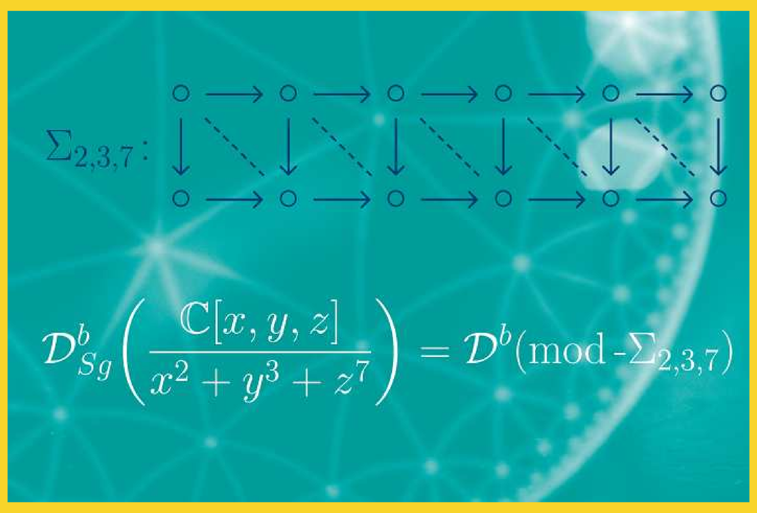

Computational Approach to the Artinian Conjecture

Phillip Linke
Universität Bielefeld
Advisor: Henning Krause

Introduction: Generic Representation Theory

Where does the name generic come from? Consider the category $\mathcal{F} = \mathrm{Fun}(\mathrm{mod}\text{-}\mathbb{K}, \mathrm{Mod}\text{-}\mathbb{K})$, where $\mathbb{K}$ is a finite field of characteristic p with $q = p^s$ elements. $F(V)$ becomes a $\mathbb{K}[\mathrm{GL}(V)]$-module for all vector spaces V. So every functor gives generically rise to representations of the general linear groups over $\mathbb{K}$.
Remark: We will often abbreviate $\mathrm{Hom}(-,-)$ by just $(-,-)$.

Structure of the simples and projectives

As described in [3] the indecomposable projectives as well as the simple functors of the category $\mathcal{F}$ are parameterized by q-restricted weights. These are finite sequences of non-negative integers less than q. This is analogue to the construction of simple $\mathbb{F}_q[M_n(\mathbb{F}_q)]$-modules M_λ for a fixed n. A simple functor F_λ is the unique top of the projective P_λ defined via $F_\lambda(\mathbb{F}_q^n) = M_\lambda$ where n is s.t. $\lambda(m) = 0$ if $m > n$ (this n will be noted as $n(\lambda)$). A projective $(\mathbb{F}_q, -)$ has a complete splitting $\bigoplus_\lambda d_\lambda P_\lambda$ with $d_\lambda = \dim M_\lambda$.

The Artinian Conjecture

Want to prove the Artinian Conjecture. Or at least get closer to it.
$Conjecture\ (L.\ Schwartz):$ The representable functors, so the projectives of finite type, in the category $\mathcal{F}$ are noetherian. Dually: The corresponding injectives are artinian.
Such a projective will look like this:

$$\bigoplus_{i=1}^{n}(V_i,-),\ V_i \in \mathrm{mod}\text{-}\mathbb{K}$$

Where the V_i need not be distinct.
The conjecture has several equivalent formulations. The one that is most practical to do calculations with is the following: Every finitely generated object in $\mathcal{F}$ has a resolution by projectives of finite type. So:

$$\text{If } F \in \mathcal{F} \text{ s.t. } \exists \phi : \bigoplus_{i=1}^{n}(V_i,-) \twoheadrightarrow F \to 0, \text{ with } \dim V_i < \infty \forall i,$$

then

$$\exists m \in \mathbb{N} \text{ and } W_j \in \mathrm{mod}\text{-}\mathbb{K}, \text{s.t.} \psi : \bigoplus_{j=1}^{m}(W_j,-) \twoheadrightarrow \mathrm{Ker}(\phi) \to 0$$

A First Approach

First try to prove that the projectives of finite type are coherent. That is: If we got a finitely presented functor, then it admits a resulution by projectives of finite type.

$$\text{Given } \bigoplus_{j=1}^{m}(W_j,-) \xrightarrow{([f],-)} \bigoplus_{i=1}^{n}(V_i,-) \twoheadrightarrow F \to 0,$$

then $\mathrm{Ker}([f],-)$ is again finitely generated.

$([f],-)$ comes from $[f] : \bigoplus_{i=1}^{n} V_i \to \bigoplus_{j=1}^{m} W_j$. So, we are on the search for a cokernel. But a weak cokernel will do already. A weak cokernel has basically the same properties as a cokernel, it just does not need to be unique.
Now why is the homomorphism f in those brackets? We have:

$$\mathcal{F} = \mathrm{Fun}(\mathrm{mod}\text{-}\mathbb{K}, \mathrm{Mod}\text{-}\mathbb{K}) \cong \mathrm{Add}(\mathbb{K}[\mathrm{mod}\text{-}\mathbb{K}], \mathrm{Mod}\text{-}\mathbb{K})$$

The Objects in $\mathbb{K}[\mathrm{mod}\text{-}\mathbb{K}]$ are the same as in $\mathrm{mod}\text{-}\mathbb{K}$ but the morphism spaces are defined as follows:

$$\mathrm{Hom}_{\mathbb{K}[\mathrm{mod}\text{-}\mathbb{K}]}(V,W) = \mathbb{K}[\mathrm{Hom}_{\mathbb{K}}(V,W)] = \left\{ \sum_{fin}[f_i] \,\middle|\, f_i : V \to W \right\}$$

This means in terms of dimension for $\mathbb{K} = \mathbb{F}_q$: $\dim \mathrm{Hom}_{\mathbb{K}[\mathrm{mod}\text{-}\mathbb{K}]}(\mathbb{F}_q^s, \mathbb{F}_q^t) = q^{st}$.

Algorithm

We first assume $n = 1$ in $\bigoplus_{i=1}^{n}(V_i,-)$ for reasons of simplicity.
1. Fix the input-data: $p, q, W_j \forall j, V, [f]$ and a testspace U.
2. Translate $([f],U)$ in matrix form and calculate its kernel.
3. Decompose the kernel into direct summands if possible.
4. Calculate the dimension that is needed to provide projective covering of the kernel.
5. Calculate the space of all $([g],U)$, s.t. $([f],U) \circ ([g],U) = 0$.
6. Then try to find one with $\dim \mathrm{Im}([g],U) = \dim \mathrm{Ker}([f],U)$.
7. If that fails, try other decomposition of the kernel or other dimension of the cover.

Problems

1. The instances get very big. Namely q^{st}-dimensional, for every projective.
2. This leads to very small test-examples.
3. The map $[g]$ needs to be independent of the testspace U.
4. The algorithm does not need to terminate since it is not clear if a weak cokernel exits.

Examples

It is a first easy result from the algorithm that the weak cokernel of a map $[f]$, in the case $(\mathbb{F}_q^m, \mathbb{F}_q^t) \xrightarrow{([f],\mathbb{F}_q^t)} (\mathbb{F}_q^n, \mathbb{F}_q^t)$, where $[f]$ is just a basis vector and the rank of the matrix f is r , is given by $(\mathbb{F}_q^m, \mathbb{F}_q^t)$ and a map given by the sum $[g_1] + (p-1)[g_2]$. With $[g_1]$ a map of full rank and $[g_2]$ a map of rank $n - r$. For sums of basis vektors or direct sums of projectives, this is not as easy.

Partial Results

Schwartz showed this resolution to exist if the functor F has only finitely many simple composition factors or when the difference endofunctor ∇ ($\nabla(F)$ is defined via the split exact sequence $0 \to F \to \Delta F \to \nabla F \to 0$, with $\Delta(F)(V) = F(V \oplus \mathbb{F}_q)$) returns a projective after finitely many iterations [5].
$(\mathbb{F}_q, -)$ known to be noetherian for all q [1].
In the case $p = q = 2$ Powell proved the conjecture to be true for factor modules of $(\mathbb{F}_2^2, -)$ [2].

Resolutions for simple Functors

Since it is not clear whether a weak cokernel exists for a given representable functor F we turn to simple functors first. The work of Auslander [6] gives us not only that a resolution of a simple functor always exists, but also how it looks like: F is simple then

$$0 \to (C,-) \to (B,-) \to (A,-) \to F \to 0$$

is a minimal projective resolution if

$$0 \to A \to B \to C \to 0$$

is an almost split or Auslander-Reiten sequence in the underlying category.
In our case this means we get a sequence

$$0 \to P_\beta \to \bigoplus_\nu P_\nu \to P_\mu \to F_\mu \to 0.$$

Given this we can use homological algebra to determine the extension groups between two given simple functors. We get

$$\mathrm{Ext}^1_{\mathcal{F}}(F_\mu, F_\lambda) \cong \mathrm{Hom}_{\mathcal{F}}(\Omega F_\mu, F_\lambda)$$

The dimension of the latter is just the number of copies of F_λ in $\mathrm{top}(\Omega F_\mu)$. The structure of the projectives allows us to simplify even further

$$\mathrm{Hom}_{\mathcal{F}}(\Omega F_\mu, F_\lambda) \cong \mathrm{Hom}_{\mathcal{F}}\left(\bigoplus_\nu F_\nu, F_\lambda\right)$$

Our problem with this is now that the decomposition of a projective functor $(\mathbb{F}_q^n, -)$ is not that canonical that we can just plug it in like that. We will need to work from the smallest Functor $(\mathbb{F}_q^n, -)$ thats top contains F_μ. By definition this is $(\mathbb{F}_q^{n(\mu)}, -)$. Using the properties of AR-sequences, the second projective in the projective presentation of F_μ will be $(\mathbb{F}_q^{n(\mu)+1}, -)$.
So we will need to find a map $[f] : \mathbb{F}_q^{n(\mu)} \to \mathbb{F}_q^{n(\mu)+1}$ s.t. the cokernel of $([f],-)$ is F_μ. Once we got that we can use the algorithm for calculation of the weak cokernels to get the next projective in this resolution. A result from [4] shows that for a simple functor F the dimension of $F(\mathbb{F}_q^k)$ is always a polynomial in k. This gives us that the dimension of the kernel of $([f], \mathbb{F}_q^k)$ will be “close to a polynomial function“. So basic combinatorics gives that we will only need to look at a finite number of test spaces $\mathbb{F}_q^k$ to get indeed a weak cokernel that will hold for all $\mathbb{F}_q^k$.

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