Coxeter Transformations and the Gelfand-Kirillov Dimension

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Motivation

The *Gelfand-Kirillov dimension* measures the *growth rate* of a *filtered K-algebra*.

Aim

We want to compute $GK\,\dim$ for preprojective algebras.

filtration of an algebra

Let K be a field and R a K-algebra with a filtration $\{R_n\}$, i.e.

- $R_n \subseteq R$ subgroup of $(R, +) \quad \forall n$
- $R_i \subseteq R_j$ for $i < j$
- $R_i R_j \subseteq R_{i+j} \quad \forall i, j$
- $\bigcup R_n = R$

$\{R_n\}$ is called

- **standard**, if $(R_1)^n = R_n \quad \forall n$.
- **finite-dimensional**, if $R_0 = K$ and $\dim_K R_n < \infty \quad \forall n$.

filtration of a module

Let M be a right R-module filtration $\{M_n\}$, i.e.

- $M_n \subseteq M$ subgroup $\forall n$
- $M_i \subseteq M_j$ for $i < j$
- $M_i R_j \subseteq M_{i+j} \quad \forall i, j$
- $\bigcup M_n = M$

$\{M_n\}$ is called

- **standard**, if $M_n = M_0 R_n \quad \forall n$.
- **finite-dimensional**, if $\dim_K M_n < \infty \quad \forall n$.

special cases: R affine; M finitely generated

R **affine**: R finitely generated by a finite-dimensional subspace V as a K-algebra.

- V is called **generating subspace** of R.

Set $R_0 := V^0 = K$ and $R_n := \sum_{i=0}^{n} V^i$. Then $\{R_n\}$ is a standard finite-dimensional filtration of R.

M finitely generated $\Rightarrow$ M has a finite-dimensional subspace M_0 s.th. $M_0 R = M$.

- M_0 is called **generating subspace** for M.

Set $M_n := M_0 R_n$. Then $\{M_n\}$ is a standard finite-dimensional filtration of M.

growth rate

Let $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be any function.
f has **polynomially bounded growth** if

$$\exists v \in \mathbb{R} \text{ s.th. } f(n) \leq n^v \text{ for } n >> 0$$

The **growth rate** of f is then defined as

$$\gamma(f) = \inf\{v | f(n) \leq n^v \text{ for } n >> 0\}$$

For f not polynomially bounded, set $\gamma(f) := \infty$.

growth rate independent

Set $f(n) := \dim R_n$ and $g(n) := \dim M_n$.

Lemma

The growth rates $\gamma(f)$ of R and $\gamma(g)$ of M are independent of the choices of V and M_0.

Proof.

Let V', M_0' be other generating subspaces, $\{R_n'\}, \{M_n'\}$ the corresponding filtrations and $f'(n) = \dim R_n', g'(n) = \dim M_n'$.

- $\bigcup R_n = R$ and $R_i \subseteq R_{i+1} \Rightarrow V' \subseteq R_a$ for some a
- $\bigcup M_n = M$ and $M_i \subseteq M_{i+1} \Rightarrow M_0' \subseteq M_b$ for some b

Proof (Cont.)

- $\bigcup R_n = R$ and $R_i \subseteq R_{i+1} \Rightarrow V' \subseteq R_a$ for some a
- $\bigcup M_n = M$ and $M_i \subseteq M_{i+1} \Rightarrow M'_0 \subseteq M_b$ for some b

$\Rightarrow M'_n = M'_0 R'_n = M'_0 (V' + K)^n \subseteq M_b R_{an} \subseteq M_{an+b}$

$\Rightarrow g'(n) \leq g(an+b)$.

Let $\gamma(g) = v, \varepsilon > 0$. Then $g(n) < n^{v+\varepsilon}$ for $n >> 0$.

$g'(n) \leq g(an+b) < (an+b)^{v+\varepsilon} = n^{v+\varepsilon}(a + \frac{b}{n})^{v+\varepsilon} < n^{v+2\varepsilon}$.

$\Rightarrow \gamma(g') \leq v = \gamma(g)$.

Proof (Cont.)

Switching the roles of V, M_0 and V', M_0' gives $\gamma(g') = \gamma(g)$.

If we choose $M = R$ and $M_0 = K$, then $g = f$. □

$$f(n) \leq n^v \quad \forall n >> 0 \Leftrightarrow \frac{\log f(n)}{\log n} \leq v \quad \forall n >> 0$$

$$\Leftrightarrow \limsup \frac{\log f(n)}{\log n} \leq v$$

Therefore

$$\gamma(f) = \limsup \frac{\log(f(n))}{\log(n)}$$

Gelfand-Kirillov dimension

Definition

$GK\dim(R) := \gamma(f)$, where $f(n) = \dim R_n$, is the **Gelfand-Kirillov dimension** of the algebra R.

Remark

If $1 \in V$ then $V^n = R_n$, hence $GK\dim(R) = \gamma(\dim V^n)$.

Coxeter transformation

Let Q be a quiver without loops and $C \in GL(\mathbb{C}^n)$ the corresponding Cartan matrix, i.e.

- $c_{ii} = 2 \quad \forall i$
- $c_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
- $c_{ij} \neq 0 \Leftrightarrow c_{ji} \neq 0$.

Let r_s be the reflection at vertex s with respect to C, i.e. $r_s x = y$ where

- $y_s = -x_s - \sum_{i \neq s} c_{is} x_i$
- $y_i = x_i$ for all $i \neq s$.

Coxeter transformation

Definition

The element $c := r_n ... r_2 r_1$ is called a **Coxeter transformation**.

Remark (e.g. in [dIPT90], 1.1)

X indecomposable non-injective $\Rightarrow \dim \tau^{-1} X = c^{-1} \dim X$.

spectral radius

Definition

The **spectral radius** ρ of a linear transformation c is the maximal $|\lambda|$, where λ is an eigenvalue of c.

Proposition ([DR81], Proposition 1)

ρ is an eigenvalue of c.

Lemma (seen in [dlPT90], 1.6)

If Q is neither Dynkin nor Euclidean, then $\rho > 1$.

Choosing a filtration

Let Q be a quiver without loops.

Choose $V^m := \bigoplus_{i=0}^{m} \tau^{-i} KQ$.

$V^m \cdot V^1 = V^{m+1}$ by defining $\tau^{-k} P(s) \cdot P(t) := \tau^{-k-1} P(t)$. Then V is a generating subspace of the filtration $\{R_n = \sum_{i=0}^{n} V^i\}$.

$GK \dim(\Pi Q)$ - Q Dynkin

Q Dynkin $\Leftrightarrow R := \Pi Q$ finite-dimensional

$\Rightarrow \dim R_n$ is constant for n large.

Thus

$GK \dim(\Pi Q) = 0$ if Q Dynkin.

$GK\dim(\Pi Q)$ - Q Euclidean

Proposition ([DR76],Proposition 1.5)

If Q is Euclidean, there exists an $h \geq 1$ such that $\dim \tau^{-h} P = \dim P + \partial_c(\dim P)\delta$ for preprojective P.

where ∂_c is a linear functional called **defect**, corresponding to the Coxeter transformation c, and δ is the "canonic" vector generating the radical subspace of all dimension vectors which are stable under the action of the Coxeter group.

∂_c and δ are stated explicitly in [DR76, Section 6].

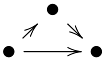
$GK \dim(\Pi Q)$ - Q Euclidean - pt.2

Set $d(n) := \sum_{i=0}^{n-1} \dim \tau^{-1} KQ = \dim V^{n-1}$.

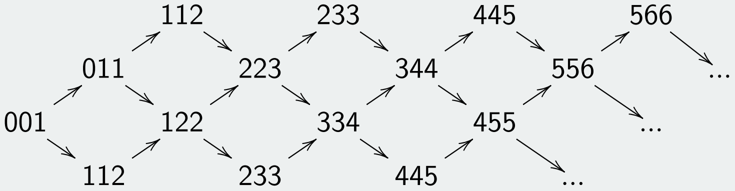
Proposition

$d(hn) = nd(h) + \binom{n}{2} \partial_c (\dim V^{h-1}) \delta$

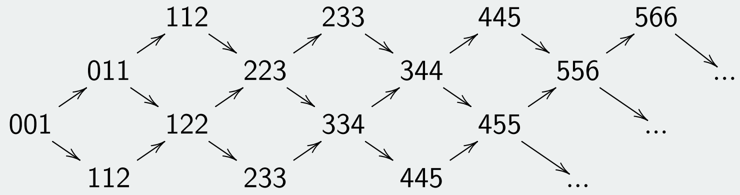
An example

Let Q be the quiver  of type $\tilde{A}_2$. Then $h = 2$, $\delta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

and $\partial_c = (-3 \quad 0 \quad 3)$. The AR-quiver looks like



with first and fourth row identified.



Choose $n = 2$. $d(4) = \begin{pmatrix} 30 \\ 36 \\ 42 \end{pmatrix}$, $d(2) = \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix}$,

$$\partial_c(\dim V^{h-1})\delta = (-3 \quad 0 \quad 3) \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix}.$$

$$\Rightarrow nd(h) + \binom{n}{2}\partial_c(\dim V^{h-1})\delta = 2\begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix} + \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = \begin{pmatrix} 30 \\ 36 \\ 42 \end{pmatrix} = d(hn)$$

$d(hn) = nd(h) + \binom{n}{2}\partial_c(\dim V^{h-1})\delta$

Proof.

- $\dim \tau^{-h}KQ = \dim KQ + \partial_c(\dim KQ)\delta$
  $\Rightarrow \dim \tau^{-kh}KQ = \dim KQ + k\partial_c(\dim KQ)\delta$
- $$\begin{aligned} d(hn) &= \sum_{k=0}^{n-1}\sum_{i=kh}^{(k+1)h-1} \dim \tau^{-i}KQ \\ &= \sum_{k=0}^{n-1}\sum_{i=0}^{h-1} \dim \tau^{-kh}\tau^{-i}KQ \\ &= \sum_{k=0}^{n-1}\sum_{i=0}^{h-1} (\dim \tau^{-i}KQ + k\,\overbrace{\partial_c(\dim \tau^{-i}KQ)}^{=\partial_c(\dim KQ)}\,\delta) \\ &= \underbrace{\sum_{k=0}^{n-1}\sum_{i=0}^{h-1} \dim \tau^{-i}KQ}_{nd(h)} + \underbrace{\sum_{k=0}^{n-1} k}_{\binom{n}{2}} \underbrace{\sum_{i=0}^{h} \partial_c(\dim KQ)\delta}_{\partial_c(\dim V^{h-1})\delta} \end{aligned}$$

□

$GK\dim(\Pi Q)$ - Q Euclidean - pt.3

- $GK\dim(\Pi Q) = \gamma(d) = \inf\{v \mid \|d(n)\| \leq n^v \text{ for } n >> 0\}.$
- $d(hn) = nd(h) + \binom{n}{2}\partial_c(\dim V^{h-1})\delta$
- $\binom{n}{2} = \frac{n(n-1)}{2} < \frac{n^2}{2}$
- for n large enough, $\|d(h)\| \leq n$ and $\|\binom{n}{2}\partial_c(\dim V^{h-1})\delta\| \leq n^2$

$\Rightarrow \|d(hn)\| \leq 2n^2$ for n sufficiently large $\Rightarrow \|d(hn)\| \leq n^2$ for $n >> 0$

$GK\dim(\Pi Q) = 2$ for Q Euclidean.

$GK\dim(\Pi Q)$ - Q neither Dynkin nor Euclidean

Proposition ([DR81], Proposition 2)

$$\rho = \lim_{m\to\infty} \sqrt[m]{\|\dim \tau^{-m} KQ\|}$$

$$\frac{\log \|\dim \tau^{-m} KQ\|}{\log m} = \frac{m}{\log m} \cdot \frac{\log \|\dim \tau^{-m} KQ\|}{m}$$

$$= \underbrace{\frac{m}{\log m}}_{\xrightarrow[m\to\infty]{} \infty} \cdot \underbrace{\log \sqrt[m]{\|\dim \tau^{-m} KQ\|}}_{\xrightarrow[m\to\infty]{} \log \rho > 0}$$

Hence $\gamma(f) = \infty$, so

$GK\dim(\Pi Q) = \infty$ if Q is neither Dynkin nor Euclidean.

Result

We have established the following

Theorem

$$GK\dim(\Pi Q) = \begin{cases} 0\ ,\ \textit{if } Q \textit{ is Dynkin} \\ 2\ ,\ \textit{if } Q \textit{ Euclidean} \\ \infty\ ,\ \textit{otherwise} \end{cases}$$

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