Matrix factorisations

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Outline

Main Theorem (Eisenbud)

Let $R = S/(f)$ be a hypersurface. Then

$$\underline{\mathrm{MCM}}(R) \sim \underline{\mathrm{MF}}_S(f)$$

Survey of the talk:

1. Define hypersurfaces. Explain, why they fit in our setting.
2. Define Matrix factorisations. Prove, that they form a Frobenius category.
3. Prove the equivalence.

Regular local rings

Definition

A Noetherian local ring S is called **regular** if

$$\operatorname{Kdim} S = \operatorname{gldim} S < \infty.$$

Example

- $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$
- $k[[X_1, \ldots, X_n]]$

Lemma

A regular local ring S is Gorenstein.

Proof.

$\operatorname{gldim} S < \infty \Rightarrow \operatorname{Ext}^i_S(k, S) = 0$ for $i \gg 0 \Rightarrow \operatorname{indim} S < \infty$ □

Quotients of Gorenstein rings

Let S be regular local. Let $0 \neq f \in S$ be a non-unit.

Lemma

A regular local ring is a domain.

Proposition

Let S be a Noetherian local ring. Let $f \in S$ be a non-unit, non-zero divisor. Then

$$\operatorname{indim}_{S/(f)} S/(f) = \operatorname{indim}_S S - 1.$$

Corollary

S Gorenstein $\Rightarrow S/(f)$ Gorenstein.

Hypersurfaces

Definition

Let S be a regular local ring and $0 \neq f \in S$ be a non-unit. Then $R := S/(f)$ is called a **hypersurface**.

Main Theorem (Eisenbud)

Let $R = S/(f)$ be a hypersurface. Then

$$\underline{\mathrm{MCM}}(R) \sim \underline{\mathrm{MF}}_S(f)$$

Matrix factorisations

Definition

Let S be a ring, $f \in S$. A **matrix factorisation** of f is a pair of maps of free S-modules

$$F \xrightarrow{\ \varphi\ } G \xrightarrow{\ \psi\ } F$$

such that

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[rr, bend left, red, "f \cdot 1_F"] & G \arrow[r, "\psi"'] \arrow[rr, bend right, red, "f \cdot 1_G"'] & F \arrow[r, "\varphi"] & G \end{tikzcd}$$

Examples

Example

Let $S = \mathbb{C}[[x, y]]$, $f = y^2 - x^2$. Then a matrix factorisation is given by

$$S^2 \xrightarrow{\begin{pmatrix} y & x \\ -x & -y \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} y & x \\ -x & -y \end{pmatrix}} S^2$$

Example

$$F \xrightarrow{1_F} F \xrightarrow{f \cdot 1_F} F$$

Morphisms of matrix factorisations

Definition

A morphism between two matrix factorisations of f is given by:

$$\begin{array}{ccccc} F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & F \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \alpha \\ F' & \xrightarrow{\varphi'} & G' & \xrightarrow{\psi'} & F' \end{array}$$

Composition of two morphisms is given componentwise.Denote the category of matrix factorisations by $\mathrm{MF}_S(f)$.

Additive Categories

Recall

A category $\mathcal{A}$ is called **additive** if

- $\mathcal{A}$ has a zero object.
- Every Hom-Set is an abelian group and composition is bilinear.
- There exists a coproduct of every two objects.

Additive Categories

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Proposition

The category $\mathrm{MF}_S(f)$ is **additive**:

- $\mathcal{A}$ has a zero object.
  - $0 \longrightarrow 0 \longrightarrow 0$ is a zero object.
- Every Hom-Set is an abelian group and composition is bilinear.
  - $(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta')$.
- There exists a coproduct of every two objects.
  - $\left(S_0 \xrightarrow{\varphi_S} S_1 \xrightarrow{\psi_S} S_0 \right) \oplus \left(T_0 \xrightarrow{\varphi_T} T_1 \xrightarrow{\psi_T} T_0 \right)$

$$:= \quad S_0 \oplus T_0 \xrightarrow{\begin{pmatrix} \varphi_S & 0 \\ 0 & \varphi_T \end{pmatrix}} S_1 \oplus T_1 \xrightarrow{\begin{pmatrix} \psi_S & 0 \\ 0 & \psi_T \end{pmatrix}} S_0 \oplus T_0$$

Important Lemmas

Lemma

Let $(\varphi, \psi) \in \mathsf{MF}_S(f)$. *Then* φ *is a monomorphism.*

Proof.

- Let $\varphi(z) = 0$.
- Then $0 = \psi\varphi(z) = f \cdot z$.
- Thus $z = 0$.

□

Matrix factorisations and exact sequences

Lemma

Let $F \xrightarrow{\varphi} G \xrightarrow{\psi} F$ *be a matrix factorisation. Then*

$$\cdots \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\varphi}} \overline{G} \xrightarrow{\overline{\psi}} \overline{F} \xrightarrow{\overline{\varphi}} \cdots$$

is exact, where $\overline{F} = F/(fF)$.

Proof.

- $\varphi\psi = f \cdot 1_G \Rightarrow \overline{\varphi}\overline{\psi} = 0$
- Let $\overline{z} \in \ker \overline{\varphi}$.Then $\varphi(z) \in fG = \varphi\psi G \Rightarrow z \in \mathrm{Im}(\psi)$.

□

Proposition

Up to isomorphism one can assume $F = G$.

Proof.

Let $F \xrightarrow{\varphi} G \xrightarrow{\psi} F$ be a matrix factorisation. Then

$$\cdots \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\varphi}} \overline{G} \xrightarrow{\overline{\psi}} \overline{F} \xrightarrow{\overline{\varphi}} \cdots$$

is exact.

Localize at a minimal prime $\mathfrak{p}$, you get artinian modules. Thus

$$\ell(\overline{G}_{\mathfrak{p}}) = \ell(\operatorname{Im} \overline{\psi}_{\mathfrak{p}}) + \ell(\ker \overline{\psi}_{\mathfrak{p}}) = \ell(\overline{F}_{\mathfrak{p}}).$$

But here

$$\operatorname{rank}(\overline{G}_{\mathfrak{p}}) = \ell(\overline{G}_{\mathfrak{p}})/\ell(R_{\mathfrak{p}}).$$

Proof (continued).

Thus $F \cong G$, say via A.Then the following provides an isomorphism between matrix factorisations:

$$\begin{array}{ccccc}
F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & F \\
\downarrow {\scriptstyle 1_F} & & \downarrow {\scriptstyle A} & & \downarrow {\scriptstyle 1_F} \\
F & \xrightarrow{A\varphi} & F & \xrightarrow{\psi A^{-1}} & F
\end{array}$$

□

Exact categories

Recall

An additive category $\mathcal{A}$ with a class of short exact sequences

$\{ \ A \rightarrowtail B \twoheadrightarrow C \ \}$ is called **exact** if

- $B \overset{1_B}{\rightarrowtail} B \twoheadrightarrow 0$, $\quad 0 \rightarrowtail B \overset{1_B}{\twoheadrightarrow} B$
- Composition of admissible monomorphisms is admissible monomorphism
- Composition of admissible epimorphisms is addmisible epimorphism
- Pushout of an admissible monomorphism and an arbitrary map yields an admissible monomorphism.
- Pullback of an admissible epimorphism and an arbitrary map yields an admissible epimorphism.

Exact sequences

Proposition

The category $\mathrm{MF}_S(f)$ *is an exact category with exact sequences*

given by:

$$\begin{array}{ccccc} S_0 & \rightarrowtail & T_0 & \twoheadrightarrow & U_0 \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ S_1 & \rightarrowtail & T_1 & \twoheadrightarrow & U_1. \end{array}$$

This means in particular:

admissible epimorphism = all epimorphisms = split epimorphisms
admissible monomorphism = split monomorphisms

$\mathrm{MF}_S(f)$ is exact

Proof.

- $$\begin{array}{ccccc} 0 & \rightarrowtail & T_0 & \overset{1_{T_0}}{\twoheadrightarrow} & T_0 \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ 0 & \rightarrowtail & T_1 & \overset{1_{T_1}}{\twoheadrightarrow} & T_1 \end{array}$$ is exact.
- Same for the dual situation.
- "Composition of admissible monomorphisms is admissible monomorphism" follows from "Composition of split monomorphisms is split monomorphism"
- Same for dual situation.

$\mathrm{MF}_S(f)$ is exact (continued)

Proof (continued).

$$\begin{array}{ccc} S_0 & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & S_0 \oplus S_0' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}} \\ T_0 & \xrightarrow[\begin{pmatrix}1\\0\end{pmatrix}]{} & T_0 \oplus S_0' \end{array}$$

is the pushout of the "canonical" split map and an arbitrary map μ. That this holds for $\mathrm{MF}_S(f)$ as well follows componentwise. □

Exact subcategories of abelian categories

Remark (added after the talk, after remarks from R. Buchweitz, M. Kalck and M. Severitt)

- Any exact category can be embedded into an abelian category by an abstract "abelian hull"-construction.
- In the case of $\mathrm{MF}_S(f)$ there are several possibilities to embed it into:
  - The abelian category of chain complexes, by mapping (φ, ψ) to $(\dots, 0, \varphi, 0, \psi, 0, \dots)$.
  - The category of modules over $S(\; 1 \rightleftarrows 2 \;)$.
  - The category of graded modules over the $\mathbb{Z}_2$-graded algebra $S[y]/(y^2 = f)$, where $|y| = 1$.

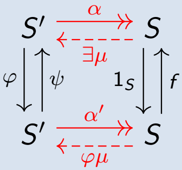
Projective objects in $\mathrm{MF}_S(f)$

Lemma

With respect to the defined exact structure $(1, f)$ is projective.

Proof.

We can assume that $S_0 = S_1 =: S'$.
Have to show every admissible epimorphism to $(1, f)$ splits.



□

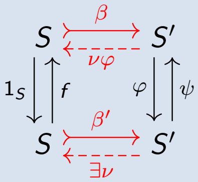
Injective objects in $\mathrm{MF}_S(f)$

Lemma

With respect to the defined exact structure $(1, f)$ is injective.

Proof.

Have to show every admissible monomorphism from $(1, f)$ splits.



Note: Every admissible monomorphism splits. □

Frobenius categories

Recall

An exact category is called **Frobenius category** if projective and injective objects coincide and the category has enough projectives and injectives.

Proposition

$\mathrm{MF}_S(f)$ *is a Frobenius category with prinjective objects* $\mathrm{add}(1, f) \oplus (f, 1)$.

$\mathrm{MF}_S(f)$ is a Frobenius category

Proof.

- We have already seen that $(1, f) \oplus (f, 1)$ is prinjective.
- The following diagram shows enough prinjectives:

$$\begin{array}{ccccc}
S_0 & \xrightarrow{\begin{pmatrix}-\varphi\\ 1\end{pmatrix}} & S_0 \oplus S_0 & \xrightarrow{(-1,-\varphi)} & S_0 \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \begin{pmatrix}1 & 0\\ 0 & f\end{pmatrix}} & & \downarrow{\scriptstyle -\psi} \\
S_0 & \xrightarrow{\begin{pmatrix}-1\\ \psi\end{pmatrix}} & S_0 \oplus S_0 & \xrightarrow{(\psi,1)} & S_0 \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \begin{pmatrix}f & 0\\ 0 & 1\end{pmatrix}} & & \downarrow{\scriptstyle -\varphi} \\
S_0 & \xrightarrow{\begin{pmatrix}-\varphi\\ 1\end{pmatrix}} & S_0 \oplus S_0 & \xrightarrow{(-1,-\varphi)} & S_0
\end{array}$$

The Main Theorem again

Theorem

Let $R = S/(f)$ be a hypersurface. Then we have:

$$\mathrm{MF}_S(f)/\operatorname{add}(1, f) \sim \mathrm{CM}(R)$$

and

$$\underline{\mathrm{MF}}_S(f) \sim \underline{\mathrm{CM}}(R).$$

The proof

Proof I: $\mathrm{MF}_S(f)/\operatorname{add}(1,f) \to \mathrm{CM}(R)$

Let

$$\begin{aligned} \mathcal{X} : \mathrm{MF}_S(f)/\operatorname{add}(1,f) &\to \mathrm{CM}(R) \\ (\varphi,\psi) &\mapsto \operatorname{Coker}\varphi \end{aligned}$$

This is well-defined:

- $$0 \longrightarrow F \xrightarrow{\varphi} G \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0$$
  $fF = \varphi\psi F \subseteq \varphi G$, hence $f \underbrace{\operatorname{Coker}\varphi}_{\text{an } R\text{-module}} = 0$.
- $\operatorname{Coker}(1,f) = 0$.
- The key lemma before shows: $\operatorname{Coker}\varphi$ has a periodic free R-resolution.

The proof (continued)

Recall

R Cohen-Macaulay of dimension d. Then for any $n \geq d$:

$$\Omega^n(M) = 0 \text{ or } \Omega^n(M) \text{ is Cohen-Macaulay.}$$

Proof I (continued).

- $\operatorname{Coker}\varphi \cong \Omega^{2n}\operatorname{Coker}\varphi$ is Cohen-Macaulay.
- $$\begin{array}{ccccccc} F & \xrightarrow{\varphi} & G & \longrightarrow & \operatorname{Coker}\varphi & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\exists} & & \\ F' & \xrightarrow{\varphi'} & G' & \longrightarrow & \operatorname{Coker}\varphi' & \longrightarrow & 0 \end{array}$$
  defines the functor on morphisms.

A small amount of commutative algebra

Proposition

Let $R = S/(f)$ be a hypersurface.

$$M \in \mathrm{CM}(R) \Rightarrow \operatorname{prdim}_S M = 1.$$

Proof.

$$\begin{aligned} \operatorname{prdim}_S M &= \operatorname{Kdim}(S) - \operatorname{depth}(M) && \text{Auslander-Buchsbaum formula} \\ &= \operatorname{Kdim}(S) - \operatorname{Kdim}(R) && M \in \mathrm{CM}(R) \\ &= 1 && \text{Krull's principal ideal theorem} \end{aligned}$$

□

The other direction of the proof

Proof II: $\mathrm{CM}(R) \to MF_S(f)/\operatorname{add}(1,f)$

Let $M \in \mathrm{CM}(R)$, then $\operatorname{prdim}_S M = 1$:

$$0 \to F \xrightarrow{\varphi} G \to M \to 0.$$

- $fM = 0 \Rightarrow fS_1 \subseteq \varphi S_0$.
- $\forall x \in S_1 \, \exists! y \in S_0 \colon fx = \varphi y$
- Set $\psi \colon S_1 \to S_0 \colon x \mapsto y$.

(φ, ψ) gives a matrix factorisation. Define $\mathcal{Y}(M) := (\varphi, \psi)$.

- $fx = \varphi y, fx' = \varphi y' \Rightarrow f(x + x') = \varphi(y + y') \Rightarrow \psi(x + x') = y + y'$
- $s \in S \Rightarrow fxs = \varphi(ys) \Rightarrow \psi(xs) = ys$.

Proof II (continued).

$M, M' \in \mathrm{CM}(R)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow \exists & & \downarrow \exists & & \downarrow & & \\
0 & \longrightarrow & F' & \xrightarrow{\varphi'} & G' & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

Minimal projective resolutions are unique up to isomorphism.

$\rightsquigarrow$ Gives a functor $\mathrm{CM}(R) \to \mathrm{MF}_S(f)/\operatorname{add}(1, f)$.

Equivalence is now obvious.
$\mathcal{X}(f, 1) = \mathrm{Coker}(f) = R$, hence $\underline{\mathrm{MF}}_S(f) \sim \underline{\mathrm{CM}}(R)$. $\square$

Interchanging φ and ψ

Corollary

$M = \mathcal{X}(\varphi, \psi) \Rightarrow \Omega M = \mathcal{X}(\psi, \varphi)$.

Proof.

- $\cdots \longrightarrow R_1 \xrightarrow{\overline{\psi}} R_0 \xrightarrow{\overline{\varphi}} R_0 \longrightarrow \cdots$
- $0 \longrightarrow \mathcal{X}(\psi, \varphi) \longrightarrow R \longrightarrow \mathcal{X}(\varphi, \psi) \longrightarrow 0$

□

Frobenius categories and homotopy

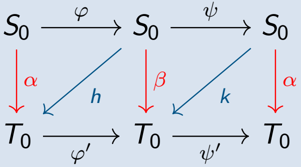
Proposition

The equivalence before can also be written as:

$$H^0(\mathrm{MF}_S(f)) \sim \underline{\mathrm{CM}}(R).$$

Proof.

null-homotopic $\Leftrightarrow$ factors through prinjective



with $\beta = \varphi' h + k\psi$ and $\alpha = \psi' k + h\varphi$.

Proof of homotopy $\Rightarrow$ factoring

Proof (continued)

$\beta = \varphi' h + k\psi$ and $\alpha = \psi' k + h\varphi$.

$$\begin{array}{ccccc}
S_0 & \xrightarrow{\quad\varphi\quad} & S_0 & \xrightarrow{\quad\psi\quad} & S_0 \\
\downarrow{\scriptstyle \begin{pmatrix}1\\ \varphi\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix}\psi\\ 1\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix}1\\ \varphi\end{pmatrix}} \\
S_0 \oplus S_0 & \xrightarrow{\begin{pmatrix}f & 0\\ 0 & 1\end{pmatrix}} & S_0 \oplus S_0 & \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & f\end{pmatrix}} & S_0 \oplus S_0 \\
\downarrow{\scriptstyle (\psi' k, h)} & & \downarrow{\scriptstyle (k, \varphi' h)} & & \downarrow{\scriptstyle (\psi' k, h)} \\
T_0 & \xrightarrow{\quad\varphi'\quad} & T_0 & \xrightarrow{\quad\psi'\quad} & T_0
\end{array}$$

Proof of factoring $\Rightarrow$ homotopy

Proof (continued)

$\alpha = \alpha_2\alpha_1 + \alpha_2'\alpha_1'$ and $\beta = \beta_2\beta_1 + \beta_2'\beta_1'$

$$\begin{array}{ccccc}
S_0 & \xrightarrow{\varphi} & S_0 & \xrightarrow{\psi} & S_0 \\
\downarrow{\scriptstyle \begin{pmatrix}\alpha_1 \\ \alpha_1'\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix}\beta_1 \\ \beta_1'\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix}\alpha_1 \\ \alpha_1'\end{pmatrix}} \\
U_0 \oplus U_0 & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}} & U_0 \oplus U_0 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}} & U_0 \oplus U_0 \\
\downarrow{\scriptstyle (\alpha_2, \alpha_2')} & & \downarrow{\scriptstyle (\beta_2, \beta_2')} & & \downarrow{\scriptstyle (\alpha_2, \alpha_2')} \\
T_0 & \xrightarrow{\varphi'} & T_0 & \xrightarrow{\psi'} & T_0
\end{array}$$

The end

Proof.

$$\alpha = \alpha_2\alpha_1 + \alpha_2'\alpha_1' \qquad \beta = \beta_2\beta_1 + \beta_2'\beta_1'$$

$$\begin{pmatrix} f\alpha_1 \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} \beta_1\varphi \\ \beta_1'\varphi \end{pmatrix} \qquad \begin{pmatrix} \beta_1 \\ \beta_1' f \end{pmatrix} = \begin{pmatrix} \alpha_1\psi \\ \alpha_1'\psi \end{pmatrix}$$

$$(\varphi'\alpha_2, \varphi'\alpha_2') = (\beta_2 f, \beta_2') \qquad (\psi'\beta_2, \psi'\beta_2') = (\alpha_2, \alpha_2' f)$$