Some Quivers Describing the Derived Categories of the Toric del Pezzos

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Following [Rud90, Bon90, BP94, KO95], the derived category over del Pezzo surfaces can be described in the following way. Assume there exists a strongly exceptional collection of vector bundles on the del Pezzo X of maximal length:

$$\mathcal{E}_1, \ldots, \mathcal{E}_n$$

which means that $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all i, j and all $k > 0$ and $\mathrm{Ext}^0(\mathcal{E}_i, \mathcal{E}_i) = \mathbb{C}$ for all i, then the endomorphism group of the direct sum $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{E}_i$ can be described as a *path algebra* A associated to some quiver with relations. Denote $D^b(\mathrm{mod}-A)$ the bounded derived category of right A-modules and $D^b(X)$ the derived category of coherent sheaves over X, then there exists an equivalence of categories

$$D^b(\mathrm{mod}-A) \cong D^b(X).$$

We are going to describe explicitely quivers of the algebra A for smooth toric del Pezzo surfaces. It is a well-known fact (see, for example, [Oda88]), that there exist precisely five toric del Pezzo surfaces, namely $\mathbb{P}_2, \mathbb{P}_1 \times \mathbb{P}_1$, and $\mathbb{P}_2$ blown up in one, two, or three points. We remark that the quivers of the first two surfaces, $\mathbb{P}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$, already occured in [Rud90, Bon90] and other references, but the latter three are new as far as we know. The description can be derived in three steps:

1. Find an exceptional collection $\mathcal{E}_1, \ldots, \mathcal{E}_n$.

2. Calculate $A := \mathrm{End}(\oplus_i \mathcal{E}_i)$.

3. Write down the quiver for A.

These calculations are an application of the description of X via the *homogeneous coordinate ring*. We fix some notation: T is the dense torus in X, M is the character lattice of T, $n := \#\Delta(1)$ is the number of rays in the fan Δ of X. Consider the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \xrightarrow{\pi} \mathrm{Pic}(X) \longrightarrow 0,$$

then the ring $S := \mathbb{C}[x_1, \dots x_n]$ becomes $\mathrm{Pic}(X)$-graded by setting $\deg(x_i) := \pi(e_i)$, where e_i is the i-th basis vector of $\mathbb{Z}^{\Delta(1)}$. There is a decomposition

$$S = \bigoplus_{\alpha \in \mathrm{Pic}(X)} S_\alpha$$

such that there exists an isomorphism

$$S_\alpha = \Gamma(X, \mathcal{O}(\alpha))$$

where α represents a divisor class in $\mathrm{Pic}(X)$ and $\mathcal{O}(\alpha)$ is its associated line bundle. In particular, for $\alpha, \beta \in \mathrm{Pic}(X)$, we have $\mathcal{O}(\alpha) \otimes \mathcal{O}(\beta) \cong \mathcal{O}(\alpha + \beta)$ and $\mathcal{H}om(\mathcal{O}(\alpha), \mathcal{O}(\beta)) \cong \mathcal{O}(-\alpha) \otimes \mathcal{O}(\beta) \cong \mathcal{O}(\beta - \alpha)$ and thus $\mathrm{Hom}(\mathcal{O}(\alpha), \mathcal{O}(\beta)) = \Gamma(X, \mathcal{O}(\beta - \alpha))$. Knowing this, the three steps above can be described more precisely as follows:

First step: We assume that the exceptional collections we are interested in consist of line bundles $\mathcal{O}(\alpha_i)$ and we make use of the identity $\mathrm{Ext}^i(\mathcal{O}(\alpha_i), \mathcal{O}(\alpha_j)) = H^i(X, \mathcal{O}(\alpha_j - \alpha_i))$. Calculating h^0 and h^2 (by Serre duality) of a line bundle is a textbook task (cf. [Ful93, Oda88]) and h^1 then can be calculated via Riemann-Roch. We then choose a nice basis for $\mathrm{Pic}(X)$ and test small linear combinations. It suffices to find n line bundles.

Second step: The algebra A decomposes:

$$A = \mathrm{End}(\bigoplus_{i=1}^{n} \mathcal{O}(\alpha_i)) = \mathrm{Hom}(\bigoplus_{i=1}^{n} \mathcal{O}(\alpha_i), \bigoplus_{i=1}^{n} \mathcal{O}(\alpha_i)) = \bigoplus_{i,j} \Gamma(X, \mathcal{O}(\alpha_i - \alpha_j)).$$

The result is a matrix algebra with entries in the homogeneous coordinate ring S.

Third step: Having identified A with some matrix algebra, one can write down the quiver and its relations.

Now we give the table for all five toric del Pezzos. For each case the following data are listed:

1. the fan with the rays marked by numbers $1, \dots n-2$ mark the T-invariant divisors chosen as basis of $\mathrm{Pic}(X)$,

2. the description of the toric variety given by a matrix whose columns contain the primitive vectors of the rays in counterclockwise order, starting with the ray labeled with 1

3. the exceptional collection found in terms of the above basis,

4. the quiver,

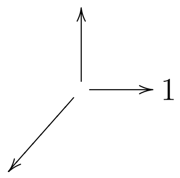
5. the relations of the quiver,

6. the matrix $(\chi(\mathcal{O}(\alpha_i), \mathcal{O}(\alpha_j)))_{i,j}$, which in all our cases coincides with the matrix $(h^0\mathcal{O}(\alpha_j - \alpha_i))_{i,j}$.

One observes that in the case of $\mathbb{P}_2$ blown up in one, two and three points, respectively, the relations of the quivers are not quadratic and thus are not Koszul.

$\mathbb{P}_2$:

Fan:

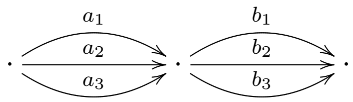


Toric data:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$

Quiver:



Relations:

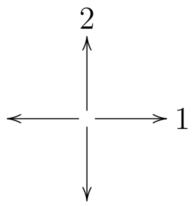
$$a_i b_j = a_j b_i$$

Matrix:

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{P}_1 \times \mathbb{P}_1$:

Fan:

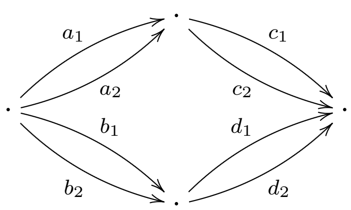


Toric data:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)$

Quiver:



Relations:

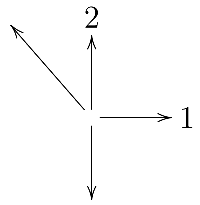
$$a_i c_j = b_j d_i$$

Matrix:

$$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{P}_2$ blown up in one point (the Hirzebruch surface $\mathbb{F}_1$):

Fan:

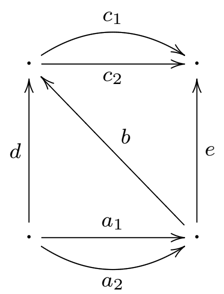


Toric data:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,1)$

Quiver:



Relations:

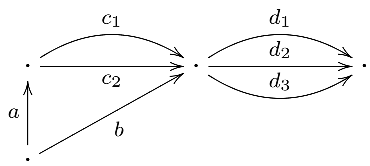
$$dc_i = a_i e, i = 1, 2$$
$$a_2 b c_1 = a_1 b c_2$$

Matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,1), \mathcal{O}(2,2)$

Quiver:



Relations:

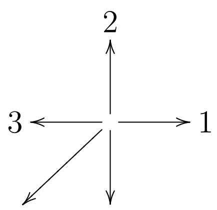
$$c_1 d_3 = c_2 d_1$$
$$a c_1 d_2 = b d_1$$
$$a c_2 d_2 = b d_3$$

Matrix:

$$\begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{P}_2$ blown up in two points:

Fan:

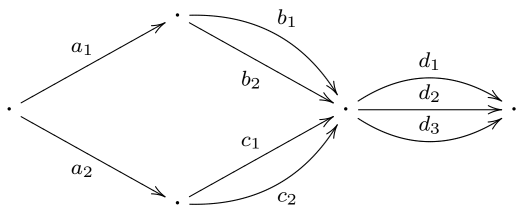


Toric data:

$$\begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(0,0,1), \mathcal{O}(-1,1,1), \mathcal{O}(0,1,1), \mathcal{O}(0,2,2)$

Quiver: Relations: Matrix:

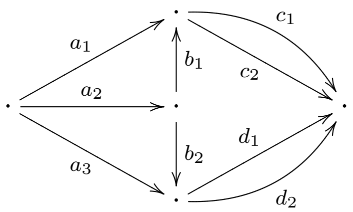


$$a_1b_1 = a_2c_1$$
$$b_1d_3 = b_2d_2$$
$$c_1d_1 = c_2d_2$$
$$a_1b_1d_1 = a_2c_2d_2$$
$$a_1b_2d_1 = a_2c_2d_3$$
$$a_1b_2d_3 = a_2c_1d_2$$

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 6 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(1,0,-1), \mathcal{O}(1,0,0), \mathcal{O}(0,1,0), \mathcal{O}(1,1,0)$

Quiver: Relations: Matrix:

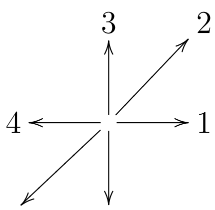


$$b_1c_2 = b_2d_2$$
$$a_1c_1 = a_3d_1$$
$$a_2b_1c_1 = a_3d_2$$
$$a_2b_2d_1 = a_1c_2$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{P}_2$ blown up in three points:

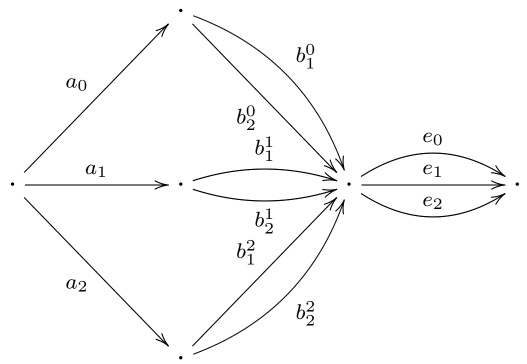
Fan: Toric data:



$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

Exceptional collection: $\mathcal{O}, \mathcal{O}(0,1,0,0), \mathcal{O}(0,0,0,1), \mathcal{O}(-1,0,1,1), \mathcal{O}(0,1,1,1), \mathcal{O}(0,2,2,2)$

Quiver: Relations: Matrix:



$$
\begin{aligned}
a_i b_1^i &= a_{i+1} b_2^{i+1} \\
b_1^i e_{i+1} &= b_2^i e_{i+2} \\
a_i b_1^i e_i &= a_{i+1} b_1^{i+1} e_{i+2} \\
a_i b_1^i e_i &= a_{i+2} b_2^{i+2} e_{i+2} \\
a_i b_2^i e_i &= a_{i+2} b_2^{i+2} e_{i+1} \\
i &\in \mathbb{Z}_3
\end{aligned}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 3 & 6 \\
0 & 1 & 0 & 0 & 2 & 5 \\
0 & 0 & 1 & 0 & 2 & 5 \\
0 & 0 & 0 & 1 & 2 & 5 \\
0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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[BP94] A. I. Bondal and A. E. Polishchuk. Homological properties of associative algebras: the method of helices. *Russ. Acad. Sci., Izv. Math.*, 42(2):219–260, 1994.

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[Oda88] Tadao Oda. *Convex Bodies and Algebraic Geometry.* Springer, 1988.

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