Computations in generic representation theory

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Where does the name generic come from? Consider the category $\mathcal{F} = \mathrm{Fun}(\text{mod-}\mathbb{K}, \text{Mod-}\mathbb{K})$, where $\mathbb{K}$ is a finite field of characteristic p with $q = p^s$ elements. $F(V)$ becomes a $\mathbb{K}[\mathrm{GL}(V)]$-module for all vector spaces V. So every functor F gives generically rise to representations of the general linear groups over $\mathbb{K}$.

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then

$$\exists m \in \mathbb{N} \text{ and } W_j \in \text{mod-}\mathbb{K}, \text{s.t.} \psi : \bigoplus_{j=1}^{m} (W_j, -) \twoheadrightarrow \mathrm{Ker}(\phi) \to 0$$

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In the case $p = q = 2$ Powell proved the conjecture to be true for factor modules of $(\mathbb{F}_2, -)$ and $(\mathbb{F}_2^2, -)$

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$$\text{Given } \bigoplus_{j=1}^{m}(W_j,-) \xrightarrow{([f],-)} \bigoplus_{i=1}^{n}(V_i,-) \twoheadrightarrow F \to 0,$$

then $\mathrm{Ker}([f],-)$ is again finitely generated.

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$([f], -)$ comes from $[f] : \bigoplus_{i=1}^{n} V_i \to \bigoplus_{j=1}^{m} W_j$. So, we are on the search for a cokernel. But a weak cokernel will do already.
A weak cokernel has basically the same properties as a cokernel, it just does not need to be unique.

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This means in terms of dimension for $\mathbb{K} = \mathbb{F}_q$:
$\dim \mathrm{Hom}_{\mathbb{K}[\mathsf{mod}\text{-}\mathbb{K}]}(\mathbb{F}_q^s, \mathbb{F}_q^t) = q^{st}$.

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7. If that fails, try other decomposition of the kernel or other dimension of the cover.

Examples

It is a first easy result from the algorithm that the weak cokernel of a map $[f]$, in the case $(\mathbb{F}_q^m, \mathbb{F}_q^t) \xrightarrow{([f],\mathbb{F}_q^t)} (\mathbb{F}_q^n, \mathbb{F}_q^t)$, where $[f]$ is just a basis vector and the rank of the matrix f is r

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3. The map $[g]$ needs to be independent of the testspace U.

Solution for the last problem maybe in combinatorics. To reduce memory and time consumption, look at the fibers of

$$\mathrm{Hom}_{\mathcal{F}}((V,-),(W,-)) \cong \mathrm{Hom}_{\mathbb{K}[\mathsf{mod}\text{-}\mathbb{K}]}(V,W) \twoheadrightarrow \mathrm{Hom}_{\mathbb{K}}(V,W).$$

The End

Thank you for you attention!