{mod} \arrow[dl] \\ & \mathbf{MSS} & \end{tikzcd} \tag{16}$$

Thus, uniqueness of left adjoints implies there exists a unique isomorphism

$$\Phi(\mathrm{M}\mathbb{Z} \wedge E) \cong E^{\mathrm{tr}}. \tag{17}$$

In particular, we get $\Phi(\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m U_+) \cong \mathrm{Fr}_m U^{\mathrm{tr}}$. This allows us to identify the unit η of the adjunction (Φ, Ψ) for free $\mathrm{M}\mathbb{Z}$-modules $\mathrm{M}\mathbb{Z} \wedge A$, where now A is a motivic space. The unit $\eta\colon \mathrm{M}\mathbb{Z} \wedge A \longrightarrow \Psi\Phi(\mathrm{M}\mathbb{Z} \wedge A)$ is determined by the map of underlying motivic spectra.

Diagram (16) shows there is a commutative diagram of motivic spectra:

$$\begin{tikzcd}
A = \mathrm{Fr}_0 A \arrow[rr] \arrow[dr] & & \mathscr{U}_\Sigma \mathrm{Fr}_0 A^{\mathrm{tr}} \cong \Psi\Phi(\mathrm{M}\mathbb{Z} \wedge A) \\
& \mathrm{M}\mathbb{Z} \wedge A \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Thus, from equation (15), the n-th level of η is determined by the map

$$\mathscr{U}(T^{\wedge n})^{\mathrm{tr}} \wedge A \longrightarrow \mathscr{U}\big((T^{\mathrm{tr}})^{\otimes^{\mathrm{tr}} n} \otimes^{\mathrm{tr}} A^{\mathrm{tr}}\big).$$

Clearly, a similar analysis applies to modules of the form $\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m A$.

Lemma 2.38 *The left adjoint Φ is strict symmetric monoidal and the right adjoint Ψ is lax symmetric monoidal.*

Proof The claim concerning Ψ follows because $\mathscr{U}_\Sigma\colon \mathbf{MSS}^{\mathrm{tr}} \longrightarrow \mathbf{MSS}$ is lax symmetric monoidal. We also note that $\Psi(\mathbb{I}^{\mathrm{tr}}) = \mathrm{M}\mathbb{Z}$. By (17) there exists an isomorphism $\Phi(\mathrm{M}\mathbb{Z}) \longrightarrow \mathbb{I}^{\mathrm{tr}}$. Moreover, Φ acquires an op-lax symmetric monoidal structure via the natural map

$$\Phi(E \wedge_{\mathrm{M}\mathbb{Z}} F) \longrightarrow \Phi(E) \otimes \Phi(F). \tag{18}$$

We note that (18) is an isomorphism: If A is a motivic space, equation (17) shows the isomorphism holds for $\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m A$. Since every $\mathrm{M}\mathbb{Z}$-module is a colimit of $\mathrm{M}\mathbb{Z}$-modules of this form, we are done. □

3 Toward the Quillen equivalence

Our basic object of study is the Quillen adjoint pair:

$$\Phi\colon \mathrm{M}\mathbb{Z} - \mathbf{mod} \rightleftarrows \mathbf{MSS}^{\mathrm{tr}} \colon \Psi$$

Up to unique isomorphism, the left adjoint is characterized by the property that it maps the free $\mathrm{M}\mathbb{Z}$-module $\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m A$ of a shifted motivic symmetric suspension spectrum $\mathrm{Fr}_m A$ to the motivic symmetric spectrum with transfers $\mathrm{Fr}_m A^{\mathrm{tr}}$. We fix functorial fibrant replacement functors $E \overset{\sim}{\rightarrowtail} E^f \twoheadrightarrow *$ for motivic symmetric spectra and motivic symmetric spectra with transfers.

Lemma 3.1 *The pair (Φ, Ψ) is a Quillen equivalence if and only if for every cofibrant $\mathrm{M}\mathbb{Z}$-module E, the unit map*

$$\eta_E\colon E \longrightarrow \Psi\big(\Phi(E)^f\big)$$

is a weak equivalence of motivic symmetric spectra.

Proof This follows from Lemma 2.37. □

Proposition 2.36 explicates a set of generators for the homotopy category of $\mathrm{M}\mathbb{Z}-\mathbf{mod}$. Next we show that in fact it suffices to consider cofibrant generating $\mathrm{M}\mathbb{Z}$-modules obtained from smooth quasi-projective schemes.

Lemma 3.2 *The pair (Φ,Ψ) is a Quillen equivalence if and only if for every smooth quasi-projective S-scheme U and all integers $m\geq 0$, the unit map*

$$\mathrm{M}\mathbb{Z}\wedge \mathrm{Fr}_m U_+ \longrightarrow \Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge \mathrm{Fr}_m U_+)^f\big)$$

is a weak equivalence of motivic symmetric spectra.

Proof Let $\mathcal{QP}_S$ denote smooth quasi-projective S-schemes, and set

$$I := \{\mathrm{M}\mathbb{Z}\wedge \mathrm{Fr}_m\big(U_+\times(\partial\Delta^n \hookrightarrow \Delta^n)_+\big)\}^{m,n\geq 0}_{U\in\mathcal{QP}_S}.$$

Suppose E is an $\mathrm{M}\mathbb{Z}$-module. Applying the small object argument to the set I and $*\longrightarrow E$, we obtain an I-cell complex E' and a map $E'\longrightarrow E$. Since every smooth S-scheme admits an open covering by smooth quasi-projective S-schemes, the latter map is a weak equivalence [23, Lemma 1.1.16]. Hence, since I-cell complexes are cofibrant and Φ is a left Quillen functor, it suffices to prove the unit is a weak equivalence for I-cell complexes.

The functors Ψ and Φ preserve filtered colimits, and weak equivalences of $\mathrm{M}\mathbb{Z}$-modules are closed under filtered colimits. Thus, transfinite induction shows it suffices to consider I-cell complexes F obtained by attaching a single cell to E. This case follows from elementary considerations since (Φ,Ψ) is a Quillen adjoint pair of stable model structures. In effect, suppose there is a commutative diagram:

$$\begin{tikzcd}
E \arrow[r, rightarrowtail] \arrow[d] & F \arrow[r] \arrow[d] & \mathrm{M}\mathbb{Z}\wedge \mathrm{Fr}_m(U_+\wedge \Delta^n/\partial\Delta^n) \arrow[d] \\
\Psi\big(\Phi(E)^f\big) \arrow[r] & \Psi\big(\Phi(F)^f\big) \arrow[r] & \Psi\Big(\Phi\big(\mathrm{M}\mathbb{Z}\wedge \mathrm{Fr}_m(U_+\wedge \Delta^n/\partial\Delta^n)\big)^f\Big)
\end{tikzcd} \tag{19}$$

The upper row of diagram (19) is a cofiber sequence of cofibrant $\mathrm{M}\mathbb{Z}$-modules. By the remark above, the lower row is a homotopy cofiber sequence. It follows that if two of the vertical maps in the diagram above are weak equivalences, then so is the third. Using the cofiber sequences $\partial\Delta^n \rightarrowtail \Delta^n \longrightarrow \Delta^n/\partial\Delta^n$, one reduces to the case $n=0$. This concludes the proof. □

The previous reduction step is somewhat standard. This is not the case for our next reduction, which shows that it suffices to consider free modules of motivic symmetric suspension spectra rather than shifted suspension spectra. Moreover, the fibrant replacement turns out to be irrelevant for suspension spectra (although this is possibly not the case for shifted suspension spectra). We identify a motivic space with its motivic symmetric suspension spectrum, and an object of Sm with its representable motivic space. In the proof of the next result we use techniques employed in the proof of Theorem 2.26.

Lemma 3.3 *The pair (Φ,Ψ) is a Quillen equivalence if and only if for every smooth quasi-projective S-scheme U, the unit map*

$$\mathrm{M}\mathbb{Z}\wedge U_+ \longrightarrow \Psi\Phi(\mathrm{M}\mathbb{Z}\wedge U_+)$$

is a weak equivalence of motivic symmetric spectra.

Proof Suppose that E and F are fibrant and cofibrant motivic symmetric spectra, respectively, and there exists a weak equivalence $F \xrightarrow{\sim} \mathbf{M}(T,E)$. Then there is a commutative diagram with horizontal weak equivalences:

$$\begin{array}{ccc} \mathrm{M}\mathbb{Z}\wedge F & \xrightarrow{\sim} & \mathbf{M}\big(T,(\mathrm{M}\mathbb{Z}\wedge E)^f\big) \\ {\scriptstyle \mathrm{unit}}\downarrow & & \downarrow{\scriptstyle \mathbf{M}(T,\mathrm{unit})} \\ \Psi\Phi(\mathrm{M}\mathbb{Z}\wedge F) & & \mathbf{M}\big(T,\Psi\Phi(\mathrm{M}\mathbb{Z}\wedge E)^f\big) \\ \downarrow & & \downarrow \\ \Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge F)^f\big) & \xrightarrow{\sim} & \mathbf{M}\big(T,\Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge E)^f\big)^f\big) \end{array} \tag{20}$$

This relies on the natural transformations $\mathrm{M}\mathbb{Z}\wedge\mathbf{M}(T,E) \longrightarrow \mathbf{M}(T,\mathrm{M}\mathbb{Z}\wedge E)$, $\Phi\big(\mathbf{M}(T,E)\big) \longrightarrow \mathbf{M}\big(T,\Phi(E)\big)$ and $\Psi\big(\mathbf{M}(T,E)\big) \xrightarrow{\cong} \mathbf{M}\big(T,\Psi(E)\big)$ which are induced by the monoidality of $\mathrm{M}\mathbb{Z}\wedge -$ and Φ; cp. Lemma 2.1 and Lemma 2.38. The diagram (20) implies

$$\mathrm{M}\mathbb{Z}\wedge F \xrightarrow{\mathrm{unit}} \Psi\Phi(\mathrm{M}\mathbb{Z}\wedge F) \longrightarrow \Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge F)^f\big)$$

is a weak equivalence if and only if

$$\mathrm{M}\mathbb{Z}\wedge E \xrightarrow{\mathrm{unit}} \Psi\Phi(\mathrm{M}\mathbb{Z}\wedge E) \longrightarrow \Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge E)^f\big)$$

is a weak equivalence.

We may apply this to the map $\mathrm{Fr}_m U_+ \longrightarrow \mathbf{M}(T^{\wedge m}, U_+^f)$ which is adjoint to the canonical weak equivalence

$$\mathrm{Fr}_m U_+ \wedge T^{\wedge m} \cong \mathrm{Fr}_m(U_+ \wedge T^{\wedge m}) \longrightarrow U_+ \overset{\sim}{\rightarrowtail} U_+^f$$

of motivic symmetric spectra. It follows that (Φ,Ψ) is a Quillen equivalence if and only if the unit map $\mathrm{M}\mathbb{Z}\wedge U_+ \longrightarrow \Psi\big((\Phi\mathrm{M}\mathbb{Z}\wedge U_+)^f\big)$ is a weak equivalence for every smooth quasi-projective U over S.

It remains to prove the fibrant replacement

$$\Phi(\mathrm{M}\mathbb{Z}\wedge U_+) \overset{\sim}{\rightarrowtail} \big(\Phi(\mathrm{M}\mathbb{Z}\wedge U_+)\big)^f$$

induces a weak equivalence

$$\Psi\Big(\Phi(\mathrm{M}\mathbb{Z}\wedge U_+) \overset{\sim}{\rightarrowtail} \big(\Phi(\mathrm{M}\mathbb{Z}\wedge U_+)\big)^f\Big)$$

of motivic symmetric spectra.

To proceed, we shall shift attention from the fibrant replacement $(-)^f$ to the enriched fibrant replacement functor $R\colon \mathbf{M}^{\mathrm{tr}} \longrightarrow \mathbf{M}^{\mathrm{tr}}$ we introduced in [6, Section 3.3.2]. The latter functor extends degreewise to a levelwise fibrant replacement functor $R\colon \mathbf{MSS}^{\mathrm{tr}} \longrightarrow \mathbf{MSS}^{\mathrm{tr}}$. Define $\mathbb{T} := \mathbf{MSS}^{\mathrm{tr}}(\mathrm{Fr}_1 T^{\mathrm{tr}}, -)$. The map $\mathrm{Fr}_1 T^{\mathrm{tr}} \longrightarrow \mathrm{Fr}_0 S^{\mathrm{tr}}_+$ adjoint to $\mathrm{id}_{T^{\mathrm{tr}}}$ induces a natural transformation $\mathrm{Id}_{\mathbf{MSS}^{\mathrm{tr}}} \longrightarrow \mathbb{T}$. We may iterate and define $\mathbb{T}^\infty(E) := \mathrm{colim}_m\, \mathbb{T}^m(E)$.

For every cofibrant motivic space A we claim the natural map

$$A^{\mathrm{tr}} = \mathrm{Fr}_0 A^{\mathrm{tr}} \longrightarrow \mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)$$

is a fibrant replacement of motivic symmetric spectra with transfers, and

$$\mathscr{U}_{\Sigma}\Big(\mathrm{Fr}_0 A^{\mathrm{tr}} \longrightarrow \mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)\Big)$$

is a weak equivalence. Since $\Phi(\mathrm{M}\mathbb{Z} \wedge U_+)$ is naturally isomorphic to $\mathrm{Fr}_0 U^{\mathrm{tr}}$ this would in turn prove the lemma.

Note that

$$\mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)_n = \mathrm{colim}_m\, \mathbf{M}^{\mathrm{tr}}\big((T^{\wedge m})^{\mathrm{tr}}, R(A \wedge T^{\wedge m})^{\mathrm{tr}}\big).$$

Therefore, the adjoint structure maps

$$\mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)_n \longrightarrow \mathbf{M}^{\mathrm{tr}}\Big(T^{\mathrm{tr}}, \mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)_{n+1}\Big)$$

are weak equivalences. Theorem 2.9 shows $\mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)$ is stably fibrant. The computation subsequent to the proof of [15, Theorem 10.1] shows

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{MSS}^{\mathrm{tr}})}\big(\mathrm{Fr}_0 B, \mathrm{Fr}_0 A^{\mathrm{tr}} \longrightarrow \mathbb{T}^{\infty} R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)$$

is an isomorphism for all cofibrant motivic spaces with transfers B. Now since $\mathbf{M}^{\mathrm{tr}}(T, -)$ is a Quillen equivalence we deduce using the weak equivalence

$$\mathrm{Fr}_n B \xrightarrow{\sim} \mathbf{M}^{\mathrm{tr}}\big(T^n, (\mathrm{Fr}_0 B)^f\big)$$

that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{MSS}^{\mathrm{tr}})}\big(\mathrm{Fr}_n B, \mathrm{Fr}_0 A^{\mathrm{tr}} \longrightarrow \mathbb{T}^{\infty} R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)$$

for all n and all cofibrant B. Since the objects $\mathrm{Fr}_n B$ generate the homotopy category $\mathrm{Ho}(\mathbf{MSS}^{\mathrm{tr}})$, it follows that $\mathrm{Fr}_0 A^{\mathrm{tr}} \longrightarrow \mathbb{T}^{\infty}\big(R(\mathrm{Fr}_0 A^{\mathrm{tr}})\big)$ is a weak equivalence. This shows we are dealing with a fibrant replacement.

Finally, to prove the weak equivalence of motivic symmetric spectra, it suffices to establish that the map is a weak equivalence of motivic spectra when forgetting transfers and symmetric group actions, see Theorem 2.16. Note that $\mathscr{U}_{\Sigma}\mathbf{MSS}^{\mathrm{tr}}(\mathrm{Fr}_1 T^{\mathrm{tr}}, -) \cong \mathbf{MSS}(\mathrm{Fr}_1 T, \mathscr{U}_{\Sigma} -)$ and $\mathscr{U}_{\Sigma}$ detects and preserves levelwise weak equivalences. The result follows since for any levelwise fibrant motivic spectrum E, the maps $E_n \longrightarrow \mathrm{colim}_m\, \mathbf{M}(T^{\wedge m}, E_{n+m})$ yield a weak equivalence of motivic spectra [15, Section 4]. □

To summarize, the results in this section show that Theorem 1.1 follows provided the unit map

$$\mathrm{M}\mathbb{Z} \wedge U_+ \longrightarrow \Psi\Phi(\mathrm{M}\mathbb{Z} \wedge U_+)$$

is a weak equivalence for all smooth quasi-projective S-schemes U.

4 Motivic functors and duality

Let $\mathbf{MF}_S$ be the category of motivic functors, i.e. $\mathbf{M}_S$-enriched functors from the category $\mathbf{fM}_S$ of finitely presentable motivic spaces to the category $\mathbf{M}_S$ of motivic spaces over S. When no confusion is likely to occur, we suppress the base scheme in the notation. The category $\mathbf{MF}$ has a ready-made highly structured smash product and acquires several monoidal model structures [7]. This section recalls the assembly map which we explore in the context of duality, and introduces some model structures on motivic functors which are not treated in our joint papers with Dundas [6], [7].

4.1 The assembly map

Since $\mathbf{MF}$ is a category of enriched functors with bicomplete codomain, it is bicomplete and enriched over $\mathbf{M}$. We denote the motivic space of maps in $\mathbf{MF}$ from X to Y by $\mathbf{M}_{\mathbf{MF}}(X,Y)$.

Let A be a finitely presentable motivic space. Denote the corresponding representable motivic functor by

$$\mathbf{M}(A,-)\colon \mathbf{fM} \longrightarrow \mathbf{M}$$
$$B \longmapsto \mathbf{M}(A,B).$$

The enriched Yoneda lemma holds in $\mathbf{MF}$, and every motivic functor can be expressed in a canonical way as a colimit of representable motivic functors.

Theorem 4.1 *The category* $\mathbf{MF}$ *is closed symmetric monoidal with unit the inclusion* $\mathbb{I}\colon \mathbf{fM} \hookrightarrow \mathbf{M}$.

Denote the monoidal product of the motivic functors X and Y by $X \wedge Y$. We sketch the basic idea behind the proof of Theorem 4.1, which is due to Day [2]. Using colimits, it suffices to describe the monoidal product of representable motivic functors

$$\mathbf{M}(A,-) \wedge \mathbf{M}(B,-) := \mathbf{M}(A \wedge B,-).$$

Internal hom objects of motivic functors are defined by

$$\mathbf{MF}(X,Y)(A) := \mathbf{M}_{\mathbf{MF}}\big(X, Y(- \wedge A)\big).$$

A special feature of motivic functors, which makes the monoidal product more transparent, is that motivic functors can be composed. Every motivic functor $X\colon \mathbf{fM} \longrightarrow \mathbf{M}$ can be extended – via enriched left Kan extension along the full inclusion $\mathbb{I}\colon \mathbf{fM} \hookrightarrow \mathbf{M}$ – to an $\mathbf{M}$-functor $\mathbb{I}_*X\colon \mathbf{M} \longrightarrow \mathbf{M}$ such that $\mathbb{I}_*X \circ \mathbb{I} = X$. The composition of motivic functors X and Y is defined by

$$X \circ Y := \mathbb{I}_*X \circ Y.$$

There is a natural assembly map $X \wedge Y \longrightarrow X \circ Y$. It is an isomorphism when Y is representable [6, Corollary 2.8]. If both X and Y are representable, then the assembly map is the natural adjointness isomorphism

$$\mathbf{M}(A,-) \wedge \mathbf{M}(B,-) = \mathbf{M}(A \wedge B,-) \cong \mathbf{M}\big(A, \mathbf{M}(B,-)\big).$$

Example 4.2 The assembly map for motivic cohomology induced by

$$\mathbf{M} \xrightarrow{\mathbb{Z}^{\mathrm{tr}}} \mathbf{M}^{\mathrm{tr}} \xrightarrow{\mathscr{U}} \mathbf{M}$$

is described in Section 2.4 (15).

The assembly maps show there exists a lax symmetric monoidal functor $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathbf{MSS}$ defined by

$$X \longmapsto \mathrm{ev}(X) = \big(X(S_+), X(T), X(T^{\wedge 2}), \ldots\big).$$

We denote by $L\mathrm{ev}$ the strict symmetric monoidal left adjoint of the evaluation functor [6, Section 2.6].

4.2 Model structures on motivic functors

Definition 4.3 Let $\mathbf{cM}$ denote the full subcategory of $\mathbf{M}$ of cofibrant finitely presentable motivic spaces. A map $f\colon X \longrightarrow Y$ of motivic functors is a $\mathbf{c}$-wise weak equivalence or a $\mathbf{c}$-wise fibration if $f(A)\colon X(A) \longrightarrow Y(A)$ is a motivic weak equivalence or a motivic fibration for all A in $\mathbf{cM}$. Maps with the left lifting property with respect to $\mathbf{c}$-wise acyclic fibrations are called $\mathbf{c}$-cofibrations.

Techniques in [6] show that these classes define a monoidal model on $\mathbf{MF}$. The homotopy functor and stable models arise as localizations of this model. To describe these, let $R\colon \mathbf{M} \longrightarrow \mathbf{M}$ denote the enriched fibrant replacement functor introduced in [6, Section 3.3.2]. It is an enriched functor equipped with a natural weak equivalence $A \longrightarrow RA$ such that RA is fibrant for every motivic space A. We will not make a notational distinction between R and its restriction to subcategories of $\mathbf{M}$. In addition, we use a $\mathbf{c}$-cofibrant replacement functor $(-)^{\mathbf{c}}\colon \mathbf{MF} \longrightarrow \mathbf{MF}$.

Definition 4.4 A map $f\colon X \longrightarrow Y$ is a $\mathbf{c}$hf-equivalence if $f^{\mathbf{c}} \circ R$ is a $\mathbf{c}$-wise weak equivalence. A $\mathbf{c}$-wise fibration $X \longrightarrow Y$ is a $\mathbf{c}$hf-fibration if for every motivic weak equivalence $A \longrightarrow B$ of cofibrant finitely presentable motivic spaces, the diagram

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & X(B) \arrow[d] \\ Y(A) \arrow[r] & Y(B) \end{tikzcd}$$

is a homotopy pullback square of motivic spaces.

As a consequence of the methods developed in [6, Section 5], the classes of $\mathbf{c}$hf-equivalences, $\mathbf{c}$hf-fibrations and $\mathbf{c}$-cofibrations form a monoidal model. Note that $\mathbf{c}$hf-fibrant motivic functors preserve weak equivalences between cofibrant motivic spaces.

Let $\mathbb{T}^{\infty}(X)$ denote the colimit of the sequence

$$X \longrightarrow \mathbf{M}\big(T, X(- \wedge T)\big) \longrightarrow \mathbf{M}\big(T^{\wedge 2}, X(- \wedge T^{\wedge 2})\big) \longrightarrow \cdots .$$

Definition 4.5 A map $f\colon X \longrightarrow Y$ is a $\mathbf{c}$-stable equivalence if the map $\mathbb{T}^\infty(R \circ f^{\mathbf{c}} \circ R)$ is a $\mathbf{c}$-wise weak equivalence. A $\mathbf{c}$hf-fibration $f\colon X \longrightarrow Y$ is a $\mathbf{c}$-stable fibration if for every cofibrant and finitely presentable motivic space A, the diagram

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & \mathbf{M}\big(T, X(A \wedge T)\big) \arrow[d] \\ X(A) \arrow[r] & \mathbf{M}\big(T, X(A \wedge T)\big) \end{tikzcd}$$

is a homotopy pullback square of motivic spaces.

We obtain the following theorem using results in [6, Section 6].

Theorem 4.6 *The classes of* $\mathbf{c}$*-stable equivalences,* $\mathbf{c}$*-cofibrations and* $\mathbf{c}$*-stable fibrations form a monoidal model structure.*

In Theorem 4.6, we may replace $\mathbf{cM}$ by any of the following categories:

1. $\mathbf{fM}$ - the finitely presentable motivic spaces. This gives the stable model introduced in [7, Section 3.5].
2. $\mathbf{wM}$ - the full subcategory of finitely presentable motivic spaces which are weakly equivalent to some object in $\mathbf{cM}$.
3. $\mathbf{tM}$ - the full subcategory of finitely presentable motivic spaces which are weakly equivalent to some smash power of T. This gives the spherewise model introduced in [7, Section 3.6].
4. $\mathbf{ctM}$ - the intersection of $\mathbf{tM}$ and $\mathbf{cM}$.

The stable models obtained using $\mathbf{wM}$ and $\mathbf{cM}$ are Quillen equivalent, and likewise for $\mathbf{tM}$ and $\mathbf{ctM}$. Recall that the adjoint functor pair $(L\mathrm{ev}, \mathrm{ev})$ induces strict symmetric monoidal Quillen equivalences between the latter models and Jardine's model for motivic symmetric spectra [7, Theorem 3.32].

Remark 4.7 For the work in [7], it would be of interest to compare $\mathbf{fM}$ and $\mathbf{wM}$. Corollary 5.3 implies that $\mathbf{cM}$ and $\mathbf{ctM}$ furnish Quillen equivalent model structures for fields of characteristic zero.

4.3 Duality

We choose the spherewise model on motivic functors as our model for the motivic stable homotopy category $\mathrm{SH}(S)$. The smash product in $\mathbf{MF}$ induces a monoidal product $\star$ in $\mathrm{SH}(S)$. In the motivic stable homotopy category, we denote the unit by $\mathbb{I}$ and internal hom objects by $[-,-]$.

Definition 4.8 A motivic functor X is dualizable if the canonical map

$$[X, \mathbb{I}] \star X \longrightarrow [X, X]$$

is an isomorphism. The object $DX := [X, \mathbb{I}]$ is the dual of X.

An important impetus in what follows is a duality result formulated by Voevodsky in [35].

Theorem 4.9 *Suppose k is a field of characteristic zero and U is a smooth quasi-projective scheme in* Sm_k*. Then U_+ is dualizable in the motivic stable homotopy category* $\mathrm{SH}(k)$*.*

Proof In [17, Appendix] it is shown that any smooth projective variety over a field k is dualizable in the motivic stable homotopy category of k. Suppose now that U is an open subscheme of a smooth projective scheme X and the reduced closed complement $Z = X \smallsetminus U$ is smooth. Denote by $N(i) \longrightarrow Z$ the normal bundle of the closed embedding $i\colon Z \hookrightarrow X$ with zero section $z\colon Z \hookrightarrow N(i)$. The homotopy purity theorem [23, Theorem 3.2.23] furnishes a cofiber sequence

$$U_+ \longrightarrow X_+ \longrightarrow N(i)/N(i) \smallsetminus z(Z) \simeq \mathbb{P}(N(i) \oplus \mathbb{A}^1_Z)/\mathbb{P}N(i),$$

which induces a cofibration sequence of suspension spectra. It follows that U_+ is dualizable in $\mathrm{SH}(k)$ since if two out of three objects in a cofiber sequence are dualizable, then so is the third.

Suppose U is an open subscheme of a smooth projective scheme X and the reduced closed complement $i\colon Z \hookrightarrow X$ is a divisor with strict normal crossings, i.e. the irreducible components $Z_1, \ldots, Z_m$ of Z are smooth and intersect transversally. We obtain an elementary distinguished square:

$$\begin{tikzcd} X \smallsetminus Z \arrow[r] \arrow[d] & X \smallsetminus Z_1 \arrow[d] \\ X \smallsetminus \bigcup_{j>1}^{m} Z_j \arrow[r] & X \smallsetminus \left(Z_1 \cap \bigcup_{j>1}^{m} Z_j\right) \end{tikzcd}$$

Thus using induction on the number of connected components and the special case proved in the previous paragraph implies that U_+ is dualizable.

In the general case, choose an open embedding $U \hookrightarrow X$ in some reduced projective scheme. The reduced closed complement $i\colon Z = X \smallsetminus U \hookrightarrow X$ has then codimension at least two. By Hironaka's theorem on resolutions of singularities [11] there exists a pullback square

$$\begin{tikzcd} \widetilde{Z} \arrow[r, hook, "\widetilde{i}"] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

where the projective map p induces an isomorphism $\widetilde{X} \smallsetminus \widetilde{Z} \cong X \smallsetminus Z = U$, $\widetilde{X}$ is smooth over k, and moreover $\widetilde{i}$ is a divisor with strict normal crossings. This implies the claim that U_+ is dualizable. $\square$

If A is a motivic space, the evaluation of the motivic functor $- \wedge A$ is the suspension spectrum $\Sigma_T^\infty A$ and there is an isomorphism $L\mathrm{ev}\Sigma_T^\infty A \cong - \wedge A$. If A is cofibrant, then $\mathbf{MF}(- \wedge A, \mathbb{I}^f)$ is a model for the dual of the cofibrant motivic functor $- \wedge A$. Here $\mathbb{I}^f$ is a fibrant replacement of $\mathbb{I}$ in the spherewise model. Since $\mathbb{I}$ is a homotopy functor, it follows that $\mathbb{T}^\infty(R \circ \mathbb{I})$ is a fibrant replacement for $\mathbb{I}$. In addition, when A is a finitely presentable motivic space we have

$$\mathbf{MF}\big(- \wedge A, \mathbb{T}^\infty(R \circ \mathbb{I})\big) \cong \mathbf{M}\big(A, \mathbb{T}^\infty(R \circ \mathbb{I})\big) \cong \mathbb{T}^\infty\big(\mathbf{M}(A, R \circ \mathbb{I})\big).$$

Up to weak equivalence, it follows that whenever A is a cofibrant finitely presentable motivic space, then $\mathbf{M}(A, -)$ is a dual of $- \wedge A$.

In the next lemma, we prove a slightly more general statement.

Lemma 4.10 *Suppose A and B are finitely presentable motivic spaces and A is cofibrant. Then $\mathbf{M}\big(A, R \circ (- \wedge B)\big)$ and $[- \wedge A, - \wedge B]$ are isomorphic in the motivic stable homotopy category* $\mathrm{SH}(S)$.

Proof Note that $\mathbf{MF}\big(- \wedge A, (- \wedge B)^f\big)$ is a model for $[- \wedge A, - \wedge B]$ and we may choose $(- \wedge B)^f = \mathbb{T}^\infty\big(R \circ (- \wedge B)\big)$. Hence

$$\mathbf{MF}\big(-\wedge A, (-\wedge B)^f\big) \cong \mathbf{M}\Big(A, \mathbb{T}^\infty\big(R\circ(-\wedge B)\big)\Big) \cong \mathbb{T}^\infty\Big(\mathbf{M}\big(A, R\circ(-\wedge B)\big)\Big).$$

The canonical map $\mathbf{M}\big(A, R \circ (- \wedge B)\big) \longrightarrow \mathbb{T}^\infty\Big(\mathbf{M}\big(A, R \circ (- \wedge B)\big)\Big)$ is a weak equivalence; this completes the proof. $\square$

5 End of the proof of Theorem 1.1

Our general goal is now to decide under which conditions the assembly map $X \wedge Y \longrightarrow X \circ Y$ is a weak equivalence. We start with a special case.

Lemma 5.1 *Suppose A is a cofibrant finitely presentable motivic space and B is a finitely presentable motivic space such that $- \wedge B$ is dualizable in* $\mathrm{SH}(S)$. *Then the evaluation of the assembly map*

$$\mathbf{M}\big(A, R(-)\big) \wedge B \longrightarrow \mathbf{M}\big(A, R \circ (- \wedge B)\big)$$

is a weak equivalence between motivic symmetric spectra.

Proof By Lemma 4.10, the map $\mathbf{M}\big(A, R(-)\big) \wedge B \longrightarrow \mathbf{M}\big(A, R \circ (- \wedge B)\big)$ descends to the canonical map

$$[- \wedge A, \mathbb{I}] \star (- \wedge B) \longrightarrow [- \wedge A, \mathbb{I} \star (- \wedge B)] \cong [- \wedge A, - \wedge B].$$

Since $- \wedge B$ is dualizable, this map is an isomorphism in the motivic stable homotopy category. $\square$

Proposition 5.2 *Suppose X is a* $\mathbf{c}$*-cofibrant motivic functor and B is a cofibrant finitely presentable motivic space such that $- \wedge B$ is dualizable in* $\mathrm{SH}(S)$*. Then the evaluation of the assembly map*

$$(X \circ R) \wedge B \longrightarrow X \circ R \circ (- \wedge B)$$

is a weak equivalence between motivic symmetric spectra.

Proof The proof proceeds by transfinite induction on the number of cells in X. The limit ordinal case follows since all the constructions involved are preserved by filtered colimits. The successor ordinal case is proven as follows: Suppose Y is the pushout of the upper row in the diagram

$$\begin{tikzcd}
\mathbf{M}(A,-) \wedge ti \arrow[d] & \mathbf{M}(A,-) \wedge si \arrow[l, tail, "\mathbf{M}(A,-)\wedge i"'] \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbf{M}\big(A,R(-)\big) \wedge ti & \mathbf{M}\big(A,R(-)\big) \wedge si \arrow[l, "\mathbf{M}(A,R(-))\wedge i"'] \arrow[r] & X \circ R
\end{tikzcd}$$

where $i\colon si \rightarrowtail ti$ is a cell, i.e. a generating cofibration in $\mathbf{MF}$. Then $Y \circ R$ is the pushout of the lower row since pushouts in $\mathbf{MF}$ are formed pointwise. The vertical maps are $\mathbf{c}$hf-equivalences. Although it is not a cofibration, the map $\mathbf{M}\big(A,R(-)\big) \wedge i$ is a pointwise monomorphism. The pushout of the lower row is a homotopy pushout square since pointwise weak equivalences are closed under pushouts along pointwise monomorphisms.

Next we consider the diagram of assembly maps:

$$\begin{tikzcd}
\mathbf{M}\big(A,R(-)\big) \wedge ti \wedge B \arrow[d] & \mathbf{M}\big(A,R(-)\big) \wedge si \wedge B \arrow[l, "\mathbf{M}(A,R(-))\wedge i \wedge B"'] \arrow[r] \arrow[d] & (X \circ R) \wedge B \arrow[d] \\
\mathbf{M}\big(A,R(- \wedge B)\big) \wedge ti & \mathbf{M}\big(A,R(- \wedge B)\big) \wedge si \arrow[l, "\mathbf{M}(A,R(-\wedge B))\wedge i"'] \arrow[r] & (X \circ R) \circ (- \wedge B)
\end{tikzcd}$$

The induced map of pushouts is the assembly map

$$(Y \circ R) \wedge B \longrightarrow Y \circ R \circ (- \wedge B).$$

Using Lemma 5.1 and the induction hypothesis (applied to the right hand side vertical map), the evaluations of the vertical maps are weak equivalences between motivic symmetric spectra. But, as noted above, both pushouts are homotopy pushouts squares. Hence the evaluation of the induced map of pushouts is a weak equivalence between motivic symmetric spectra. □

Corollary 5.3 *Suppose X is a motivic functor and B is a cofibrant finitely presentable motivic space such that $- \wedge B$ is dualizable in* $\mathrm{SH}(S)$*. When X preserves weak equivalences of cofibrant finitely presentable motivic spaces, then the evaluation of the assembly map*

$$X \wedge B \longrightarrow X \circ (- \wedge B)$$

is a weak equivalence between motivic symmetric spectra.

Proof This follows from Proposition 5.2 using a **c**-cofibrant replacement. Since B is a cofibrant finitely presentable motivic space, we get that $- \wedge B$ and $\circ(- \wedge B)$ preserve **c**-wise weak equivalences. □

We claim the assumption on the motivic functor in Corollary 5.3 holds for motivic cohomology

$$\mathrm{M}\mathbb{Z} : \mathbf{cM} \hookrightarrow \mathbf{M} \xrightarrow{\mathbb{Z}^{\mathrm{tr}}} \mathbf{M}^{\mathrm{tr}} \xrightarrow{\mathscr{U}} \mathbf{M}.$$

The forgetful functor $\mathscr{U}$ preserves both weak equivalences and fibrations, by Lemma 2.7. Hence its left adjoint $\mathbb{Z}^{\mathrm{tr}}$ preserves weak equivalences between cofibrant motivic spaces.

Next we compare the assembly map in Corollary 5.3 with the unit of the adjunction relating $\mathrm{M}\mathbb{Z} - \mathbf{mod}$ to $\mathbf{MSS}^{\mathrm{tr}}$.

Lemma 5.4 *The evaluation of the assembly map*

$$\mathrm{M}\mathbb{Z} \wedge B \longrightarrow \mathrm{M}\mathbb{Z} \circ (- \wedge B)$$

coincides with the unit

$$\mathrm{M}\mathbb{Z} \wedge B \longrightarrow \Psi\Phi(\mathrm{M}\mathbb{Z} \wedge B).$$

Proof It suffices to prove the maps coincide as maps between motivic spectra. This follows from the construction of Φ in Section 2.4. □

We are ready to prove:

Theorem 5.5 *Suppose k is a field of characteristic zero. Then there is a strict symmetric monoidal Quillen equivalence:*

$$\Phi \colon \mathrm{M}\mathbb{Z} - \mathbf{mod} \rightleftarrows \mathbf{MSS}^{\mathrm{tr}} : \Psi$$

Proof Use Lemma 3.3, Theorem 4.9, Corollary 5.3, and Lemma 5.4. □

Remark 5.6 Theorem 5.5 holds more generally for base schemes S for which $\mathrm{SH}(S)$ is generated by dualizable motivic spectra. It is not known whether all smooth quasi-projective schemes over fields of positive characteristic are dualizable.

Remark 5.7 Combining Theorems 2.9 and 5.5 implies Theorem 1.1 stated in the introduction.

6 Tate motives

We let **DM** denote the homotopy category of symmetric $\mathbb{G}_m^{\mathrm{tr}}[1]$-spectra of non-connective chain complexes with transfers introduced in Theorem 2.9. Let $\mathcal{P}$ denote the essentially small category of smooth projective S-schemes. Then every object U in $\mathcal{P}$ is dualizable in $\mathrm{SH}(S)$ according to [17, Appendix]. Thus, we have weak equivalences

$$\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m U_+ \xrightarrow{\sim} \Psi\Phi(\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m U_+). \tag{21}$$

Theorem 6.1 *The localizing subcategory of* $\mathrm{Ho}(\mathrm{M}\mathbb{Z} - \mathbf{mod})$ *generated by*

$$\{\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m U_+\}_{U \in \mathcal{P}}^{m \geq 0}$$

is equivalent to the localizing subcategory of **DM** *generated by*

$$\{\mathrm{Fr}_m U^{\mathrm{tr}}\}_{U \in \mathcal{P}}^{m \geq 0}.$$

Proof Let $\mathcal{C}$ be the localizing subcategory of $\mathrm{Ho}(\mathrm{M}\mathbb{Z} - \mathbf{mod})$ generated by the $\mathrm{M}\mathbb{Z}$-modules $\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m U_+$. Let $\mathcal{D}$ be the localizing subcategory of **DM** generated by the motives $\mathrm{Fr}_m U^{\mathrm{tr}}$. Theorem 2.9 (3) shows that $\mathcal{D}$ is equivalent to the corresponding localizing subcategory $\mathcal{D}'$ of $\mathbf{MSS}^{\mathrm{tr}}$.

The total left derived functor of Φ maps $\mathcal{C}$ to $\mathcal{D}'$. Now since (21) is a weak equivalence, the total right derived functor of Ψ maps $\mathcal{D}'$ to $\mathcal{C}$ and it detects isomorphisms by Lemma 2.37. Moreover, the unit for the derived adjunction restricts to an isomorphism on generators of $\mathcal{C}$. □

The category **MTM** of mixed Tate motives is the localizing subcategory of **DM** generated by

$$\mathbb{Z}(q) := \begin{cases} {\mathbb{G}_m^{\mathrm{tr}}}^{\otimes q} & \text{if } q \geq 0, \\ \mathrm{Fr}_{-q} S_+^{\mathrm{tr}} & \text{if } q < 0. \end{cases}$$

Let $\mathcal{T}$ denote the subset of projective spaces $\{\mathbb{P}^0, \mathbb{P}^1, \ldots\}$ contained in $\mathcal{P}$. Recall there is a homotopy cofiber sequence

$$\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n \longrightarrow S_s^n \wedge \mathbb{G}_m^n. \tag{22}$$

The localizing subcategory of **DM** generated by $\{\mathrm{Fr}_m(\mathbb{P}^n)^{\mathrm{tr}}\}^{m,n \geq 0}$ coincides with the category of mixed Tate motives by (22). This observation also follows from the projective bundle formula for the motive of $\mathbb{P}^n$.

Define the category of spherical $\mathrm{M}\mathbb{Z}$-modules as the localizing subcategory of $\mathrm{Ho}(\mathrm{M}\mathbb{Z} - \mathbf{mod})$ generated by the free $\mathrm{M}\mathbb{Z}$-modules

$$\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m \mathbb{P}_+^n$$

for $m, n \geq 0$.

Corollary 6.2 *The category* **MTM** *of mixed Tate motives is equivalent to the category of spherical* $\mathrm{M}\mathbb{Z}$*-modules.*

Proof This follows from the proof of Theorem 6.1 since the total derived adjoint pair of (Φ, Ψ) restricts appropriately. □

Remark 6.3 Lemma 2.38 implies that the equivalences in Theorem 6.1 and Corollary 6.2 are strict monoidal.

We refer to [1] and [20] for alternate interpretations of mixed Tate motives. Note that the above descriptions hold for base schemes such as number rings and arithmetic schemes in general. We shall give another module-theoretic description of mixed Tate motives **MTM** which relies on Morita theory for stable model structures.

The category **MSS** of motivic symmetric spectra is enriched over ordinary symmetric spectra **SS**. Let $\mathbf{SS}_{\mathbf{MSS}}(E, F)$ denote the symmetric spectrum of maps between E and F. We may choose a set $\mathcal{T}^f$ of fibrant replacements for the $\mathrm{M}\mathbb{Z}$-modules

$$\{\mathrm{M}\mathbb{Z} \wedge \mathrm{Fr}_m \mathbb{P}^n_+\}^{m,n\geq 0}.$$

Denote by $\mathrm{End}(\mathcal{T}^f)$ the **SS**-category with objects motivic symmetric spectra E, F in $\mathcal{T}^f$ and with morphisms $\mathbf{SS}_{\mathbf{MSS}}(E, F)$. We shall view this category as a symmetric ring spectrum with many objects [29]. By [6, Theorem 4.4] there exists a pointwise model structure on the enriched category of modules over $\mathrm{End}(\mathcal{T}^f)$, i.e. the category of **SS**-functors

$$\mathrm{End}(\mathcal{T}^f) \longrightarrow \mathbf{SS}.$$

Theorem 6.4 *The category of mixed Tate motives* **MTM** *is equivalent to the pointwise homotopy category of* $\mathrm{End}(\mathcal{T}^f)$*-modules.*

Proof Follows from Corollary 6.2 and [29, Theorem 3.9.3]. □

Work of Dugger [4] associates to the additive model category of motivic symmetric spectra with transfers an endomorphism DGA with many objects. In the same paper it is demonstrated that, unlike endomorphism ring spectra, endomorphism DGA's are not Quillen invariant in general.

7 Rational coefficients

In this section, we shall extend Theorem 5.5 to perfect fields and coefficients in $\mathbb{Q}$-algebras. We use rational coefficients only for notational convenience. Throughout this section the base scheme is a perfect field k. The assumption on k implies every reduced k-scheme Z has a stratification

$$\emptyset = Z^{-1} \hookrightarrow Z^0 \hookrightarrow \cdots \hookrightarrow Z^{n-1} \hookrightarrow Z^n = Z, \tag{23}$$

where Z^i is a k-scheme of dimension i and $Z^i \smallsetminus Z^{i-1}$ is smooth over k for all $0 \leq i \leq n$. If d is a non-negative integer, let $\mathrm{SH}_d(k)$ denote the localizing subcategory of $\mathrm{SH}(k)$ containing all shifted suspension spectra $\mathrm{Fr}_n T^{\wedge m} \wedge U_+$ of T-suspensions of smooth connected k-schemes of dimension $\leq d$. We prove the next result using an argument analogous to the proof of [22, Lemma 3.3.7].

Lemma 7.1 *Let* $j\colon U \hookrightarrow X$ *be a non-empty open subset of a smooth connected* k*-scheme* X *of dimension* d*. Then the quotient* X/U *is contained in* $\mathrm{SH}_{d-1}(k)$*.*

Proof Let $i\colon Z \hookrightarrow X$ be the $n < d$ dimensional reduced closed complement of $U \hookrightarrow X$. If Z is smooth then homotopy purity [23, Theorem 3.2.23] yields a weak equivalence between X/U and the Thom space of the normal bundle of i. Induction on the number of elements in an open covering trivializing the normal bundle implies the suspension spectrum of the Thom space is contained in $\mathrm{SH}_{d-1}(k)$.

If X is not smooth, choose a stratification as in (23). There is an induced sequence of motivic spaces

$$* = X \smallsetminus Z^n/U \xrightarrow{s_n} X \smallsetminus Z^{n-1}/U \xrightarrow{s_{n-1}} \cdots \xrightarrow{s_1} X \smallsetminus Z^0/U \xrightarrow{s_0} X/U, \quad (24)$$

and homotopy cofiber sequences

$$X \smallsetminus Z^m/U \xrightarrow{s_m} X \smallsetminus Z^{m-1}/U \longrightarrow X \smallsetminus Z^{m-1}/X \smallsetminus Z^m.$$

Since the closed complement of $X \smallsetminus Z^m$ in $X \smallsetminus Z^{m-1}$ is the smooth k-scheme $Z^m \smallsetminus Z^{m-1}$, the argument above shows that $X \smallsetminus Z^{m-1}/X \smallsetminus Z^m$ is contained in $\mathrm{SH}_{d-1}(k)$. Thus, an inductive argument using (24) implies the result. □

The other ingredient in the proof of Theorem 7.3 is the following lemma.

Lemma 7.2 *Let $g\colon V \longrightarrow U$ be a finite étale map of degree d where U is a smooth connected k-scheme. In the homotopy category of $\mathrm{M}\mathbb{Z}$-modules over U there is map $g^\tau\colon \mathrm{M}\mathbb{Z}_U \wedge U_+ \longrightarrow \mathrm{M}\mathbb{Z}_U \wedge V_+$ such that the composition*

$$\mathrm{M}\mathbb{Z}_U \wedge U_+ \xrightarrow{g^\tau} \mathrm{M}\mathbb{Z}_U \wedge V_+ \xrightarrow{\mathrm{M}\mathbb{Z}_U \wedge g_+} \mathrm{M}\mathbb{Z}_U \wedge U_+$$

coincides with multiplication by d. It follows that the free module $\mathrm{M}\mathbb{Q}_k \wedge U_+$ is a retract of $\mathrm{M}\mathbb{Q}_k \wedge V_+$ in the homotopy category of $\mathrm{M}\mathbb{Q}$-modules over k.

Proof Associated to the graph $\Gamma(g) = V \hookrightarrow V \times_U U = V$ of g is its transpose $\Gamma(g)^\tau = V \hookrightarrow U \times_U V = V$ considered as a finite correspondence over U. A straightforward computation in Cor_U shows that

$$\Gamma(g) \circ \Gamma(g)^\tau = d \cdot [U] \in \mathrm{Cor}_U(U,U) \cong \mathbb{Z}.$$

The canonical additive functor

$$\gamma_U\colon \mathrm{Cor}_U \longrightarrow \mathbf{ChSS}_U^{\mathrm{tr}} \longrightarrow \mathbf{DM}_U$$

induces a homomorphism $\gamma_U'\colon \mathrm{Cor}_U(U,U) \longrightarrow \mathrm{Hom}_{\mathbf{DM}_U}(U,U)$. If U is a field, then this is an isomorphism by Voevodsky's cancellation theorem [34]. Since γ_U is compatible with pullback along the map $\mathrm{Spec}\big(F(U)\big) \longrightarrow U$ obtained from the function field of U, the map γ_U' is injective.

The map $g\colon V \longrightarrow U$ is projective. By Theorem 6.1, the transpose of $\Gamma(g)$ determines a map $g^\tau\colon \mathrm{M}\mathbb{Z}_U \wedge U_+ \longrightarrow \mathrm{M}\mathbb{Z}_U \wedge V_+$ in such a way that the composition $\mathrm{M}\mathbb{Z}_U \wedge g_+ \circ g^\tau$ coincides with multiplication by d. In particular, $\mathrm{M}\mathbb{Q}_U \wedge U_+$ is a retract of $\mathrm{M}\mathbb{Q}_U \wedge V_+$ in the homotopy category of $\mathrm{M}\mathbb{Q}$-modules over U. For any smooth map $f\colon S \longrightarrow S'$ of base schemes, there is a smooth base change functor $f_\sharp\colon \mathbf{MSS}_S \longrightarrow \mathbf{MSS}_{S'}$ left adjoint to the pullback f^*. The projection formula $f_\sharp\big(f^*(E) \wedge F\big) \cong E \wedge f_\sharp(F)$ in [24] shows $f_\sharp$ lifts to $\mathrm{M}\mathbb{Q}$-modules and preserves free modules. Via the total left derived of the smooth base change functor $\mathrm{M}\mathbb{Q}_U - \mathbf{mod} \longrightarrow \mathrm{M}\mathbb{Q}_k - \mathbf{mod}$, one gets that $\mathrm{M}\mathbb{Q}_k \wedge U_+$ is a retract of $\mathrm{M}\mathbb{Q}_k \wedge V_+$ considered in the homotopy category of $\mathrm{M}\mathbb{Q}$-modules over k. □

Theorem 7.3 *If k is a perfect field there is a strict symmetric monoidal Quillen equivalence:*

$$\Phi\colon \mathrm{M}\mathbb{Q} - \mathbf{mod} \rightleftarrows \mathbf{MSS}^{\mathrm{tr}}_{\mathbb{Q}} : \Psi$$

Proof By Theorem 2.9 and Lemma 2.27, it remains to prove that the unit map $\eta_X : \mathrm{M}\mathbb{Q} \wedge X_+ \longrightarrow \Psi\Phi(X_+)$ is a weak equivalence for any smooth quasi-projective connected k-scheme X. We proceed by induction on the dimension d of X. The case $d = 0$ follows from the results in Section 5 since any smooth zero-dimensional k-scheme is smooth projective.

Now assume that η_E is a weak equivalence for all $E \in \mathrm{SH}_{d-1}(k)$, and choose an open embedding $j : X \hookrightarrow Y$ into an integral projective k-scheme. If j is the identity, then η_E is a weak equivalence by our results in Section 5. In general, de Jong's theorem on alterations [19] shows there exists a connected smooth projective k-scheme Y' and a map $f : Y' \longrightarrow Y$ such that

- $X' := f^{-1}(X) \hookrightarrow Y'$ is the complement of a smooth normal crossings divisor, and
- for some non-empty open subset $U \hookrightarrow X$, f restricts to a finite étale map $g\colon V := f^{-1}(U) \longrightarrow U$.

Using homotopy purity [23, Theorem 3.2.23] and induction on the number of irreducible components we get that X' is dualizable in $\mathrm{SH}(k)$. So $\eta_{X'}$ is a weak equivalence. Lemma 7.1, the cofiber sequence $V_+ \hookrightarrow X'_+ \longrightarrow X'/V$ and induction imply that η_V is a weak equivalence. Since $\mathrm{M}\mathbb{Q} \wedge U_+$ is a retract of $\mathrm{M}\mathbb{Q} \wedge V_+$ by Lemma 7.2, it follows that η_U is a weak equivalence. Thus η_X is a weak equivalence by the induction hypothesis and the cofiber sequence $U_+ \hookrightarrow X_+ \longrightarrow X/U$. □

Acknowledgements We gratefully acknowledge valuable comments on a preprint version of this paper and [26]. The second author is pleased to acknowledge the hospitality and support of the Institute for Advanced Study.

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