Specifications and their Gibbs States

Chris Preston

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In 1975 I published *Random Fields* [13], which gave a mathematical treatment of some models used in Statistical Physics. Around 1980 I started to revise and expand these notes, but after about two years and a two hundred page hand-written manuscript I lost all interest in this project. In 1998 I was persuaded to give a course on the stuff, and in order to prepare for the course I TEXed part of the manuscript. The present account is a further reworking of this material.

These notes are not at all suitable for learning about Gibbs states. They are aimed rather at those who are already acquainted with the subject and present a general and rather abstract framework which can be used to treat the usual lattice and particle models in Statistical Physics. However, I have tried to give detailed proofs of all the results, and so in this sense it is not assumed that the reader is an expert in the field.

Anyone (with a mathematical background) wanting to learn about Gibbs states is recommended to start with Georgii's book *Gibbs Measures and Phase Transitions* [11]. Georgii works with explicit models and it is these which show how interesting the subject can be.

The results on measures and kernels which will be needed are taken from *Some Notes on Measure Theory* [14], which should therefore be looked at before reading these notes. This is all the more necessary because our approach (and notation) in [14] is somewhat eccentric. References to [14] are marked with an 'M', thus for example Proposition M.14.4 is a reference to Proposition 14.4 in Chapter 14 of [14].

The choice of material presented here is determined simply by how far I have got with reworking the original manuscript. It is possible that further chapters will appear from time to time.

Chris Preston
Pankow
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Contents

1 A lattice model

In this chapter and the next we look at two examples of structures which are the prototypes for the theory we will develop in the following chapters. The first involves a countable product of measurable spaces and is the basic set-up for dealing with lattice models in statistical mechanics. The second, which is discussed in the following chapter, involves a framework for dealing with models of particles (usually moving in some continuous space such as $\mathbb{R}^d$).

It is assumed that the reader is familiar with the notation employed in *Some Notes on Measure Theory* [14]. References to [14] are marked with an 'M', thus for example Proposition M.14.4 is a reference to Proposition 14.4 in Chapter 14 of [14]. Some of the topics we address are rather technical. For example, we are interested in conditions which ensure that the Kolmogorov extension property holds, and this will be achieved using substandard Borel spaces (our cheap version of standard Borel spaces introduced in Chapter M.18). It is probably enough to start by just glancing through these two chapters and then to return to them after studying Chapter 4.

Let S be a countably infinite set and for each $s \in S$ let $(X_s, \mathcal{F}_s)$ be a measurable space; put $X = \prod_{s\in S} X_s$ and $\mathcal{F} = \prod_{s\in S} \mathcal{F}_s$. Also let $\mathcal{N}$ denote the set of finite subsets of S; thus $\varnothing \in \mathcal{N}$ and both $\Lambda \cup \Delta$ and $\Lambda \setminus \Delta$ are in $\mathcal{N}$ for all Λ, $\Delta \in \mathcal{N}$.

For each $\Lambda \in \mathcal{N}$ let $\mathcal{R}^\Lambda$ denote the set of all measurable rectangles in X having the form $\prod_{s\in S} F_s$ with $F_s \in \mathcal{F}_s$ for each $s \in \Lambda$ and $F_s = X_s$ for all $s \in S \setminus \Lambda$. Put $\mathcal{F}^\Lambda = \sigma(\mathcal{R}^\Lambda)$, thus $\mathcal{F}^\Lambda$ is a sub-σ-algebra of $\mathcal{F}$. In particular, $\mathcal{F}^\varnothing$ is the trivial σ-algebra $\{\varnothing, X\}$.

We consider X as the basic set of configurations and for each $\Lambda \in \mathcal{N}$ the σ-algebra $\mathcal{F}^\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on in Λ.

Proposition 1.1 *$\mathcal{F}^{\Lambda\cup\Delta} = \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$ holds for all Λ, $\Delta \in \mathcal{N}$, where $\mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$ denotes the smallest σ-algebra containing both $\mathcal{F}^\Lambda$ and $\mathcal{F}^\Delta$.*

Proof Clearly $\mathcal{R}^\Lambda \subset \mathcal{R}^{\Lambda\cup\Delta}$ and hence $\mathcal{F}^\Lambda = \sigma(\mathcal{R}^\Lambda) \subset \sigma(\mathcal{R}^{\Lambda\cup\Delta}) = \mathcal{F}^{\Lambda\cup\Delta}$. In the same way $\mathcal{F}^\Delta \subset \mathcal{F}^{\Lambda\cup\Delta}$ and thus $\mathcal{F}^\Lambda \vee \mathcal{F}^\Delta = \sigma(\mathcal{F}^\Lambda \cup \mathcal{F}^\Delta) \subset \sigma(\mathcal{F}^{\Lambda\cup\Delta}) = \mathcal{F}^{\Lambda\cup\Delta}$. But if $R \in \mathcal{R}^{\Lambda\cup\Delta}$ then there exist $R_1 \in \mathcal{R}^\Lambda$ and $R_2 \in \mathcal{R}^\Delta$ with $R = R_1 \cap R_2$ and therefore $R \in \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$, and this implies that $\mathcal{F}^{\Lambda\cup\Delta} \subset \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$. □

Note that if Λ, $\Delta \in \mathcal{N}$ with $\Lambda \subset \Delta$ then by Proposition 1.1 $\mathcal{F}^\Lambda \subset \mathcal{F}^\Delta$, since $\mathcal{F}^\Delta = \mathcal{F}^\Lambda \vee \mathcal{F}^{\Delta\setminus\Lambda}$. Put $\mathcal{A} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{F}^\Lambda$; then $\mathcal{A}$ is an algebra (since the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ is increasing) and $\sigma(\mathcal{A}) = \mathcal{F}$, because $\mathcal{R} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{R}^\Lambda$ is the set of all measurable rectangles in X, and so by definition $\mathcal{F} = \sigma(\mathcal{R})$.

There is another way to obtain the σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$: For each $\Lambda \in \mathcal{N}$ put $X_\diamond^\Lambda = \prod_{s\in\Lambda} X_s$ and $\mathcal{F}_\diamond^\Lambda = \prod_{s\in\Lambda} \mathcal{F}_s$. Let $\mathrm{p}_\Lambda : X \to X_\diamond^\Lambda$ be the projection mapping with $\mathrm{p}_\Lambda(\{x_s\}_{s\in S}) = \{x_s\}_{s\in\Lambda}$. Then clearly $\mathrm{p}_\Lambda^{-1}(\mathcal{R}_\diamond^\Lambda) = \mathcal{R}^\Lambda$, where $\mathcal{R}_\diamond^\Lambda$ denotes the set of measurable rectangles in $X_\diamond^\Lambda$. Therefore by Proposition M.2.4

$$\mathcal{F}^\Lambda = \sigma(\mathcal{R}^\Lambda) = \sigma(\mathrm{p}_\Lambda^{-1}(\mathcal{R}_\diamond^\Lambda)) = \mathrm{p}_\Lambda^{-1}(\sigma(\mathcal{R}_\diamond^\Lambda)) = \mathrm{p}_\Lambda^{-1}(\mathcal{F}_\diamond^\Lambda) ,$$

since by definition $\mathcal{F}_\diamond^\Lambda = \sigma(\mathcal{R}_\diamond^\Lambda)$, i.e., we have $\mathcal{F}^\Lambda = \mathrm{p}_\Lambda^{-1}(\mathcal{F}_\diamond^\Lambda)$.

The mapping $\mathrm{p}_\Lambda : X \to X_\diamond^\Lambda$ is clearly surjective. Thus for each $F \in \mathcal{F}^\Lambda$ there is a unique element $F_\diamond \in \mathcal{F}_\diamond^\Lambda$ with $\mathrm{p}_\Lambda^{-1}(F_\diamond) = F$, and in fact $F_\diamond = \mathrm{p}_\Lambda(F)$ (since if $F = \mathrm{p}_\Lambda^{-1}(F_\diamond)$ then $\mathrm{p}_\Lambda(F) = \mathrm{p}_\Lambda(\mathrm{p}_\Lambda^{-1}(F_\diamond)) = F_\diamond$, because p_Λ is surjective).

Proposition 1.2 *Let $\Lambda,\ \Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$. Then $E \cap F$ is non-empty whenever $E \in \mathcal{F}^\Lambda$ and $F \in \mathcal{F}^\Delta$ are both non-empty.*

Proof Let $E \in \mathcal{F}^\Lambda$ and $F \in \mathcal{F}^\Delta$ be non-empty. Since $\mathcal{F}^\Lambda = \mathrm{p}_\Lambda^{-1}(\mathcal{F}_\diamond^\Lambda)$, there exists a non-empty set $E_\diamond \in \mathcal{F}_\diamond^\Lambda$ with $E = \mathrm{p}_\Lambda^{-1}(E_\diamond)$ and in the same way there exists a non-empty set $F_\diamond \in \mathcal{F}_\diamond^\Delta$ with $F = \mathrm{p}_\Delta^{-1}(F_\diamond)$. Now take $u = \{x_s\}_{s\in\Lambda} \in E_\diamond$ and $v = \{x_s\}_{s\in\Delta} \in F_\diamond$ and then choose arbitrary points x_s, $s \in S \setminus (\Lambda \cup \Delta)$, to obtain an element $x = \{x_s\}_{s\in S} \in X$. Then $\mathrm{p}_\Lambda(x) = u$ and hence $x \in E = \mathrm{p}_\Lambda^{-1}(E_\diamond)$, and in the same way $x \in F = \mathrm{p}_\Delta^{-1}(F_\diamond)$, since $\mathrm{p}_\Delta(x) = v$. Thus $x \in E \cap F$ which shows that $E \cap F$ is non-empty. □

Note that if $\Lambda,\ \Delta \in \mathcal{N}$ with $\Lambda \subset \Delta$ then by Proposition 1.1 (1) $\mathcal{F}^\Lambda \subset \mathcal{F}^\Delta$, since $\mathcal{F}^\Delta = \mathcal{F}^\Lambda \vee \mathcal{F}^{\Delta\setminus\Lambda}$. Put $\mathcal{A} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{F}^\Lambda$; then $\mathcal{A}$ is an algebra (since the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ is increasing) and $\sigma(\mathcal{A}) = \mathcal{F}$, because $\mathcal{R} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{R}^\Lambda$ is the set of all measurable rectangles in X, and so by definition $\mathcal{F} = \sigma(\mathcal{R})$.

For each $\Lambda \in \mathcal{N}$ let $\mu_\Lambda \in \mathrm{P}(\mathcal{F}^\Lambda)$. The family of probability measures $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ is said to be *consistent* if $\mu_\Lambda(F) = \mu_\Delta(F)$ for all $F \in \mathcal{F}^\Lambda$ whenever $\Lambda \subset \Delta$. We say that the family of σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ has the *Kolmogorov extension property* if for each consistent family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ there exists a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}$. If μ exists then it is unique, since it is determined by the family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ on the algebra $\mathcal{A}$ and $\sigma(\mathcal{A}) = \mathcal{F}$.

We are now going to show that the Kolmogorov extension property holds if $(X_s, \mathcal{F}_s)$ is a substandard Borel space for each $s \in S$. (Substandard Borel spaces are defined in Chapter M.18. In particular, by Proposition M.18.1 a standard Borel space is substandard Borel.) As a first step in this direction let us just assume that $(X_s, \mathcal{F}_s)$ is countably generated for each $s \in S$. (The facts that we need about countably generated spaces can be found in Chapter M.16.)

Lemma 1.1 *$(X, \mathcal{F}^\Lambda)$ is countably generated for each $\Lambda \in \mathcal{N}$.*

Proof By Proposition M.16.3 (4) $(X_\diamond^\Lambda, \mathcal{F}_\diamond^\Lambda)$ is countably generated and thus by Proposition M.16.3 (3) $(X, \mathcal{F})$ is countably generated since $\mathrm{p}_\Lambda^{-1}(\mathcal{F}_\diamond^\Lambda) = \mathcal{F}^\Lambda$. □

The next result is crucial for showing that the Kolmogorov extension property holds. It involves atoms, which are defined in Chapter M.16.

Lemma 1.2 *Let $\{\Lambda_n\}_{n\ge 0}$ be an increasing sequence from $\mathcal{N}$ with $S = \bigcup_{n\ge 0} \Lambda_n$ and let $\{A_n\}_{n\ge 0}$ be a decreasing sequence of atoms with $A_n \in \mathrm{A}(\mathcal{F}^{\Lambda_n})$ for each $n \ge 0$. Then $\bigcap_{n\ge 0} A_n \ne \varnothing$.*

Proof For each $n \ge 0$ let $A'_n = \mathrm{p}_{\Lambda_n}(A_n)$, so A'_n is the unique element of $\mathcal{F}_\diamond^{\Lambda_n}$ with $\mathrm{p}_{\Lambda_n}^{-1}(A'_n) = A_n$. Thus by Lemma M.16.4 $A'_n \in \mathrm{A}(\mathcal{F}_\diamond^{\Lambda_n})$. For the moment fix $n \ge 0$ and let $x = \{x_s\}_{s\in\Lambda_n} \in A'_n$, $y = \{y_s\}_{s\in\Lambda_{n+1}} \in A'_{n+1}$. Consider the element $z = \{z_s\}_{s\in\Lambda_{n+1}} \in X_\diamond^{\Lambda_{n+1}}$ with $z_s = x_s$ if $s \in \Lambda_n$ and $z_s = y_s$ when $s \in \Lambda_{n+1} \setminus \Lambda_n$. Then $z \in A'_{n+1}$. (Let $\Delta = \Lambda_{n+1} \setminus \Lambda_n$, and let $B \in \mathrm{A}(\mathcal{F}_\diamond^\Delta)$ be the atom containing $\{y_s\}_{s\in\Delta}$. Therefore $z \in A'_n \times B$ and by Proposition M.16.9 $A'_n \times B \in \mathrm{A}(\mathcal{F}_\diamond^{\Lambda_{n+1}})$, since $\mathcal{F}_\diamond^{\Lambda_{n+1}} = \mathcal{F}_\diamond^{\Lambda_n} \times \mathcal{F}_\diamond^\Delta$. But $y \in (A'_n \times B) \cap A'_{n+1}$, since $A_{n+1} \subset A_n$, which is only possible if $A'_n \times B = A'_{n+1}$ and hence $z \in A'_{n+1}$.) Now for each $n \ge 0$ let $\{x_s^n\}_{s\in\Lambda_n} \in A'_n$. Then there is a unique element $x = \{x_s\}_{s\in S}$ in X such that $x_s = x_s^n$ for all $s \in \Lambda_n \setminus \Lambda_{n-1}$, $n \ge 0$ (with $\Lambda_0 = \varnothing$), and the above observation shows that $\{x_s\}_{s\in\Lambda_n} \in A'_n$ for all $n \ge 0$. But this implies that $x \in A_n$ for all $n \ge 0$ and hence $\bigcap_{n\ge 0} A_n \ne \varnothing$. □

Proposition 1.3 *Suppose that $(X_s, \mathcal{F}_s)$ is substandard Borel for each $s \in S$. Then $(X, \mathcal{F})$ is substandard Borel and the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ has the Kolmogorov extension property.*

Proof By Proposition M.18.2 (2) $(X_\diamond^\Lambda, \mathcal{F}_\diamond^\Lambda)$ is a substandard Borel space and thus by Lemma M.18.2 $(X, \mathcal{F}^\Lambda)$ is substandard Borel for each $\Lambda \in \mathcal{N}$, since p_Λ is surjective. Choose an increasing sequence $\{\Lambda_n\}_{n\ge 0}$ from $\mathcal{N}$ with $S = \bigcup_{n\ge 0} \Lambda_n$. Then the sequence $\{\mathcal{F}^{\Lambda_n}\}_{n\ge 0}$ is increasing with $\mathcal{F} = \sigma(\bigcup_{n\ge 0} \mathcal{F}^{\Lambda_n})$ (since for each $\Lambda \in \mathcal{N}$ there exists $n \ge 0$ with $\Lambda \subset \Lambda_n$). Moreover, Lemma 1.2 says exactly that $\mathcal{F}$ is the inverse limit of the sequence $\{\mathcal{F}^{\Lambda_n}\}_{n\ge 0}$. Therefore by Theorem M.19.1 $(X, \mathcal{F})$ is standard Borel (which also follows from Proposition M.18.2 (2)) and the sequence $\{\mathcal{F}^{\Lambda_n}\}_{n\ge 0}$ has the Kolmogorov extension property. (This means that if $\mu_n \in \mathrm{P}(\mathcal{F}^{\Lambda_n})$ for each $n \ge 0$ and the sequence $\{\mu_n\}_{n\ge 0}$ is consistent in that $\mu_n(F) = \mu_{n+1}(F)$ for all $F \in \mathcal{F}^{\Lambda_n}$, $n \ge 0$, then there exists a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_n(F)$ for all $F \in \mathcal{F}^{\Lambda_n}$, $n \ge 0$.) But then the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ also has the Kolmogorov extension property: Let $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ be a consistent family of measures. Then the sequence $\{\mu_{\Lambda_n}\}_{n\ge 0}$ is consistent and so there exists $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_{\Lambda_n}(F)$ for all $F \in \mathcal{F}^{\Lambda_n}$, $n \ge 0$.

It thus follows that $\mu(F) = \mu_\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}$, since if $\Lambda \in \mathcal{N}$ and $F \in \mathcal{F}^\Lambda$ then there exists $n \geq 0$ with $\Lambda \subset \Lambda_n$, hence $F \in \mathcal{F}^{\Lambda_n}$, which implies that $\mu(F) = \mu_{\Lambda_n}(F) = \mu_\Lambda(F)$. $\square$

We now look at product measures on $(X, \mathcal{F})$ and for this we need no assumptions on the measurable spaces $(X_s, \mathcal{F}_s)$, $s \in S$. For each $s \in S$ let ω_s be a probability measure on $(X_s, \mathcal{F}_s)$ and let ω be the corresponding product measure on $(X, \mathcal{F})$ given by Theorem M.15.3. Thus ω is the unique probability measure such that

$$\omega\Big(\prod_{s\in S} F_s\Big) = \prod_{s\in S} \omega_s(F_s)$$

for each measurable rectangle $\prod_{s\in S} F_s$.

A measure $\mu \in \mathrm{P}(\mathcal{F})$ will be called *independent* if whenever Λ, $\Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$ then $\mu(E \cap F) = \mu(E)\mu(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$. If $\mu \in \mathrm{P}(\mathcal{F})$ is independent then $\mu(fg) = \mu(f)\mu(g)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^\Delta)$ whenever Λ, $\Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$.

Proposition 1.4 *The product measure ω is independent.*

Proof If $E = \prod_{s\in S} E_s \in \mathcal{R}^\Lambda$ and $F = \prod_{s\in S} F_s \in \mathcal{R}^\Delta$ then $E_s = X_s$ for all $s \in S \setminus \Lambda$, $F_s = X_s$ for all $s \in S \setminus \Delta$ and thus $E \cap F = \prod_{s\in S} G_s$, where

$$G_s = \begin{cases} E_s & \text{if } s \in \Lambda\ , \\ F_s & \text{if } s \in \Delta\ , \\ X_s & \text{otherwise}\ . \end{cases}$$

It therefore follows that

$$\omega(E \cap F) = \prod_{s\in S} \omega_s(G_s) = \prod_{s\in S} \omega_s(E_s) \times \prod_{s\in S} \omega_s(F_s) = \omega(E)\omega(F)$$

and thus $\omega(E \cap F) = \omega(E)\omega(F)$ for all $E \in \mathcal{R}^\Lambda$, $F \in \mathcal{R}^\Delta$. Now fix $E \in \mathcal{R}^\Lambda$ and consider the finite measures $F \mapsto \omega(E \cap F)$ and $F \mapsto \omega(E)\omega(F)$ on $(X, \mathcal{F}^\Delta)$. They agree on $\mathcal{R}^\Delta$, which is closed under finite intersections, and so by they agree on $\sigma(\mathcal{R}^\Delta) = \mathcal{F}^\Delta$ (by Proposition M.3.3), i.e., $\omega(E \cap F) = \omega(E)\omega(F)$ for all $E \in \mathcal{R}^\Lambda$, $F \in \mathcal{F}^\Delta$. This time fix $F \in \mathcal{F}^\Delta$ and consider the finite measures $E \mapsto \omega(E \cap F)$ and $E \mapsto \omega(E)\omega(F)$ on $(X, \mathcal{F}^\Lambda)$. The same argument shows they are equal and hence $\omega(E \cap F) = \omega(E)\omega(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$. $\square$

For each $\Lambda \in \mathcal{N}$ let $\mathcal{A}_\Lambda = \bigcup_{\Delta \subset S\setminus\Lambda} \mathcal{F}^\Delta$ (so $\mathcal{A}_\Lambda$ is an algebra) and put $\mathcal{F}_\Lambda = \sigma(\mathcal{A}_\Lambda)$. In particular $\mathcal{F}_\varnothing = X$. We consider the σ-algebra $\mathcal{F}_\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on outside of Λ.

Note that $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda \supset \mathcal{F}^{\Delta\setminus\Lambda} \vee \mathcal{F}^\Lambda = \mathcal{F}^{\Lambda\cup\Delta} \supset \mathcal{F}^\Delta$ for each $\Delta \in \mathcal{N}$, and hence $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda = \mathcal{F}$ for each $\Lambda \in \mathcal{N}$.

In the following again let $\omega \in \mathrm{P}(\mathcal{F})$ be the product of the probability measures $\{\omega_s\}_{s\in S}$. For the moment consider $\Lambda \in \mathcal{N}_0 = \{\Lambda \in \mathcal{N} : \Lambda \neq \varnothing\}$ to be fixed, let $\omega^\Lambda = \prod_{s\in\Lambda} \omega_s \in \mathrm{P}(\mathcal{F}^\Lambda_\diamond)$ be the (finite) product of the measures $\{\omega_s\}_{s\in\Lambda}$ and define a mapping $r_\Lambda : X \times X^\Lambda_\diamond \to X$ by

$$r_\Lambda(\{x_s\}_{s\in S}, \{y_s\}_{s\in\Lambda}) = \begin{cases} x_s & \text{if } s \in S \setminus \Lambda\ , \\ y_s & \text{if } s \in \Lambda\ . \end{cases}$$

If $R \in \mathcal{F}$ is a measurable rectangle then $r_\Lambda^{-1}(R) \subset \mathcal{F}^\Delta \times \mathcal{F}^\Lambda_\diamond$ for some $\Delta \in \mathcal{N}$ with $\Delta \subset S \setminus \Lambda$; thus $r_\Lambda^{-1}(F) \subset \mathcal{F}_\Lambda \times \mathcal{F}^\Lambda_\diamond$ and hence Proposition M.2.4 implies that $r_\Lambda^{-1}(\mathcal{F}) \subset \mathcal{F}_\Lambda \times \mathcal{F}^\Lambda_\diamond$.

Now applying Lemmas M.15.1 and M.15.2 to the present situation shows that there is a probability kernel $\omega^\Lambda_\diamond : \mathrm{M}(\mathcal{F}_\Lambda \times \mathcal{F}^\Lambda_\diamond) \to \mathrm{M}(\mathcal{F}_\Lambda)$ given by

$$\omega^\Lambda_\diamond(g)(x) = \omega^\Lambda(g_x)$$

for all $g \in \mathrm{M}(\mathcal{F}_\Lambda \times \mathcal{F}^\Lambda_\diamond)$, $x \in X$, where g_x is the section defined by $g_x(y) = g(x, y)$ for all $y \in X^\Lambda_\diamond$. Let $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F}_\Lambda)$ be given by $\omega_\Lambda(f) = \omega^\Lambda_\diamond(f \circ r_\Lambda)$, thus

$$\omega_\Lambda(f)(x) = \omega^\Lambda(f(r_\Lambda(x, \cdot)))$$

for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$. Then ω_Λ is linear and continuous and hence it is a probability kernel. This kernel $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F}_\Lambda)$ will also be regarded as a probability kernel $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ such that $\omega_\Lambda(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$ for each $f \in \mathrm{M}(\mathcal{F})$. In order to increase the legibility we will often write $x_{S\setminus\Lambda}y$ instead of $r_\Lambda(x, y)$. With this notation, and exceptionally using an integral sign, we have

$$\omega_\Lambda(f)(x) = \int f(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$.

Lemma 1.3 *For each $x \in X$ the following hold:*

(1) $\omega_\Lambda(x, E \cap \mathrm{p}_\Lambda^{-1}(F)) = I_E(x)\omega^\Lambda(F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda_\diamond$.

(2) $\omega_\Lambda(x, E \cap F) = I_E(x)\omega(F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$.

(3) $\omega_\Lambda(x, E \cap F) = I_E(x)\omega_\Lambda(x, F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}$.

Proof (1) Note that if $E \in \mathcal{F}_\Lambda$ and $F \in \mathcal{F}^\Lambda_\diamond$ then $I_E(r_\Lambda(x, y)) = I_E(x)$ and $I_{F'}(r_\Lambda(x, y)) = I_F(y)$ for all $(x, y) \in X \times X^\Lambda_\diamond$, where $F' = \mathrm{p}_\Lambda^{-1}(F)$. (Consider $(x, y) \in X \times X^\Lambda_\diamond$ to be fixed; then $E \mapsto I_E(r_\Lambda(x, y))$ and $E \mapsto I_E(x)$ are both elements of $\mathrm{P}(\mathcal{F}_\Lambda)$ which agree if E is a measurable rectangle in $\mathcal{F}_\Lambda$ and so by Proposition M.3.4 they are equal. The second statement can be established in the same way.) Let $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda_\diamond$ and put $F' = \mathrm{p}_\Lambda^{-1}(F)$; then

$$\begin{aligned}\omega_\Lambda(x, E \cap F') &= \omega_\Lambda(I_{E\cap F'})(x) = \omega^\Lambda_\diamond(I_{E\cap F'} \circ r_\Lambda)(x) = \omega^\Lambda((I_{E\cap F'} \circ r_\Lambda)_x) \\ &= \omega^\Lambda((I_E \circ r_\Lambda)_x (I_{F'} \circ r_\Lambda)_x) = \omega^\Lambda(I_E(x) I_F) = I_E(x)\omega^\Lambda(F)\end{aligned}$$

for all $x \in X$.

(2) This follows from (1), since if $\omega_{|\Lambda}$ is the restriction of ω to $\mathcal{F}^\Lambda$ then ω^Λ is the image $(\mathrm{p}_\Lambda)_*\omega_{|\Lambda}$ of $\omega_{|\Lambda}$ under p_Λ.

(3) Fix $E \in \mathcal{F}_\Delta$ and $x \in X$. Then it follows from (2) that

$$\omega_\Lambda(x, E \cap F' \cap F) = I_{E\cap F'}(x)\omega_\Lambda(x, F) = I_E(x)\omega_\Lambda(x, F' \cap F)$$

for all $F' \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$. The measures $A \mapsto \omega_\Lambda(x, E\cap A)$ and $A \mapsto I_E(x)\omega_\Lambda(x, A)$ thus agree on all sets of the form $F' \cap F$ with $F' \in \mathcal{F}_\Lambda$ and $F \in \mathcal{F}^\Lambda$ and therefore by Proposition M.3.3 they are equal (since the set of all such sets is closed under finite intersections, contains X and generates the σ-algebra $\mathcal{F}$). This implies that $\omega_\Lambda(x, E \cap F) = I_E(x)\omega_\Lambda(x, F)$ for all $F \in \mathcal{F}$. □

We no longer consider $\Lambda \in \mathcal{N}_0$ to be fixed and therefore we now have a family of probability kernels $\{\omega_\Lambda\}_{\Lambda\in\mathcal{N}_0}$.

Proposition 1.5 *The family of kernels* $\{\omega_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ *is compatible in the sense that* $\omega_\Delta\omega_\Lambda = \omega_\Delta$ *for all* Λ, $\Delta \in \mathcal{N}_0$ *with* $\Lambda \subset \Delta$.

Proof Let Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$. Then for all $E \in \mathcal{F}_\Delta$, $F' \in \mathcal{F}^{\Delta\setminus\Lambda}$ and $F \in \mathcal{F}^\Lambda$ it follows from Lemma 1.3 (2) and Proposition 1.4 that

$$\begin{aligned}(\omega_\Delta\omega_\Lambda)(x, E \cap F' \cap F) &= \omega_\Delta(\omega_\Lambda(I_{E\cap F'\cap F}))(x) = \omega_\Delta(I_{E\cap F'}\omega(F))(x) \\ &= \omega(F)\omega_\Delta(I_{E\cap F'})(x) = \omega(F)I_E(x)\omega(F') \\ &= I_E(x)\omega(F')\omega(F) = I_E(x)\omega(F' \cap F) \\ &= \omega_\Delta(x, E \cap F' \cap F)\end{aligned}$$

for each $x \in X$. Thus for each $x \in X$ the probability measures $(\omega_\Delta\omega_\Lambda)(x, \cdot)$ and $\omega_\Delta(x, \cdot)$ agree on all sets of the form $E \cap F' \cap F$ with $E \in \mathcal{F}_\Delta$, $F' \in \mathcal{F}^{\Delta\setminus\Lambda}$ and $F \in \mathcal{F}^\Lambda$, and therefore by Proposition M.3.3 they are equal (since the set of all such sets is closed under finite intersections, contains X and it generates the σ-algebra $\mathcal{F}$). Hence $(\omega_\Delta\omega_\Lambda)(x, \cdot) = \omega_\Delta(x, \cdot)$ for all $x \in X$, i.e., $\omega_\Delta\omega_\Lambda = \omega_\Delta$. □

Consider a family $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ from $\mathrm{M}(\mathcal{F})$. For each $\Lambda \in \mathcal{N}_0$ define a mapping $\omega^v_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ by letting

$$\omega^v_\Lambda(f)(x) = \begin{cases} (\omega_\Lambda(v_\Lambda)(x))^{-1}\omega_\Lambda(v_\Lambda f)(x) & \text{if } 0 < \omega_\Lambda(v_\Lambda)(x) < \infty\ , \\ 0 & \text{otherwise.}\end{cases}$$

for each $f \in \mathrm{M}(\mathcal{F})$. Those who prefer integral signs can write this as

$$\omega_\Lambda^v(f)(x) = I_{G_\Lambda}(x)\Big(\int v_\Lambda(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y)\Big)^{-1} \int v_\Lambda(x_{S\setminus\Lambda}y) f(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y) ,$$

where $G_\Lambda = \big\{x \in X : 0 < \int v_\Lambda(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y) < \infty\big\}$. Then ω_Λ^v is linear and continuous and is thus a kernel. It is what is called a *quasi-probability kernel*, meaning that for each $x \in X$ the measure $\omega_\Lambda^v(x, \cdot)$ is either a probability measure or 0. Moreover, $\omega_\Lambda^v(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$ for each $f \in \mathrm{M}(\mathcal{F})$ and

$$\omega_\Lambda^v(x, E \cap F) = I_E(x)\omega_\Lambda^v(x, F)$$

for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$, $x \in X$.

The family $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ will be called *multiplicative* if for each Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ there exists $v_{\Delta,\Lambda} \in \mathrm{M}(\mathcal{F}_\Lambda)$ such that $v_\Delta = v_{\Delta,\Lambda} v_\Lambda$.

Proposition 1.6 *If $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is multiplicative then the family of kernels $\{\omega_\Lambda^v\}_{\Lambda\in\mathcal{N}_0}$ is compatible, again in the sense that $\omega_\Delta^v \omega_\Lambda^v = \omega_\Delta^v$ for all Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$.*

Proof This is a special case of Theorem 4.1. □

In order to apply Proposition 1.6 we require a multiplicative family and we now look at how such families typically arise in lattice models in statistical physics.

Multiplicative families are usually obtained by exponentiating additive families, and for this we need the extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$, which we denote by $\mathbb{R}^\Diamond$. These, however, will only ever be used as a parameterisation of $\mathbb{R}^+_\infty$: The mapping $a \mapsto \exp(-a)$ maps $\mathbb{R}$ bijectively onto $\mathbb{R}^+ \setminus \{0\}$, and defining $\exp(-\infty) = 0$ and $\exp(\infty) = \infty$ then gives a bijection between $\mathbb{R}^\Diamond$ and $\mathbb{R}^+_\infty$. Each element of $\mathbb{R}^+_\infty$ thus has a unique representation in the form $\exp(-a)$ with $a \in \mathbb{R}^\Diamond$. One reason for using this representation is that it corresponds to the usual notation in statistical physics. Moreover, it also increases the legibility since it converts multiplicative expressions into additive ones. This is because $\exp(-(a + b)) = \exp(-a)\exp(-b)$ holds for all a, $b \in \mathbb{R}^\Diamond$, provided we define $-\infty + a = a + (-\infty) = -\infty$ for all $a \neq \infty$ and $\infty + a = a + \infty = \infty$ for all $a \in \mathbb{R}^\Diamond$. (The fact that $-\infty + \infty = \infty$ is irrelevant in practice, since in any given situation at most one of the values $-\infty$ and ∞ can occur.)

The set of all mappings from X to $\mathbb{R}^\Diamond$ will be denoted by $\mathrm{M}^\Diamond(X)$. Thus for each $v \in \mathrm{M}^\Diamond(X)$ there is the mapping $\exp(-v) \in \mathrm{M}(X)$, and if u, $v \in \mathrm{M}^\Diamond(X)$ then $\exp(-(u+v)) = \exp(-u)\exp(-v)$. Finally, we need a σ-algebra of subsets of $\mathbb{R}^\Diamond$. Let $\mathcal{S}$ be the subset of $\mathcal{P}(\mathbb{R}^\Diamond)$ consisting of all sets of the form $\{t \in \mathbb{R}^\Diamond : t \le a\}$ with $a \in \mathbb{R}^\Diamond$ and put $\mathcal{B}^\Diamond = \sigma(\mathcal{S})$. If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then the set of all measurable mappings from $(X, \mathcal{E})$ to $(\mathbb{R}^\diamond, \mathcal{B}^\diamond)$ will be denoted by $\mathrm{M}^\diamond(\mathcal{E})$. It is easy to see that if $v \in \mathrm{M}^\diamond(X)$ then $\exp(-v) \in \mathrm{M}(\mathcal{E})$ if and only if $v \in \mathrm{M}^\diamond(\mathcal{E})$.

Now for each $\Lambda \in \mathcal{N}_0$ let $e_\Lambda \in \mathrm{M}^\diamond(\mathcal{F})$; then the family $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is said to be *additive* if for each $\Lambda, \Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ there is a mapping $e_{\Delta,\Lambda} \in \mathrm{M}^\diamond(\mathcal{F}_\Lambda)$ such that $e_\Delta = e_{\Delta,\Lambda} + e_\Lambda$. If $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is additive and $v_\Lambda = \exp(-e_\Lambda)$ for each $\Lambda \in \mathcal{N}_0$ then $v = \{v_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is clearly multiplicative. Additive families can be obtained using potentials.

A *potential* is a family $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$, where $\Phi_\Lambda \in \mathrm{M}^\diamond(X^\Lambda_\diamond, \mathcal{F}^\Lambda_\diamond)$ for each $\Lambda \in \mathcal{N}_0$. Given such a potential Φ then for each $\Lambda \in \mathcal{N}_0$ the corresponding *conditional energy* $E^\Phi_\Lambda : X \to \mathbb{R}^\diamond$ is defined by

$$E^\Phi_\Lambda = \sum^{*}_{\Delta \cap \Lambda \neq \varnothing} \Phi_\Delta \circ \mathrm{p}_\Delta \ ,$$

where $\Delta \cap \Lambda \neq \varnothing$ means $\{\Delta \in \mathcal{N}_0 : \Delta \cap \Lambda \neq \varnothing\}$ and the sum is defined pointwise. Moreover, the following convention has been used: If T is a countable set and $f : T \to \mathbb{R}^\diamond$ is any mapping then

$$\sum^{*}_{t \in T} f(t) = \begin{cases} \sum_{t \in T} f(t) & \text{if } \sum_{t \in T} |f(t)| < \infty \ , \\ \infty & \text{otherwise} \ , \end{cases}$$

(so in particular the value $-\infty$ cannot occur). Note that $\Phi_\Lambda \circ \mathrm{p}_\Delta \in \mathrm{M}^\diamond(X, \mathcal{F}^\Delta)$ for each $\Delta \in \mathcal{N}_0$, which implies that $E^\Phi_\Lambda \in \mathrm{M}^\diamond(X, \mathcal{F})$ for each $\Lambda \in \mathcal{N}_0$.

Proposition 1.7 *If Φ is a potential then the family $\{E^\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is additive.*

Proof Let $\Lambda, \Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ and define $E^\Phi_{\Delta,\Lambda} : X \to \mathbb{R}^\diamond$ by

$$E^\Phi_{\Delta,\Lambda} = \sum^{*}_{\Gamma \cap \Lambda = \varnothing, \Gamma \cap \Delta \neq \varnothing} \Phi_\Gamma \circ \mathrm{p}_\Gamma \ ,$$

where $\Gamma \cap \Lambda = \varnothing$, $\Gamma \cap \Delta \neq \varnothing$ means $\{\Gamma \in \mathcal{N}_0 : \Gamma \cap \Lambda = \varnothing \text{ and } \Gamma \cap \Delta \neq \varnothing\}$. Then $E^\Phi_{\Delta,\Lambda} \in \mathrm{M}^\diamond(X, \mathcal{F}_\Lambda)$ and it is easily checked that $E^\Phi_\Delta = E^\Phi_{\Delta,\Lambda} + E^\Phi_\Lambda$. This implies that the family $\{\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is additive. $\square$

A mapping $u \in \mathrm{M}^\diamond(\mathcal{F})$ will be called *quasi-local* if for each $\varepsilon > 0$ there exists $\Lambda \in \mathcal{N}$ and a bounded mapping $v \in \mathrm{M}^\diamond(\mathcal{F}^\Lambda)$ such that $\|u - v\| \le \varepsilon$, where if $g : Y \to \mathbb{R}^\diamond$ is a mapping then $\|g\| = \sup\{|g(y)| : y \in Y\}$. In particular, a quasi-local mapping is bounded.

Proposition 1.8 *Let* $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ *be a potential with*

$$\sum_{|\Delta|>1,\, s\in\Delta} \|\Phi_\Delta\| < \infty$$

for each $s \in S$*, where* $|\Delta| > 1,\ s \in \Delta$ *means* $\{\Delta \in \mathcal{N}_0 : |\Delta| > 1 \text{ and } s \in \Delta\}$*. Then* E^Φ_Λ *is quasi-local for each* $\Lambda \in \mathcal{N}_0$*.*

Proof For each $\Delta \in \mathcal{N}_0$ put $b_\Delta = \|\Phi_\Delta\|$ and let $\Lambda \in \mathcal{N}_0$. Then

$$\sum_{\Delta\cap\Lambda\neq\varnothing} \|\Phi_\Delta \circ \mathrm{p}_\Delta\| \le \sum_{\Delta\cap\Lambda\neq\varnothing} b_\Delta \le \sum_{s\in\Lambda}\ \sum_{|\Delta|>1,\, s\in\Delta} b_\Delta < \infty$$

and thus $E^\Phi_\Lambda = \sum_{\Delta\cap\Lambda\neq\varnothing} \Phi_\Delta \circ \mathrm{p}_\Delta$, with the convergence uniform and absolute. Let $\varepsilon > 0$; then there exists $\Gamma \in \mathcal{N}_0$ with $\Gamma \supset \Lambda$ such that $\sum_{\Delta\setminus\Gamma\neq\varnothing,\, s\in\Delta} b_\Delta < \varepsilon/|\Lambda|$ for each $s \in \Lambda$. Put $v = \sum_{\Delta\cap\Lambda\neq\varnothing,\, \Delta\subset\Gamma} \Phi_\Delta \circ \mathrm{p}_\Delta$, and the same calculation as above shows that the convergence here is again uniform and absolute, and so in particular v is bounded. Moreover $\Phi_\Delta \circ \mathrm{p}_\Delta \in \mathrm{M}^\Diamond(\mathcal{F}^\Gamma)$ for each $\Delta \subset \Gamma$ and thus $v \in \mathrm{M}^\Diamond(\mathcal{F}^\Gamma)$ (since the result corresponding to Lemma M.9.4 holds). But

$$\begin{aligned} \|E^\Phi_\Lambda - v\| &\le \sum_{\Delta\cap\Lambda\neq\varnothing,\, \Delta\setminus\Gamma\neq\varnothing} \|\Phi_\Delta \circ \mathrm{p}_\Delta\| \le \sum_{s\in\Lambda}\ \sum_{\Delta\setminus\Gamma\neq\varnothing,\, s\in\Delta} \|\Phi_\Delta \circ \mathrm{p}_\Delta\| \\ &\le \sum_{s\in\Lambda}\ \sum_{\Delta\setminus\Gamma\neq\varnothing,\, s\in\Delta} b_\Delta < \sum_{s\in\Lambda} \varepsilon/|\Lambda| = \varepsilon\ , \end{aligned}$$

and this shows that E^Φ_Λ is quasi-local. □

2 A particle model

Let $(S, \mathcal{S})$ be a fixed measurable space and let $\mathcal{N}$ be a non-empty subset of $\mathcal{S}$ such that $\Lambda \cup \Delta$ and $\Lambda \setminus \Delta$ are elements of $\mathcal{N}$ for all $\Lambda,\ \Delta \in \mathcal{N}$. This implies that also $\Lambda \cap \Delta \in \mathcal{N}$ for all $\Lambda,\ \Delta \in \mathcal{N}$, since $\Lambda \cap \Delta = \Lambda \setminus (\Lambda \setminus \Delta)$. We further suppose there exists a countable subset $\mathcal{N}_c$ of $\mathcal{N}$ such that $\bigcup_{\Lambda \in \mathcal{N}_c} \Lambda = S$. This implies that there also exists an increasing sequence $\{\Lambda_n\}_{n \geq 1}$ from $\mathcal{N}$ with $\bigcup_{n \geq 1} \Lambda_n = S$.

Put $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Let X be the set of all measures x on $(S, \mathcal{S})$ taking values only in the set $\mathbb{N}_\infty$ and for which $x(\Lambda) < \infty$ for all $\Lambda \in \mathcal{N}$. For $\Lambda \in \mathcal{N}$ let $\mathcal{S}_\Lambda = \{A \in \mathcal{S} : A \subset \Lambda\}$ and let $\mathcal{R}^\Lambda$ be the set of all subsets of X having the form $\{x \in X : x(A) = n\}$ with $A \in \mathcal{S}_\Lambda$ and $n \in \mathbb{N}$. Let $\mathcal{F}^\Lambda = \sigma(\mathcal{R}^\Lambda)$.

We consider X as the basic set of configurations and for each $\Lambda \in \mathcal{N}$ the σ-algebra $\mathcal{F}^\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on in Λ.

Proposition 2.1 $\mathcal{F}^{\Lambda \cup \Delta} = \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$ *holds for all* $\Lambda,\ \Delta \in \mathcal{N}$.

Proof Clearly $\mathcal{R}^\Lambda \subset \mathcal{R}^{\Lambda \cup \Delta}$, since $\mathcal{S}_\Lambda \subset \mathcal{S}_{\Lambda \cup \Delta}$, and hence $\mathcal{F}^\Lambda \subset \mathcal{F}^{\Lambda \cup \Delta}$. In the same way $\mathcal{F}^\Delta \subset \mathcal{F}^{\Lambda \cup \Delta}$ and thus $\mathcal{F}^\Lambda \vee \mathcal{F}^\Delta = \sigma(\mathcal{F}^\Lambda \cup \mathcal{F}^\Delta) \subset \sigma(\mathcal{F}^{\Lambda \cup \Delta}) = \mathcal{F}^{\Lambda \cup \Delta}$. Now let $R \in \mathcal{R}^{\Lambda \cup \Delta}$; then $R = \{x \in X : x(A) = n\}$ for some $A \in \mathcal{S}_{\Lambda \cup \Delta}$ and some $n \in \mathbb{N}$. Put $B = \Lambda \cap A$ and $B' = (\Delta \setminus \Lambda) \cap A$; then $B \in \mathcal{S}_\Lambda$, $B' \in \mathcal{S}_\Delta$, and since A is the disjoint union of B and B' it follows that

$$R = \bigcup_{k=0}^{n} \{x \in X : x(B) = k\} \cap \{x \in X : x(B') = n - k\} \ .$$

Therefore $R \in \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$, and this implies that $\mathcal{F}^{\Lambda \cup \Delta} \subset \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$. □

Note that if $\Lambda,\ \Delta \in \mathcal{N}$ with $\Lambda \subset \Delta$ then by Proposition 2.1 $\mathcal{F}^\Lambda \subset \mathcal{F}^\Delta$, since $\mathcal{F}^\Delta = \mathcal{F}^\Lambda \vee \mathcal{F}^{\Delta \setminus \Lambda}$. Let $\mathcal{A} = \bigcup_{\Lambda \in \mathcal{N}} \mathcal{F}^\Lambda$; then $\mathcal{A}$ is an algebra (since the family $\{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}}$ is increasing). Put $\mathcal{F} = \sigma(\mathcal{A})$; then $(X, \mathcal{F})$ will be the basic measurable space.

For each $x \in X$, $\Lambda \in \mathcal{N}$ let $x_\Lambda \in X$ be given by $x_\Lambda(B) = x(\Lambda \cap B)$ for each $B \in \mathcal{S}$.

Lemma 2.1 *Let* $\Lambda \in \mathcal{N}$, $F \in \mathcal{F}^\Lambda$ *and* $x \in X$*; then* $x \in F$ *if and only if* $x_\Lambda \in F$.

Proof Let $\mathcal{C} = \{E \subset X : x \in E$ if and only if $x_\Lambda \in E\}$; then it is easily checked that $\mathcal{C}$ is a σ-algebra. But if $A \in \mathcal{S}_\Lambda$ and $n \in \mathbb{N}$ then $\{x \in X : x(A) = n\} \in \mathcal{C}$, thus $\mathcal{R}^\Lambda \subset \mathcal{C}$ and hence $\mathcal{F}^\Lambda \subset \mathcal{C}$. Therefore if $F \in \mathcal{F}^\Lambda$ and $x \in X$ then $x \in F$ if and only if $x_\Lambda \in F$. □

Proposition 2.2 *Let $\Lambda, \Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$. Then $E \cap F$ is non-empty whenever $E \in \mathcal{F}^\Lambda$ and $F \in \mathcal{F}^\Delta$ are both non-empty.*

Proof Let $E \in \mathcal{F}^\Lambda$ and $F \in \mathcal{F}^\Delta$ be non-empty. Choose $x \in E$, $y \in F$ and let $z = x_\Lambda + y_\Delta$, thus $x \in X$ and $z_\Lambda = x_\Lambda$, $z_\Delta = y_\Delta$. But by Lemma 2.1 $z \in E$, since $z_\Lambda = x_\Lambda$ and $x_\Lambda \in E$, and in the same way $z \in F$, since $z_\Delta = y_\Delta$ and $y_\Delta \in F$. Thus $z \in E \cap F$ and so $E \cap F$ is non-empty. □

As in the lattice model there is another way to obtain the σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}}$. Put $\mathcal{N}_0 = \{\Lambda \in \mathcal{N} : \Lambda \neq \varnothing\}$; since $\mathcal{F}^\varnothing = \{\varnothing, X\}$ is the trivial σ-algebra there is no harm in restricting our attention to $\{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}_0}$. For each $\Lambda \in \mathcal{N}_0$ let $\mathcal{S}_{|\Lambda}$ be the trace σ-algebra of $\mathcal{S}$ on Λ. Since $\Lambda \in \mathcal{S}$ this means $\mathcal{S}_{|\Lambda}$ is the σ-algebra of subsets of Λ consisting of those sets $B \in \mathcal{S}$ with $B \subset \Lambda$. (Note the subtle difference between $\mathcal{S}_\Lambda$ and $\mathcal{S}_{|\Lambda}$; they both consist of the same sets, but in the former these sets are considered as subsets of S and in the latter as subsets of Λ.)

Let $X^\Lambda_\diamond$ be the set of all measures x on $(\Lambda, \mathcal{S}_{|\Lambda})$ taking values only in the set $\mathbb{N}$, let $\mathcal{R}^\Lambda_\diamond$ be the set of all subsets of $X^\Lambda_\diamond$ having the form $\{x \in X^\Lambda_\diamond : x(A) = n\}$ with $A \in \mathcal{S}_{|\Lambda}$ and $n \in \mathbb{N}$, and put $\mathcal{F}^\Lambda_\diamond = \sigma(\mathcal{R}^\Lambda_\diamond)$. (Using the notation introduced in Chapter M.13 this means that $(X^\Lambda_\diamond, \mathcal{F}^\Lambda_\diamond)$ is the measurable space $(\Lambda_\triangleleft, (\mathcal{S}_{|\Lambda})_\triangleleft)$.)

Define a mapping $\mathrm{p}_\Lambda : X \to X^\Lambda_\diamond$ by $\mathrm{p}_\Lambda(x)\,(B) = x(B)$ for each $B \in \mathcal{S}_{|\Lambda}$.

Lemma 2.2 *The mapping p_Λ is surjective and $\mathrm{p}_\Lambda^{-1}(\mathcal{F}^\Lambda_\diamond) = \mathcal{F}^\Lambda$.*

Proof If $x \in X^\Lambda_\diamond$ then $x = \mathrm{p}_\Lambda(x')$, where x' is the element of X defined by $x'(B) = x(B \cap \Lambda)$ for each $B \in \mathcal{S}$; thus p_Λ is surjective. Now let $A \in \mathcal{S}_{|\Lambda}$ (and so also $A \in \mathcal{S}_\Lambda$) and let $n \in \mathbb{N}$; since $\mathrm{p}_\Lambda(x)(A) = x(A)$ for all $x \in X$ it follows that

$$\mathrm{p}_\Lambda^{-1}(\{y \in X^\Lambda_\diamond : y(A) = n\}) = \{x \in X : x(A) = n\}\ .$$

Hence $\mathrm{p}_\Lambda^{-1}(\mathcal{F}^\Lambda_\diamond) = \mathcal{F}^\Lambda$ and therefore by Proposition M.2.4

$$\mathcal{F}^\Lambda = \sigma(\mathcal{R}^\Lambda) = \sigma(\mathrm{p}_\Lambda^{-1}(\mathcal{F}^\Lambda_\diamond)) = \mathrm{p}_\Lambda^{-1}(\sigma(\mathcal{F}^\Lambda_\diamond)) = \mathrm{p}_\Lambda^{-1}(\mathcal{F}^\Lambda_\diamond)\ . \ \square$$

For each $\Lambda \in \mathcal{N}$ let $\mu_\Lambda \in \mathrm{P}(\mathcal{F}^\Lambda)$. Exactly as in the lattice model the family of probability measures $\{\mu_\Lambda\}_{\Lambda \in \mathcal{N}}$ is said to be *consistent* if $\mu_\Lambda(F) = \mu_\Delta(F)$ for all $F \in \mathcal{F}^\Lambda$ whenever $\Lambda \subset \Delta$. Also as before the family of σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}}$ has the *Kolmogorov extension property* if for each consistent family $\{\mu_\Lambda\}_{\Lambda \in \mathcal{N}}$ there exists a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}$. If μ exists then it is unique, since it is determined by the family $\{\mu_\Lambda\}_{\Lambda \in \mathcal{N}}$ on the algebra $\mathcal{A}$ and $\sigma(\mathcal{A}) = \mathcal{F}$.

We are now going to show that if $(S, \mathcal{S})$ is a substandard Borel space then $(X, \mathcal{F})$ is substandard Borel and the family of σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}}$ has the Kolmogorov extension property. As a first step in this direction let us just assume that $(S, \mathcal{S})$ is countably generated.

Lemma 2.3 *$(X, \mathcal{F}^\Lambda)$ is countably generated for each $\Lambda \in \mathcal{N}$.*

Proof By Proposition M.16.3 (2) $(\Lambda, \mathcal{S}_{|\Lambda})$ is countably generated and thus by Proposition M.16.3 (6) $(X_\diamond^\Lambda, \mathcal{F}_\diamond^\Lambda)$ is countably generated. Hence by Lemma 2.2 and Proposition M.16.3 (3) $(X, \mathcal{F}^\Lambda)$ is countably generated. □

Lemma 2.4 *Let $\{\Lambda_n\}_{n\geq 0}$ be an increasing sequence from $\mathcal{N}$ with $S = \bigcup_{n\geq 0} \Lambda_n$ and let $\{A_n\}_{n\geq 0}$ be a decreasing sequence of atoms with $A_n \in \mathrm{A}(\mathcal{F}^{\Lambda_n})$ for each $n \geq 0$. Then $\bigcap_{n\geq 0} A_n \neq \varnothing$.*

Proof For the moment fix $n \geq 0$ and let $x_n \in A_n$ with $x_n(S) = x_n(\Lambda_n)$ (and there exists such an element, since by Lemma 2.1 $x_{\Lambda_n} \in A_n$ for all $x \in A_n$). Let $y \in A_{n+1}$ and consider the element $x_{n+1} = x_n + y_\Delta$, where $\Delta = \Lambda_{n+1} \setminus \Lambda_n$; then $x_{n+1}(S) = x_{n+1}(\Lambda_{n+1})$ and $(x_{n+1})_{\Lambda_n} = x_n$. We will show that $x_{n+1} \in A_{n+1}$: The element y_Δ is in $\mathcal{F}^\Delta$, so let $B \in \mathrm{A}(\mathcal{F}^\Delta)$ be the atom containing it. Then by Lemma 2.1 $y \in B$ and hence $y \in A_{n+1} \cap B \subset A_n \cap B$. Proposition M.16.10 thus implies that $A_n \cap B \in \mathrm{A}(\mathcal{F}^{\Lambda_n} \vee \mathcal{F}^\Delta) = \mathrm{A}(\mathcal{F}^{\Lambda_{n+1}})$. But $y \in (A_n \cap B) \cap A_{n+1}$, which is only possible if $A_n \cap B = A_{n+1}$. In particular $x_{n+1} \in A_n \cap B = A_{n+1}$.

This shows that, given $x_n \in A_n$ with $x_n(S) = x_n(\Lambda_n)$, there exists $x_{n+1} \in A_{n+1}$ with $x_{n+1}(S) = x_{n+1}(\Lambda_{n+1})$ and $(x_{n+1})_{\Lambda_n} = x_n$. Therefore by induction we can construct a sequence $\{x_n\}_{n\geq 0}$ from X such that $x_n \in A_n$ with $x_n(S) = x_n(\Lambda_n)$ and $(x_{n+1})_{\Lambda_n} = x_n$ for all $n \geq 0$. There then exists $x \in X$ such that $x_{\Lambda_n} = x_n$ for all $n \geq 0$. Thus by Lemma 2.1 $x \in A_n$ for all $n \geq 0$ and hence $\bigcap_{n\geq 0} A_n \neq \varnothing$. □

Proposition 2.3 *Suppose that $(S, \mathcal{S})$ is substandard Borel. Then $(X, \mathcal{F})$ is substandard Borel and the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ has the Kolmogorov extension property.*

Proof By Proposition M.18.2 (1) $(\Lambda, \mathcal{S}_{|\Lambda})$ is a substandard Borel space, thus by Proposition M.18.2 (4) $(X_\triangleleft^\Lambda, \mathcal{F}_\triangleleft^\Lambda)$ is substandard Borel and so Lemmas M.18.2 and 2.2 imply that $(X^\Lambda, \mathcal{F}^\Lambda)$ is substandard Borel for each $\Lambda \in \mathcal{N}$, since p_Λ is surjective. The proof is now exactly the same as the proof of Proposition 1.3, but using Lemma 2.4 instead of Lemma 1.2. □

We next show how to construct probability measures on $(X, \mathcal{F})$ which are the analogues of the product measures in the lattice model. These measures are the Poisson point processes.

Let $\Lambda \in \mathcal{N}_0$ and for each $n \geq 0$ let $(\Lambda^n, \mathcal{S}_{|\Lambda}^n)$ denote the n-fold product of the measurable space $(\Lambda, \mathcal{S}_{|\Lambda})$ with itself. (In particular this means $(\Lambda^0, \mathcal{S}_{|\Lambda}^0)$ is a one-point measurable space.) Let $(\Omega_\Lambda, \mathcal{G}_\Lambda)$ be the disjoint union of the spaces $(\Lambda^n, \mathcal{S}_{|\Lambda}^n)$, $n \geq 0$; in other words, Ω_Λ is the disjoint union of the sets Λ^n, $n \geq 0$, and $\mathcal{G}_\Lambda$ is the σ-algebra $\{F \subset \Omega_\Lambda : F \cap \Lambda^n \in \mathcal{S}^n_{|\Lambda}$ for each $n \geq 0\}$. A mapping $q_\Lambda : \Omega_\Lambda \to X$ can be defined by letting

$$q_\Lambda((s_1, \ldots, s_n)) = \sum_{j=1}^{n} \varepsilon_{s_j}$$

if $(s_1, \ldots, s_n) \in \Lambda^n$ with $n \geq 1$, and letting the image of the single point in Λ^0 be the zero measure on $(S, \mathcal{S})$.

Lemma 2.5 *The mapping* $q_\Lambda : (\Omega_\Lambda, \mathcal{G}_\Lambda) \to (X, \mathcal{F}^\Lambda)$ *is measurable.*

Proof Let $A \in \mathcal{S}_\Lambda$ and $k \in \mathbb{N}$; then

$$\begin{aligned} q_\Lambda^{-1}(\{x \in X : x(A) = k\}) \cap \Lambda^n \\ = \{(s_1, \ldots, s_n) \in \Lambda^n : |\{1 \leq j \leq n : s_j \in A\}| = k\} \end{aligned}$$

for each $n \geq 0$, and this subset of Λ^n is clearly an element of $\mathcal{S}^n_{|\Lambda}$. It follows that $q_\Lambda^{-1}(\{x \in X : x(\Delta) = k\}) \in \mathcal{G}_\Lambda$, and thus that $q_\Lambda^{-1}(\mathcal{R}^\Lambda) \subset \mathcal{G}_\Lambda$. Therefore by Proposition M.2.4 $q_\Lambda^{-1}(\mathcal{F}^\Lambda) \subset \mathcal{G}_\Lambda$. $\square$

Let ϱ be a measure on $(S, \mathcal{S})$ with $\varrho(\Lambda) < \infty$ for each $\Lambda \in \mathcal{N}$. For each $\Lambda \in \mathcal{N}_0$ let ϱ_Λ be the restriction of ϱ to $(\Lambda, \mathcal{S}_{|\Lambda})$, and for each $n \geq 0$ let ϱ^n_Λ be the the n-fold product of ϱ_Λ with itself as a measure on $(\Lambda^n, \mathcal{S}^n_{|\Lambda})$. (In particular this means ϱ^0_Λ is the unit mass on the single point in Λ^0.) Define $\varrho^\natural_\Lambda \in \mathrm{P}(\mathcal{G}_\Lambda)$ by

$$\varrho^\natural_\Lambda(F) = \exp(-\varrho(\Lambda)) \sum_{n \geq 0} \frac{1}{n!} \varrho^n_\Lambda(F \cap \Lambda^n)$$

for each $F \in \mathcal{G}_\Lambda$; and let $\omega^\Lambda_\varrho = (q_\Lambda)_* \varrho^\natural_\Lambda$ be the image measure of $\varrho^\natural_\Lambda$ under q_Λ; thus ω^Λ_ϱ is the element of $\mathrm{P}(\mathcal{F}^\Lambda)$ defined by $\omega^\Lambda_\varrho(F) = \varrho^\natural_\Lambda(q_\Lambda^{-1}(F))$ for each $F \in \mathcal{F}^\Lambda$.

Proposition 2.4 *The measures* $\{\omega^\Lambda_\varrho\}_{\Lambda \in \mathcal{N}_0}$ *are consistent (i.e., if* $\Lambda, \Delta \in \mathcal{N}_0$ *with* $\Lambda \subset \Delta$ *then* $\omega^\Lambda_\varrho(F) = \omega^\Delta_\varrho(F)$ *for all* $F \in \mathcal{F}^\Lambda$*). Moreover, if* $\Lambda, \Delta \in \mathcal{N}_0$ *with* $\Lambda \cap \Delta = \varnothing$ *then* $\omega^{\Lambda \cup \Delta}_\varrho(E \cap F) = \omega^\Lambda_\varrho(E) \omega^\Delta_\varrho(F)$ *for all* $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$.

Proof For each $h : S \to \mathbb{R}^+$ let $\mathrm{supp}(h) = \{s \in S : h(s) \neq 0\}$ and for $\Lambda \in \mathcal{N}$ let B_Λ denote the set of bounded measurable mappings $h : (S, \mathcal{S}) \to (\mathbb{R}^+_\infty, \mathcal{B}^+_\infty)$ with $\mathrm{supp}(h) \subset \Lambda$. Put $\mathsf{B} = \bigcup_{\Lambda \in \mathcal{N}} \mathsf{B}_\Lambda$ and for each $h \in \mathsf{B}$ define $\xi_h : X \to \mathbb{R}^+$ by letting $\xi_h(x) = x(h)$ for each $x \in X$ (noting that $x(h) < \infty$). If $h \in \mathsf{B}_\Lambda$ then ξ_h is a measurable mapping from $(X, \mathcal{F}^\Lambda)$ to $(\mathbb{R}^+_\infty, \mathcal{B}^+_\infty)$. We need the following uniqueness result for Laplace transforms:

Lemma 2.6 *If ν_1 and ν_2 are probability measures on $(X, \mathcal{F}^\Lambda)$ such that*

$$\nu_1(\exp(-\xi_h)) = \nu_2(\exp(-\xi_h))$$

for all $h \in \mathsf{B}_\Lambda$ then $\nu_1 = \nu_2$.

Proof For each $A \in \mathcal{S}_\Lambda$ define $n_\Lambda : X \to \mathbb{N}$ by $n_\Lambda(x) = x(A)$; thus $\xi_{I_A} = n_\Lambda$. Let $\mathcal{T}$ be the set of all finite intersections of elements from $\mathcal{R}^\Lambda$. Then $\mathcal{T}$ is closed under finite intersections and $\sigma(\mathcal{T}) = \sigma(\mathcal{R}^\Lambda) = \mathcal{F}^\Lambda$, and hence by Proposition M.3.3 it is enough to show that $\nu_1(F) = \nu_2(F)$ for all $F \in \mathcal{T}$. Let $A_1, \ldots, A_n \in \mathcal{S}_\Lambda$; then for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$ the mapping $h_\lambda = \sum_{k=1}^n \lambda_k I_{A_k}$ is in B_Λ and $\xi_{h_\lambda} = \sum_{k=1}^n \lambda_k n_{A_k}$. Therefore

$$\nu_1\Big(\exp\Big(-\sum_{k=1}^n \lambda_k n_{A_k}\Big)\Big) = \nu_2\Big(\exp\Big(-\sum_{k=1}^n \lambda_k n_{A_k}\Big)\Big)$$

for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$. Thus by the uniqueness theorem for Laplace transforms in $(\mathbb{R}^+)^n$ (see, for example, Feller [9], Chapter XIII) the mappings $n_{A_1}, \ldots, n_{A_n}$ have the same joint distribution with respect to ν_1 and ν_2, meaning that for all $p_1, \ldots, p_n \in \mathbb{N}^n$

$$\begin{aligned}\nu_1(\{x \in X : n_{A_k}(x) = p_k \text{ for } k = 1, \ldots, n\}) \\ = \nu_2(\{x \in X : n_{A_k}(x) = p_k \text{ for } k = 1, \ldots, n\}) .\end{aligned}$$

But this says exactly that $\nu_1(F) = \nu_2(F)$ for all $F \in \mathcal{T}$. □

Lemma 2.7 *For each $\Lambda \in \mathcal{N}_0$ and each $h \in \mathsf{B}_\Lambda$*

$$\omega_\varrho^\Lambda(\exp(-\xi_h)) = \exp(-\varrho_\Lambda(1 - \exp(-h))) .$$

Proof Let $\eta_h = \xi_h \circ q_\Lambda$. Then by Lemma 2.5 $\eta_h \in \mathrm{M}(\mathcal{G}_\Lambda)$ and hence

$$\omega_\varrho^\Lambda(\exp(-\xi_h)) = ((q_\Lambda)_* \varrho_\Lambda^\natural)(\exp(-\xi_h)) = \varrho_\Lambda^\natural(\exp(-\xi_h) \circ q_\Lambda) = \varrho_\Lambda^\natural(\exp(-\eta_h)) .$$

But if $(s_1, \ldots, s_n) \in \Lambda^n$ with $n \geq 1$ then $\eta_h((s_1, \ldots, s_n)) = \sum_{j=1}^n h(s_j)$, and so $\eta_h = \sum_{j=1}^n h \circ \mathrm{p}_j^n$, where $\mathrm{p}_j^n : \Lambda^n \to \Lambda \subset S$ is the projection onto the j th component. Thus

$$\begin{aligned}&\varrho_\Lambda^\natural(\exp(-\eta_h)) \\ &\quad = \exp(-\varrho(\Lambda)) \Big(1 + \sum_{n \geq 1} \frac{1}{n!} \varrho_\Lambda^n \Big(\exp\big(-\sum_{j=1}^n h \circ \mathrm{p}_j^n\big)\Big)\Big) \\ &\quad = \exp(-\varrho(\Lambda)) \Big(1 + \sum_{n \geq 1} \frac{1}{n!} \varrho_\Lambda^n \Big(\prod_{j=1}^n \exp(-h \circ \mathrm{p}_j^n)\Big)\Big) \\ &\quad = \exp(-\varrho(\Lambda)) \Big(1 + \sum_{n \geq 1} \frac{1}{n!} \big(\varrho_\Lambda(\exp(-h))\big)^n\Big) \\ &\quad = \exp(-\varrho(\Lambda)) \exp(\varrho_\Lambda(\exp(-h))) = \exp(-\varrho_\Lambda(1 - \exp(-h))) .\end{aligned}$$

□

Note that if $\Lambda \in \mathcal{N}_0$ and $h \in \mathsf{B}_\Lambda$ then $1 - \exp(-h) = 0$ on $S \setminus \Lambda$, and hence the statement in Lemma 2.7 can be written as

$$\omega_\varrho^\Lambda(\exp(-\xi_h)) = \exp(-\varrho(1 - \exp(-h))) \ .$$

Now let Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$. Then $\mathsf{B}_\Lambda \subset \mathsf{B}_\Delta$ and so for all $h \in \mathsf{B}_\Lambda$

$$\omega_\varrho^\Lambda(\exp(-\xi_h)) = \exp(-\varrho(1 - \exp(-h))) = \omega_\varrho^\Delta(\exp(-\xi_h)) \ .$$

Therefore Lemma 2.6 implies that ω_ϱ^Λ is equal to the restriction of ω_ϱ^Δ to $\mathcal{F}^\Lambda$, i.e., $\omega_\varrho^\Delta(F) = \omega_\varrho^\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, which shows that the measures $\{\omega_\varrho^\Lambda\}_{\Lambda \in \mathcal{N}_0}$ are consistent.

Lemma 2.8 *Let* Λ, $\Delta \in \mathcal{N}_0$ *with* $\Lambda \cap \Delta = \varnothing$. *Then*

$$\omega_\varrho^{\Lambda\cup\Delta}(E \cap F) = \omega_\varrho^\Lambda(E)\omega_\varrho^\Delta(F)$$

for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$.

Proof By Proposition 2.1 $\mathcal{F}^{\Lambda\cup\Delta} = \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$ and by Proposition 2.2 $E \cap F$ is non-empty whenever $E \in \mathcal{F}^\Lambda$ and $F \in \mathcal{F}^\Delta$ are both non-empty. Therefore by Theorem M.15.4 there exists a unique probability measure ν on $(X, \mathcal{F}^{\Lambda\cup\Delta})$ such that $\nu(E \cap F) = \omega_\varrho^\Lambda(E)\omega_\varrho^\Delta(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$, and it then follows that $\nu(fg) = \omega_\varrho^\Lambda(f)\omega_\varrho^\Delta(g)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^\Delta)$.

Let $h \in \mathsf{B}_{\Lambda\cup\Delta}$; then there exist unique mappings $h_1 \in \mathsf{B}_\Lambda$ and $h_2 \in \mathsf{B}_\Delta$ such that $h = h_1 + h_2$. Therefore by Lemma 2.7

$$\begin{aligned}
\nu(\exp(-\xi_h)) &= \nu(\exp(-\xi_{h_1+h_2})) = \nu(\exp(-(\xi_{h_1} + \xi_{h_2}))) \\
&= \nu(\exp(-\xi_{h_1})\exp(-\xi_{h_2})) = \omega_\varrho^\Lambda(\exp(-\xi_{h_1}))\omega_\varrho^\Delta(\exp(-\xi_{h_2})) \\
&= \exp(-\varrho(1 - \exp(-h_1)))\exp(-\varrho(1 - \exp(-h_2))) \\
&= \exp(-\varrho(1 - \exp(-h_1)) - \varrho(1 - \exp(-h_2))) \\
&= \exp(-\varrho((1 - \exp(-h_1)) + (1 - \exp(-h_2)))) \\
&= \exp(-\varrho(1 - \exp(-h_1 - h_2))) = \exp(-\varrho(1 - \exp(-h))) \\
&= \omega_\varrho^{\Lambda\cup\Delta}(\exp(-\xi_h)) \ ,
\end{aligned}$$

noting that $(1 - \exp(-h_1)) + (1 - \exp(-h_2)) = (1 - \exp(-h_1)\exp(-h_2))$ holds because $\exp(-h_1) = 1$ on $S \setminus \Lambda$ and $\exp(-h_2) = 1$ on $S \setminus \Delta$. Thus by Lemma 2.6 $\nu = \omega_\varrho^{\Lambda\cup\Delta}$ and hence $\omega_\varrho^{\Lambda\cup\Delta}(E \cap F) = \nu(E \cap F) = \omega_\varrho^\Lambda(E)\omega_\varrho^\Delta(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$. □

This completes the proof of Proposition 2.4. □

Assume now that $(S, \mathcal{S})$ is substandard Borel. Then by Proposition 2.3 there exists a unique measure $\omega_\varrho \in \mathrm{P}(\mathcal{F})$ such that $\omega_\varrho(F) = \omega_\varrho^\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}_0$. Thus by Lemma 2.7 ω_ϱ is the unique element of $\mathrm{P}(\mathcal{F})$ such that

$$\omega_\varrho(\exp(-\xi_h)) = \exp(-\varrho(1 - \exp(-h)))$$

for all $h \in \mathsf{B}$. Moreover, if $\Lambda,\ \Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$ then

$$\omega_\varrho(E \cap F) = \omega_\varrho(E)\omega_\varrho(F)$$

for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$.

The measure ω_ϱ is called the *Poisson point process* corresponding to ϱ. Note that if $\Lambda \in \mathcal{N}_0$ and $n \geq 0$ then

$$\begin{aligned}\omega_\varrho(\{x \in X : x(\Lambda) = n\}) &= \varrho_\Lambda^\natural(\Lambda^n) \\ &= \exp(-\varrho(\Lambda))\frac{1}{n!}\varrho_\Lambda^n(\Lambda^n) = \exp(-\varrho(\Lambda))\frac{1}{n!}(\varrho(\Lambda))^n \ ,\end{aligned}$$

since $\Lambda^n = q_\Lambda^{-1}(\{x \in X : x(\Lambda) = n\})$. This means that, with respect to ω_ϱ, the random variable n_Λ defined by $n_\Lambda(x) = x(\Lambda)$ has a Poisson distribution with parameter $\varrho(\Lambda)$.

A measure $\omega \in \mathrm{P}(\mathcal{F})$ will be called *independent* if whenever $\Lambda,\ \Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$ then $\omega(E \cap F) = \omega(E)\omega(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$. If $\omega \in \mathrm{P}(\mathcal{F})$ is independent then $\omega(fg) = \omega(f)\omega(g)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^\Delta)$ whenever $\Lambda,\ \Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$.

Proposition 2.5 *The Poisson point process ω_ϱ is independent.*

Proof This follows immediately from Lemma 2.8. □

We should point out that the measure ω_ϱ exists without the assumption that $(S, \mathcal{S})$ is substandard Borel. Since the measures $\{\omega_\varrho^\Lambda\}_{\Lambda \in \mathcal{N}_0}$ are consistent there exists a mapping $\omega_\varrho : \mathcal{A} \to \mathbb{R}^+$ with $\omega_\varrho(F) = \omega_\varrho^\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}_0$, and ω_ϱ is clearly a finitely additive measure on $(X, \mathcal{A})$. It can then be shown directly (with $(S, \mathcal{S})$ a general measurable space but with the assumptions made about $\mathcal{N}$ still in force) that ω_ϱ is countably additive and thus it extends to a probability measure on $(X, \mathcal{F})$. The reason this is possible is because ω_ϱ can be realised as a countable product of probability measures and such a product exists without any assumptions on the underlying measurable spaces (see Theorem M.15.3).

For each $\Lambda \in \mathcal{N}$ let $\mathcal{A}_\Lambda = \bigcup_{\Delta \subset S \setminus \Lambda} \mathcal{F}^\Delta$ (so $\mathcal{A}_\Lambda$ is an algebra) and put $\mathcal{F}_\Lambda = \sigma(\mathcal{A}_\Lambda)$. In particular $\mathcal{F}_\varnothing = X$. We consider the σ-algebra $\mathcal{F}_\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on outside of Λ.

Note that $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda \supset \mathcal{F}^{\Delta\setminus\Lambda} \vee \mathcal{F}^\Lambda = \mathcal{F}^{\Lambda\cup\Delta} \supset \mathcal{F}^\Delta$ for each $\Delta \in \mathcal{N}$, and hence $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda = \mathcal{F}$ for each $\Lambda \in \mathcal{N}$.

In the following we continue to work with the Poisson point process corresponding to ϱ but now denote this measure just by ω. For the moment consider $\Lambda \in \mathcal{N}_0$ to be fixed, let $\omega^\Lambda \in \mathrm{P}(\mathcal{F}_\diamond^\Lambda)$ be the image measure $(\mathrm{p}_\Lambda)_*\omega_{|\Lambda}$ of $\omega_{|\Lambda}$ under p_Λ, where $\omega_{|\Lambda}$ is the restriction of ω to $\mathcal{F}^\Lambda$. Define a mapping $r_\Lambda : X \times X_\diamond^\Lambda \to X$ by

$$r_\Lambda(x, y) = x_{S\setminus\Lambda} + i_\Lambda(y) ,$$

where $x_{S\setminus\Lambda}(B) = x(B \setminus \Lambda)$ and $i_\Lambda(y)(B) = y(B \cap \Lambda)$ for all $B \in \mathcal{S}$. If $A \in \mathcal{N}_\Delta$ for some $\Delta \in \mathcal{N}$ and $n \in \mathbb{N}$ then

$$\begin{aligned} r_\Lambda^{-1}(\{x \in X : x(A) = n\}) \\ &= \bigcup_{k=0}^{n} r_\Lambda^{-1}(\{x \in X : x(A \setminus \Lambda) = k\} \cap \{x \in X : x(A \cap \Lambda) = n-k\}) \\ &= \bigcup_{k=0}^{n} \{x \in X : x(A \setminus \Lambda) = k\} \times \{y \in X_\diamond^\Lambda : y(A \cap \Lambda) = n-k\}) , \end{aligned}$$

which is an element of $\mathcal{F}^\Delta \times \mathcal{F}_\diamond^\Lambda \subset \mathcal{F} \times \mathcal{F}_\diamond^\Lambda$ and hence Proposition M.2.4 implies that $r_\Lambda^{-1}(\mathcal{F}) \subset \mathcal{F}_\Lambda \times \mathcal{F}_\diamond^\Lambda$.

Now applying Lemmas M.15.1 and M.15.2 to the present situation shows that there is a probability kernel $\omega_\diamond^\Lambda : \mathrm{M}(\mathcal{F}_\Lambda \times \mathcal{F}_\diamond^\Lambda) \to \mathrm{M}(\mathcal{F}_\Lambda)$ given by

$$\omega_\diamond^\Lambda(g)(x) = \omega^\Lambda(g_x)$$

for all $g \in \mathrm{M}(\mathcal{F}_\Lambda \times \mathcal{F}_\diamond^\Lambda)$, $x \in X$, where g_x is the section defined by $g_x(y) = g(x, y)$ for all $y \in X_\diamond^\Lambda$. Let $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F}_\Lambda)$ be given by $\omega_\Lambda(f) = \omega_\diamond^\Lambda(f \circ r_\Lambda)$, thus

$$\omega_\Lambda(f)(x) = \omega^\Lambda(f(r_\Lambda(x, \cdot)))$$

for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$. Then ω_Λ is linear and continuous and hence it is a probability kernel. This kernel $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F}_\Lambda)$ will also be regarded as a probability kernel $\omega_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ such that $\omega_\Lambda(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$ for each $f \in \mathrm{M}(\mathcal{F})$. In order to increase the legibility we will often write $x_{S\setminus\Lambda}y$ instead of $r_\Lambda(x, y)$. With this notation, and exceptionally using an integral sign, we have

$$\omega_\Lambda(f)(x) = \int f(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$.

Lemma 2.9 *For each $x \in X$ the following hold:*

(1) $\omega_\Lambda(x, E \cap \mathrm{p}_\Lambda^{-1}(F)) = I_E(x)\omega^\Lambda(F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}_\diamond^\Lambda$.

(2) $\omega_\Lambda(x, E \cap F) = I_E(x)\omega(F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$.

(3) $\omega_\Lambda(x, E \cap F) = I_E(x)\omega_\Lambda(x, F)$ for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}$.

Proof (1) Let $(x, y) \in X \times X_\diamond^\Lambda$, $F \in \mathcal{F}_\diamond^\Lambda$ and put $F' = \mathrm{p}_\Lambda^{-1}(F)$, thus $F' \in \mathcal{F}^\Lambda$. Then by Lemma 2.1 $r_\Lambda(x, y) \in F'$ if and only if $(r_\Lambda(x, y))_\Lambda = (i_\Lambda(y))_\Lambda \in F'$ and $(i_\Lambda(y))_\Lambda \in F'$ if and only if $y \in F$. Hence $I_{F'}(r_\Lambda(x, y)) = I_F(y)$. Let $E \in \mathcal{F}_\Lambda$; then the proof of Lemma 2.1 shows that $z \in E$ if and only if $z_{S\setminus\Lambda} \in E$, and thus $I_E(r_\Lambda(x, y)) = I_E(x)$, since $(r_\Lambda(x, y))_{S\setminus\Lambda} = x_{S\setminus\Lambda}$. With these properties the proof is now the same as the proof of Lemma 1.3 (1). □

(2) and (3) follow exactly as in the proof of Lemma 1.3 (2) and (3). □

We no longer consider $\Lambda \in \mathcal{N}_0$ to be fixed and therefore we now have a family of probability kernels $\{\omega_\Lambda\}_{\Lambda\in\mathcal{N}_0}$.

Proposition 2.6 *The family of kernels $\{\omega_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is compatible in the sense that $\omega_\Delta\omega_\Lambda = \omega_\Delta$ for all Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$.*

Proof This is exactly the same as the proof of Proposition 1.5. □

Consider a family $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ from $\mathrm{M}(\mathcal{F})$. For each $\Lambda \in \mathcal{N}_0$ define a mapping $\omega_\Lambda^v : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ by letting

$$\omega_\Lambda^v(f)(x) = \begin{cases} (\omega_\Lambda(v_\Lambda)(x))^{-1}\omega_\Lambda(v_\Lambda f)(x) & \text{if } 0 < \omega_\Lambda(v_\Lambda)(x) < \infty\ , \\ 0 & \text{otherwise.} \end{cases}$$

for each $f \in \mathrm{M}(\mathcal{F})$. Those who prefer integral signs can write this as

$$\omega_\Lambda^v(f)(x) = I_{G_\Lambda}(x)\Big(\int v_\Lambda(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y)\Big)^{-1} \int v_\Lambda(x_{S\setminus\Lambda}y) f(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y)\ ,$$

where $G_\Lambda = \{x \in X : 0 < \int v_\Lambda(x_{S\setminus\Lambda}y)\, d\omega^\Lambda(y) < \infty\}$. Then ω_Λ^v is linear and continuous and is thus a kernel. As in the lattice model it is a quasi-probability kernel. Moreover, $\omega_\Lambda^v(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$ for each $f \in \mathrm{M}(\mathcal{F})$ and

$$\omega_\Lambda^v(x, E \cap F) = I_E(x)\omega_\Lambda^v(x, F)$$

for all $E \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$, $x \in X$.

The family $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ will be called *multiplicative* if for each Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ there exists $v_{\Delta,\Lambda} \in \mathrm{M}(\mathcal{F}_\Lambda)$ such that $v_\Delta = v_{\Delta,\Lambda}v_\Lambda$.

Proposition 2.7 *If $v = \{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is multiplicative then the family of kernels $\{\omega_\Lambda^v\}_{\Lambda\in\mathcal{N}_0}$ is compatible, again in the sense that $\omega_\Delta^v\omega_\Lambda^v = \omega_\Delta^v$ for all Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$.*

Proof This is a special case of Theorem 4.1. □

In order to apply Proposition 2.7 we require a multiplicative family and we now look at how such families typically arise in lattice models in statistical physics.

For each $\Lambda \in \mathcal{N}_0$ let $e_\Lambda \in \mathrm{M}^\Diamond(\mathcal{F})$; as in the lattice model the family $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is said to be *additive* if for each Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ there exists a mapping $e_{\Delta,\Lambda} \in \mathrm{M}^\Diamond(\mathcal{F}_\Lambda)$ such that $e_\Delta = e_{\Delta,\Lambda} + e_\Lambda$. If $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is additive and $v_\Lambda = \exp(-e_\Lambda)$ then $v = \{v_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is clearly multiplicative. Additive families can be obtained using interaction potentials, but this requires some preparation.

In what follows assume that $(S, \mathcal{S})$ is separable (i.e., $\{s\} \in \mathcal{S}$ for each $s \in S$) and countably generated. For each $s \in S$ let $\varepsilon_s \in \mathrm{P}(\mathcal{S})$ be the measure defined by $\varepsilon_s(A) = I_A(s)$ for each $A \in \mathcal{S}$. If $s \neq t$ then $\varepsilon_s \neq \varepsilon_t$, since $(S, \mathcal{S})$ is separable. For each $p \in \mathbb{N}$ let $\mathbb{N}_p = \{n \in \mathbb{N}_\infty : n < p\}$.

Lemma 2.10 *The set X consists exactly of all elements of the form $\sum_{n \in \mathbb{N}_p} \varepsilon_{s_n}$, where $p \in \mathbb{N}_\infty$ and $\{s_n\}_{n \in \mathbb{N}_p}$ is a (finite or infinite) sequence from S such that $\{n \in \mathbb{N}_p : s_n \in \Lambda\}$ is finite for each $\Lambda \in \mathcal{N}_0$. Moreover, $\sum_{n \in \mathbb{N}_p} \varepsilon_{s_n} = \sum_{n \in \mathbb{N}_q} \varepsilon_{t_n}$ if and only if $p = q$ and there exists a bijective mapping $\tau : \mathbb{N}_p \to \mathbb{N}_p$ with $s_n = t_{\tau(n)}$ for all $n \in \mathbb{N}_p$.*

Proof There exists a disjoint sequence $\{\Lambda_n\}_{n \geq 1}$ from $\mathcal{N}$ with $S = \bigcup_{n \geq 1} \Lambda_n$ and the restriction to Λ_n of any $x \in X$ is a finite measure. It is therefore enough to show that the result holds for finite measures. More precisely, it is enough to show the following: If $x \in X$ with $0 < x(S) = p < \infty$ then there exist points $s_1, \ldots, s_p \in S$ (not necessarily distinct) such that $x = \sum_{k=1}^p \varepsilon_{s_k}$. Moreover, if also $x = \sum_{k=1}^p \varepsilon_{t_k}$ then there exists a permutation τ of $\{1, \ldots, p\}$ such that $s_k = t_{\tau(k)}$ for $k = 1, \ldots, p$. The second statement is easy to establish: If $x = \sum_{k=1}^p \varepsilon_{s_k}$ then $x(\{t\}) > 0$ if and only if $t = s_j$ for some j; thus if also $x = \sum_{k=1}^p \varepsilon_{t_k}$ then there exists j so that $s_1 = t_j$, which implies that $\sum_{k=2}^p \varepsilon_{s_k} = \sum_{k \neq j} \varepsilon_{t_k}$. Repeating this $p - 1$ times then produces the required permutation τ of $\{1, \ldots, p\}$.

Let $x \in X$ with $0 < x(S) = p < \infty$; to construct the points $s_1, \ldots, s_p \in S$ such that $x = \sum_{k=1}^p \varepsilon_{s_k}$ we make use of the measurable space $(M, \mathcal{B})$ introduced in Chapter M.16. Thus $M = \{0,1\}^{\mathbb{N}}$ (the space of all sequences $\{z_n\}_{n \geq 0}$ of 0's and 1's), considered as a compact metric space in the usual way and $\mathcal{B}$ is the σ-algebra of Borel subsets of M. Since $(S, \mathcal{S})$ is countably generated there exists by Proposition M.16.1 a mapping $f : S \to M$ with $f^{-1}(\mathcal{B}) = \mathcal{S}$, and since $(S, \mathcal{S})$ is separable it then follows from Proposition M.16.8 that f is injective. Consider the image measure $\nu = f_* x$ of x under f, so ν is the measure on $\mathcal{B}$ with $\nu(B) = x(f^{-1}(B))$ for each $B \in \mathcal{B}$. In particular $\nu(M) = p$ and ν only takes values in the set $\{0, \ldots, p\}$. Suppose there exist points $z_1, \ldots, z_p \in M$ such that $\nu = \sum_{k=1}^p \varepsilon_{z_k}$. Then, since $\nu(M \setminus \{z\}) = x(f^{-1}(M \setminus \{z\})) = x(S)$ for each $z \notin f(S)$, it follows that $z_k \in f(S)$ and so for each k there exists a unique point $s_k \in S$ with $f(s_k) = z_k$. Let $A \in \mathcal{S}$; then $A = f^{-1}(B)$ for some $B \in \mathcal{B}$ (since $f^{-1}(\mathcal{B}) = \mathcal{S}$) and hence

$$x(A) = \nu(B) = \sum_{k=1}^{p} \varepsilon_{z_k}(B) = \sum_{k=1}^{p} I_B(f(s_k)) = \sum_{k=1}^{p} I_A(s_k) = \sum_{k=1}^{p} \varepsilon_{s_k}(A) ,$$

which shows that $x = \sum_{k=1}^{p} \varepsilon_{s_k}$.

It is therefore enough to show that if ν is a measure on $\mathcal{B}$ taking only values in the set $\{0, \ldots, p\}$ and with $\nu(M) = p$ then there exist $z_1, \ldots, z_p \in M$ such that $\nu = \sum_{k=1}^{p} \varepsilon_{z_k}$. For each $m \geq 0$ let $\mathcal{A}_m$ consist of all sets of the form $\mathrm{q}_m^{-1}(\{w\})$ with $w \in \{0,1\}^{m+1}$, where $\mathrm{q}_m(\{z_n\}_{n\geq 0}) = (z_0, \ldots, z_m)$. Thus $\mathcal{A}_m \subset \mathcal{B}$, $\mathcal{A}_m$ contains 2^{m+1} elements and forms a partition of M; moreover, each element of $\mathcal{A}_m$ is the union of two elements of $\mathcal{A}_{m+1}$. There is thus a decreasing sequence $\{A_m\}_{m\geq 0}$ with $A_m \in \mathcal{A}_m$ such that $x(A_m) \geq 1$ for each $m \geq 0$: Choose A_0 so that $\nu(A_0) \geq 1$. (This is possible, since if A_0' is the other element of $\mathcal{A}_0$ then $\nu(A_0)$ and $\nu(A_0')$ are both elements of $\mathbb{N}$ with $\nu(A_0) + \nu(A_0') = p \geq 1$.) Suppose $A_0, \ldots, A_m$ have been chosen with these properties; then A_m is the union of two elements of $\mathcal{A}_{m+1}$ and $x(A_{m+1}) \geq 1$ must hold for at least one of these. But $\bigcap_{m\geq 0} A_m$ consists of a single point $z_1 \in M$ and by Lemma M.3.2 $\nu(\{z_1\}) = \lim_m \nu(A_m) \geq 1$. Now define $\nu_1 : \mathcal{B} \to \mathbb{N}$ by

$$\nu_1(B) = \begin{cases} \nu(B) - 1 & \text{if } z_1 \in B , \\ \nu(B) & \text{if } z_1 \notin B ; \end{cases}$$

then ν_1 is is a measure on $\mathcal{B}$ taking only values in the set $\{0, \ldots, p-1\}$ with $\nu_1(M) = p - 1$ and $\nu = \nu_1 + \varepsilon_{z_1}$. Repeating this procedure $p - 1$ times then clearly gives the result. □

We are now going to define what is meant by an m-body interaction. For the moment fix $m \geq 1$. Let $(S^m, \mathcal{S}^m)$ be the m-fold product of the measurable space $(S, \mathcal{S})$ with itself. Define $\tau_m : \mathbb{N}^m \to \mathbb{N}$ by

$$\tau_m(k_1, \ldots, k_m) = \begin{cases} 1 & \text{if } k_1, \ldots, k_m \text{ are distinct elements of } \mathbb{N} , \\ 0 & \text{otherwise} . \end{cases}$$

Let $f \in \mathrm{M}(\mathcal{S}^m)$; then by Lemma 2.10 we can define a mapping $U_f : X \to \mathbb{R}_\infty^+$ by

$$U_f\Big(\sum_{k\in\mathbb{N}_p} \varepsilon_{s_k}\Big) = \sum_{k_1\in\mathbb{N}_p} \cdots \sum_{k_m\in\mathbb{N}_p} \tau_m(k_1, \ldots, k_m) f(s_{k_1}, \ldots, s_{k_m}) .$$

Now let $\Phi \in \mathrm{M}^\diamond(\mathcal{S}^m)$; we define $U_\Phi : X \to \mathbb{R}^\diamond$ by

$$U_\Phi(x) = \begin{cases} U_{\Phi^+}(x) - U_{\Phi^-}(x) & \text{if } U_{\Phi^+}(x) < \infty \text{ and } U_{\Phi^-}(x) < \infty , \\ \infty & \text{otherwise} , \end{cases}$$

where Φ^+, $\Phi^- \in \mathrm{M}(\mathcal{S}^m)$ are given by

$$\Phi^+(u) = \begin{cases} \Phi(u) & \text{if } \Phi(u) \ge 0 \,, \\ 0 & \text{otherwise} \,, \end{cases} \quad \Phi^-(u) = \begin{cases} -\Phi(u) & \text{if } \Phi(u) \le 0 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

The mapping $\Phi \in \mathrm{M}^\Diamond(\mathcal{S}^m)$ is considered here as an *m-body interaction* and U_Φ is then the *potential energy* corresponding to Φ. However, in order to work with U_Φ we need to know that it is measurable, i.e., an element of $\mathrm{M}^\Diamond(\mathcal{F})$, and so we next establish that this is the case. Define $\alpha : \mathrm{M}(\mathcal{S}^m) \to \mathrm{M}(X)$ by $\alpha(f)(x) = U_f(x)$. For each $x \in X$ the mapping $f \mapsto \alpha(f)(x)$ (from $\mathrm{M}(\mathcal{S}^m)$ to $\mathbb{R}^+_\infty$) is linear and continuous and is thus a measure, which means that α is a pre-kernel.

Lemma 2.11 *α is a kernel, i.e., $\alpha(f) \in \mathrm{M}(\mathcal{F})$ for each $f \in \mathrm{M}(\mathcal{S}^m)$.*

Proof For each $x \in X$, $\Lambda \in \mathcal{N}_0$ again let $x_\Lambda \in X$ be given by $x(B) = x(\Lambda \cap B)$ for each $B \in \mathcal{S}$. Let $\{\Lambda_n\}_{n\ge1}$ be an increasing sequence from $\mathcal{N}_0$ with $S = \bigcup_{n\ge1} \Lambda_n$ and for each $n \ge 1$ define $\alpha_n : \mathrm{M}(\mathcal{S}^m) \to \mathrm{M}(X)$ by $\alpha_n(f)(x) = \alpha(f)(x_{\Lambda_n})$, thus α_n is a pre-kernel. But $\{\alpha_n(f)\}_{n\ge1}$ is an increasing sequence from $\mathrm{M}(X)$ with $\lim_n \alpha_n(f) = \alpha(f)$ for each $f \in \mathrm{M}(\mathcal{S}^m)$ and hence it is enough to show that α_n is a kernel for each $n \ge 1$. Fix $n \ge 1$ and put $X_q = \{x \in X : x(\Lambda_n) \le q\}$ for each $q \ge 0$. Then $X_q \in \mathcal{F}$ and $X = \bigcup_{q\ge0} X_q$; thus by Lemma M.14.5 it is now enough to show that for each $q \ge 0$ the pre-kernel $\alpha_n^q : \mathrm{M}(\mathcal{S}^m) \to \mathrm{M}(X_q)$ given by $\alpha_n^q(f)(x) = \alpha(f)(x_{\Lambda_n})$ for all $x \in X_q$ is a kernel, i.e., that $\alpha_n^q(f) \in \mathrm{M}(\mathcal{F}_q)$ for all $f \in \mathrm{M}(\mathcal{S}^m)$ for all $f \in \mathrm{M}(\mathcal{S}^m)$, with $\mathcal{F}_q$ the trace σ-algebra of $\mathcal{F}$ on X_q. The reason for reducing things to the pre-kernel α_n^q is because it is finite, since $\alpha_n^q(1)(x) = \alpha(1)(x_{\Lambda_n}) \le q^m$ for all $x \in X_q$, and therefore by Proposition M.14.5 α_n^q will be a kernel if $\alpha_n^q(I_R) \in \mathrm{M}(\mathcal{F}_q)$ for each measurable rectangle $R \in \mathcal{S}^m$. Let $R = A_1 \times \cdots \times A_m$ with $A_1, \ldots, A_m \in \mathcal{S}$ and put $A'_k = A_k \cap \Lambda_n$ for each k. Then

$$\alpha_n(I_R)\Big(\sum_{k\in\mathbb{N}_p} \varepsilon_{s_k}\Big) = \sum_{k_1\in\mathbb{N}_p} \cdots \sum_{k_m\in\mathbb{N}_p} \tau_m(k_1,\ldots,k_m) I_{A'_1}(s_{k_1}) \times \cdots \times I_{A'_m}(s_{k_m}) \,.$$

Let $\mathcal{D}$ be the finite partition of S generated by the sets $A'_1, \ldots, A'_m$ and for each k let $\mathcal{D}_k = \{D \in \mathcal{D} : D \subset A'_k\}$. Thus

$$\begin{aligned} &\alpha_n(I_R)\Big(\sum_{k\in\mathbb{N}_p} \varepsilon_{s_k}\Big) \\ &\quad = \sum_{k_1\in\mathbb{N}_p} \cdots \sum_{k_m\in\mathbb{N}_p} \tau_m(k_1,\ldots,k_m) \sum_{D_1\in\mathcal{D}_1} \cdots \sum_{D_m\in\mathcal{D}_m} I_{D_1}(s_{k_1}) \times \cdots \times I_{D_m}(s_{k_m}) \\ &\quad = \sum_{D_1\in\mathcal{D}_1} \cdots \sum_{D_m\in\mathcal{D}_m} \sum_{k_1\in\mathbb{N}_p} \cdots \sum_{k_m\in\mathbb{N}_p} \tau_m(k_1,\ldots,k_m) I_{D_1}(s_{k_1}) \times \cdots \times I_{D_m}(s_{k_m}) \end{aligned}$$

and from this it is easy to see that $\alpha_n^q(I_R)$ is constant on each set of the form $\bigcap_{D\in\mathcal{D}} \{x \in X^q : x(D) = \ell_D\}$. But $\{x \in X^q : x(D) = \ell\} \in \mathcal{F}_q$ for all $D \in \mathcal{D}$, $\ell \in \mathbb{N}$ and therefore $\alpha_n^q(I_R) \in \mathrm{M}(\mathcal{F}_q)$. □

Lemma 2.12 *$U_\Phi \in \mathrm{M}^\Diamond(\mathcal{F})$ for each $\Phi \in \mathrm{M}^\Diamond(\mathcal{S}^m)$.*

Proof By Lemma 2.11 U_{Φ^+} and U_{Φ^-} are both in $\mathrm{M}(\mathcal{F})$ and thus $U_\Phi \in \mathrm{M}^\Diamond(\mathcal{F})$. (The reader should check that if f, $g \in \mathrm{M}(\mathcal{F})$ then $(f-g)I_F+\infty(I_{X\setminus F}) \in \mathrm{M}^\Diamond(\mathcal{F})$, where $F = \{x \in X : f(x) < \infty$ and $g(x) < \infty\}$.) □

For each $\Lambda \in \mathcal{N}_0$ let $D_\Lambda = \{(s_1, \ldots, s_m) \in S^m : s_k \in \Lambda$ for at least one $k\}$; thus $D_\Lambda \in \mathcal{S}^m$ (since it is the union of m measurable rectangles). For $\Phi \in \mathrm{M}^\Diamond(\mathcal{S}^m)$ and $\Lambda \in \mathcal{N}_0$ put $\Phi_\Lambda = I_{D_\Lambda}\Phi$ (so $\Phi_\Lambda \in \mathrm{M}^\Diamond(\mathcal{S}^m)$) and define an element of $\mathrm{M}^\Diamond(\mathcal{F})$ by $E^\Phi_\Lambda = U_{\Phi_\Lambda}$. E^Φ_Λ is called the *conditional energy* corresponding to Φ and Λ.

Lemma 2.13 *Let $\Lambda \in \mathcal{N}_0$. If $f \in \mathrm{M}(\mathcal{S}^m)$ with $f(u) = 0$ for all $u \in S^m \setminus D_\Lambda$ then $\alpha(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$.*

Proof If $f \in \mathrm{M}(\mathcal{S}^m)$ with $f(u) = 0$ for all $u \in S^m \setminus D_\Lambda$ then $\alpha(f)(x) = \alpha(f)(x_{S\setminus\Lambda})$ for all $x \in X$, where $x_{S\setminus\Lambda}(A) = x(A \setminus \Lambda)$ for each $A \in \mathcal{S}$. Thus $\alpha(f) = \alpha(f) \circ r_\Lambda$, where $r_\Lambda : X \to X$ is given by $r_\Lambda(x) = x_{S\setminus\Lambda}$ for each $x \in X$. But $r_\Lambda^{-1}(\mathcal{F}) \subset \mathcal{F}_\Lambda$ and hence $\alpha(f) \in \mathrm{M}(\mathcal{F}_\Lambda)$. □

Proposition 2.8 *If $\Phi \in \mathrm{M}^\Diamond(\mathcal{S}^m)$ then the family $\{E^\Phi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is additive.*

Proof For Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$ put $\Phi_{\Delta,\Lambda} = I_{D_{\Delta,\Lambda}}\Phi$, where $D_{\Delta,\Lambda} = D_\Delta \setminus D_\Lambda$ and let $E^\Phi_{\Delta,\Lambda} = U_{\Phi_{\Delta,\Lambda}}$. Then $\Phi_\Delta = \Phi_{\Delta,\Lambda} + \Phi_\Lambda$ and from this it easily follows that $E^\Phi_\Delta = E^\Phi_{\Delta,\Lambda} + E^\Phi_\Lambda$. But $\Phi^+_{\Delta,\Lambda}(u) = \Phi^-_{\Delta,\Lambda}(u) = 0$ for all $u \in S^m \setminus D_\Lambda$ and hence by Lemma 2.13 $E^\Phi_{\Delta,\Lambda} \in \mathrm{M}^\Diamond(\mathcal{F}_\Lambda)$. Thus the family $\{E^\Phi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is additive. □

Now for each $m \geq 1$ let $\Phi_m \in \mathrm{M}^\Diamond(\mathcal{S}^m)$ be an m-body interaction (where in practice we will have $\Phi_m = 0$ for all but finitely many m). Put $\Phi = \{\Phi_m\}_{m\geq1}$ and define the *conditional energy* $E^\Phi_\Lambda \in \mathrm{M}(\mathcal{F})$ corresponding to Φ and Λ by

$$E^\Phi_\Lambda = \sum_{m\geq1}^{*} E^{\Phi_m}_\Lambda \ .$$

It easily follows from Proposition 2.8 that the family $\{E^\Phi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is additive.

3 Kernels

The basic objects of study in these notes are specifications, which are introduced in the next chapter. Let $(X, \mathcal{F})$ be a measurable space and $\{\mathcal{F}_A\}_{A\in J}$ a decreasing family of sub-σ-algebras of $\mathcal{F}$. A specification is a family of kernels $\{\pi_A\}_{A\in J}$, where π_A is a strict $\mathcal{F}_A$-measurable quasi-probability kernel. (This family must also satisfy certain compatibility conditions.) In the present chapter we thus define this special class of kernels and look at some of their properties.

We start by recalling a few definitions and facts from *Some Notes on Measure Theory* [14]. These are mostly taken from Chapter M.14 (Kernels), but they are specialised to the case where the kernels are defined over a single space, since that is what we will be dealing with here.

Let $(X, \mathcal{F})$ be a measurable space; then $\mathrm{M}(\mathcal{F})$ is the poset (partially ordered set) of all measurable mappings from $(X, \mathcal{F})$ to $(\mathbb{R}^+_\infty, \mathcal{B}^+_\infty)$; in particular $\mathrm{M}(\mathcal{F})$ is a complete subspace of $\mathrm{M}(X)$, the poset of all mappings from X to $\mathbb{R}^+_\infty$. If N is a subspace of $\mathrm{M}(X)$ then a mapping $\Phi : N \to \mathrm{M}(X)$ is said to be linear if $\Phi(af + bg) = a\Phi(f) + b\Phi(g)$ for all f, $g \in N$ and all a, $b \in \mathbb{R}^+$. Moreover, Φ is monotone if $\Phi(g) \le \Phi(f)$ whenever f, $g \in N$ with $g \le f$, and if N is a complete subspace of $\mathrm{M}(X)$ then a linear mapping $\Phi : N \to \mathrm{M}(X)$ is continuous if it is monotone and

$$\Phi\Big(\lim_{n\to\infty} f_n\Big) = \lim_{n\to\infty} \Phi(f_n)$$

for each increasing sequence $\{f_n\}_{n\ge1}$ of elements from N.

A mapping $\pi : X \times \mathcal{F} \to \mathbb{R}^+_\infty$ is said to be an $(X, \mathcal{F})$-*kernel* (or just a *kernel* if it is clear from the context what is meant) if π is an $(X, \mathcal{F})|(X, \mathcal{F})$-kernel, i.e., if $\pi(x, \cdot)$ is a measure on $(X, \mathcal{F})$ for each $x \in X$ and $\pi(\cdot, F) \in \mathrm{M}(\mathcal{F})$ for each $F \in \mathcal{F}$. The set of all $(X, \mathcal{F})$-kernels will be denoted by $\mathrm{K}(\mathcal{F})$.

If $\pi \in \mathrm{K}(\mathcal{F})$ then by Theorem M.14.3 there exists a unique continuous linear mapping $\Phi_\pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ with $\Phi_\pi(I_F) = \pi(\cdot, F)$ for all $F \in \mathcal{F}$. As in [14], however, we just write π instead of Φ_π, and consider π also as the unique continuous linear mapping $\pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ with $\pi(I_F) = \pi(\cdot, F)$ for all $F \in \mathcal{F}$.

A kernel π is said to be *finite* if $\pi(x, \cdot)$ is a finite measure (i.e., if $\pi(x, X) < \infty$) for each $x \in X$. As a special case of this, π is called a *probability kernel* if $\pi(x, \cdot)$ is a probability measure (i.e., if $\pi(x, X) = 1$) for each $x \in X$. Somewhat more generally, π is said to be a *quasi-probability kernel* if $\pi(x, X)$ is either 0 or 1 for each $x \in X$. In terms of the mapping $\pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ the kernel is finite if and only if $\pi(1)(x) < \infty$ for each $x \in X$, it is a probability kernel if and only if $\pi(1) = 1$ and finally it is a quasi-probability kernel if and only if the mapping $\pi(1)$ only takes on the values 0 and 1.

Let π, $\varrho \in \mathrm{K}(\mathcal{F})$ be kernels and μ be a measure on $\mathcal{F}$. As in Chapter M.14 there is then the measure $\mu\pi$ on $\mathcal{F}$ with $(\mu\pi)(f) = \mu(\pi(f))$ for all $f \in \mathrm{M}(\mathcal{F})$ and there is the kernel $\pi\varrho \in \mathrm{K}(\mathcal{F})$, which as a mapping $\pi\varrho : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ is given by $(\pi\varrho)(f) = \pi(\varrho(f))$ for all $f \in \mathrm{M}(\mathcal{F})$. Moreover, $\mu(\pi\varrho) = (\mu\pi)\varrho$.

Now let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$ (meaning that $\mathcal{E} \subset \mathcal{P}(X)$ is a σ-algebra with $\mathcal{E} \subset \mathcal{F}$). A kernel $\pi \in \mathrm{K}(\mathcal{F})$ is then said to be *$\mathcal{E}$-measurable* if it is an $(X,\mathcal{E})|(X,\mathcal{F})$-kernel. In other words, the kernel π is $\mathcal{E}$-measurable if and only if $\pi(\cdot, F) \in \mathrm{M}(\mathcal{E})$ for each $F \in \mathcal{F}$. Thus we here have a continuous linear mapping $\pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{E})$, i.e., $\pi(f) \in \mathrm{M}(\mathcal{E})$ for all $f \in \mathrm{M}(\mathcal{F})$.

An $\mathcal{E}$-measurable kernel π is said to be *strict* if if $\pi(x, E \cap F) = I_E(x)\pi(x, I_F)$ for all $F \in \mathcal{F}$, $E \in \mathcal{E}$, $x \in X$.

Lemma 3.1 *If π is a strict $\mathcal{E}$-measurable kernel then $\pi(gf) = g\pi(f)$ holds for all $g \in \mathrm{M}(\mathcal{E})$, $f \in \mathrm{M}(\mathcal{F})$.*

Proof First fix $E \in \mathcal{E}$ and let $N_E = \{f \in \mathrm{M}(\mathcal{F}) : \pi(I_E f) = I_E\pi(f)\}$; then N_E is a complete subspace of $\mathrm{M}(\mathcal{F})$ with $I_F \in N_E$ for each $F \in \mathcal{F}$ and thus by Proposition M.9.2 $N_E = \mathrm{M}(\mathcal{F})$. This implies $\pi(I_E f) = I_E\pi(f)$ for all $f \in \mathrm{M}(\mathcal{F})$ for each $E \in \mathcal{E}$. Now let

$$N = \{g \in \mathrm{M}(\mathcal{E}) : \pi(gf) = g\pi(f) \text{ for all } f \in \mathrm{M}(\mathcal{F})\} ;$$

then N is a complete subspace of $\mathrm{M}(\mathcal{E})$ with $I_E \in N$ for each $E \in \mathcal{E}$ and hence again applying Proposition M.9.2 $N = \mathrm{M}(\mathcal{E})$. □

The converse of Lemma 3.1 clearly holds: If π is an $\mathcal{E}$-measurable kernel with $\pi(gf) = g\pi(f)$ for all $g \in \mathrm{M}(\mathcal{E})$, $f \in \mathrm{M}(\mathcal{F})$ then π is strict.

The following fact is useful for showing that a given kernel is strict:

Lemma 3.2 *Let π be a finite kernel and let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $\mathcal{F}$ closed under finite intersections with $X \in \mathcal{T}$. If $\pi(x, A \cap B) = I_A(x)\pi(x, B)$ for all $A \in \mathcal{S}$, $B \in \mathcal{T}$ and all $x \in X$ then $\pi(gf) = g\pi(f)$ for all $g \in \mathrm{M}(\sigma(\mathcal{S}))$, $f \in \mathrm{M}(\sigma(\mathcal{T}))$.*

Proof Fix $x \in X$. Let $A \in \mathcal{S}$; then $B \mapsto \pi(x, A \cap B)$ and $B \mapsto I_A(x)\pi(x, B)$ are both finite measures on $\mathcal{F}$ with $\pi(x, A \cap X) = I_A(x)\pi(x, X)$ (since by assumption $X \in \mathcal{T}$) which agree on $\mathcal{T}$ and so by Proposition M.3.3 they agree on $\sigma(\mathcal{T})$, i.e., $\pi(x, A \cap B) = I_A(x)\pi(x, B)$ for all $B \in \sigma(\mathcal{T})$ for each $A \in \mathcal{S}$.

Now repeat the above argument with $\sigma(\mathcal{T})$ replacing $\mathcal{S}$ and $\mathcal{S}$ replacing $\mathcal{T}$, and note that $\pi(x, X \cap B) = I_X(x)\pi(x, B)$ holds for any $B \in \mathcal{F}$. This shows then that $\pi(x, A \cap B) = I_A(x)\pi(x, B)$ for all $B \in \sigma(\mathcal{T})$ and all $A \in \sigma(\mathcal{S})$. Finally, the proof of Lemma 3.1 now shows that $\pi(gf) = g\pi(f)$ for all $g \in \mathrm{M}(\sigma(\mathcal{S}))$ and all $f \in \mathrm{M}(\sigma(\mathcal{T}))$. □

If $\mathcal{D}$ and $\mathcal{E}$ are sub-σ-algebras of $\mathcal{F}$ then the smallest σ-algebra containing both $\mathcal{D}$ and $\mathcal{E}$ will be denoted by $\mathcal{D} \vee \mathcal{E}$, thus $\mathcal{D} \vee \mathcal{E} = \sigma(\mathcal{D} \cup \mathcal{E})$. (Let $\mathcal{S}$ be the set of all subsets of X having the form $D \cap E$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$; then $\mathcal{S}$ contains X and is closed under finite intersections and so Proposition M.14.4 can be applied here.)

Lemma 3.3 *(1) If π is an $\mathcal{E}$-measurable kernel then $\pi\varrho$ is $\mathcal{E}$-measurable for each kernel ϱ.*

(2) If π is a strict $\mathcal{E}$-measurable kernel and ϱ is any $\mathcal{E}$-measurable kernel then $(\pi\varrho)(f) = \pi(1)\varrho(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, and so if π is also a probability kernel then $\pi\varrho = \varrho$.

(3) If π and ϱ are strict $\mathcal{E}$-measurable kernels then for all f, $g \in \mathrm{M}(\mathcal{F})$

$$\pi(f\varrho(g)) = \pi(f)\varrho(g) = \varrho(g\pi(f)) \; .$$

(4) If π and ϱ are strict $\mathcal{E}$-measurable kernels and $\mathcal{D}$ is a sub-σ-algebra of $\mathcal{F}$ with $\mathcal{F} = \mathcal{E} \vee \mathcal{D}$ then $\pi = \varrho$ if and only if $\pi(f) = \varrho(f)$ for all $f \in \mathrm{M}(\mathcal{D})$.

(5) If π is a strict $\mathcal{E}$-measurable kernel and ϱ a strict $\mathcal{D}$-measurable kernel, where $\mathcal{D}$ is a sub-σ-algebra of $\mathcal{F}$ with $\mathcal{E} \subset \mathcal{D}$, then $\pi\varrho$ is a strict $\mathcal{E}$-measurable kernel.

Proof (1) This is clear.

(2) If $f \in \mathrm{M}(\mathcal{F})$ then $\varrho(f) \in \mathrm{M}(\mathcal{E})$ and thus

$$(\pi\varrho)(f) = \pi(\varrho(f)) = \pi(\varrho(f)1) = \pi(1)\varrho(f) \; .$$

(3) Let f, $g \in \mathrm{M}(\mathcal{F})$; then $\pi(f)$ and $\varrho(g)$ are both elements of $\mathrm{M}(\mathcal{E})$ and therefore $\pi(f\varrho(g)) = \pi(f)\varrho(g) = \varrho(g\pi(f))$.

(4) If $\pi(f) = \varrho(f)$ for all $f \in \mathrm{M}(\mathcal{D})$ then $\pi(gf) = g\pi(f) = g\varrho(f) = \varrho(gf)$ for all $f \in \mathrm{M}(\mathcal{D})$, $g \in \mathrm{M}(\mathcal{E})$ and therefore by Proposition M.14.4 $\pi(f) = \varrho(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, i.e., $\pi = \varrho$. The converse holds trivially.

(5) By (1) $\pi\varrho$ is $\mathcal{E}$-measurable. If $f \in \mathrm{M}(\mathcal{F})$, $g \in \mathrm{M}(\mathcal{E})$ then

$$(\pi\varrho)(gf) = \pi(\varrho(gf)) = \pi(g\varrho(f)) = g\pi(\varrho(f)) = g(\pi\varrho)(f) \; ,$$

since g is also an element of $\mathrm{M}(\mathcal{D})$. □

We next introduce a standard method for normalising a kernel, and first say something about the multiplicative 'inverse' of a mapping in $\mathrm{M}(X)$. For each $f \in \mathrm{M}(X)$ define $f^{-1} \in \mathrm{M}(X)$ by letting $f^{-1}(x) = (f(x))^{-1}$ for each $x \in X$, where $0^{-1} = \infty$ and $\infty^{-1} = 0$. In general this is not a real multiplicative inverse, since $f^{-1}f = ff^{-1} = I_G$, where $G = \{x \in X : 0 < f(x) < \infty\}$. The operation

$f \mapsto f^{-1}$ preserves measurability, i.e., if $f \in \mathrm{M}(\mathcal{F})$ then also $f^{-1} \in \mathrm{M}(\mathcal{F})$, since $\{x \in X : f^{-1}(x) > a\} = \{x \in X : f(x) < a^{-1}\}$ for all $a \in \mathbb{R}^+_\infty$. Now let $\pi \in \mathrm{K}(\mathcal{F})$ and let v, $w \in \mathrm{M}(\mathcal{F})$. Then we can define a linear mapping $\pi' : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ by letting $\pi'(f) = v\pi(wf)$ for each $f \in \mathrm{M}(\mathcal{F})$. Clearly π' is continuous and so it is a kernel. As a mapping $\pi' : X \times \mathcal{F} \to \mathbb{R}^+_\infty$ it is given for all $x \in X$, $F \in \mathcal{F}$ by

$$\pi'(x, F) = v(x)\pi(wI_F)(x) = v(x)\pi(x, \cdot)(wI_F) .$$

This construction will usually be applied with $v = (\pi(w))^{-1}$, Let $\pi \in \mathrm{K}(\mathcal{F})$ and $w \in \mathrm{M}(\mathcal{F})$; then the mapping $\pi^w : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ defined by

$$\pi^w(f) = (\pi(w))^{-1}\pi(wf)$$

for each $f \in \mathrm{M}(\mathcal{F})$ is a kernel. Define a mapping $\alpha^w \in \mathrm{M}(\mathcal{F})$ by

$$\alpha^w(x) = \begin{cases} (\pi(w)(x))^{-1} & \text{if } 0 < \pi(w)(x) < \infty , \\ 0 & \text{otherwise} , \end{cases}$$

If $\pi(w)(x) = 0$ then Proposition M.14.3 (1) implies that $\pi(wf)(x) = 0$ and thus $\pi^w(f)(x) = 0$ for all $f \in \mathrm{M}(\mathcal{F})$. This means that

$$\pi^w(f) = \alpha^w\pi(wf)$$

for all $f \in \mathrm{M}(\mathcal{F})$. In particular π^w is a quasi-probability kernel and $\pi^w(1)(x) = 1$ if and only if $0 < \pi(w)(x) < \infty$. Define a further mapping $j^w \in \mathrm{M}(\mathcal{F})$ by

$$j^w(x) = \begin{cases} 1 & \text{if } 0 < w(x) < \infty , \\ 0 & \text{otherwise} ; \end{cases}$$

thus $j^w = I_G$, where $G = \{x \in X : 0 < w(x) < \infty\}$. Note that if $\pi(w)(x) < \infty$ then by Proposition M.14.3 (2) $\pi(j^w wf)(x) = \pi(I_G wf)(x) = \pi(wf)(x)$ for all $f \in \mathrm{M}(\mathcal{F})$ and this means that also

$$\pi^w(f) = \alpha^w\pi(j^w wf)$$

for all $f \in \mathrm{M}(\mathcal{F})$. If the kernel π is $\mathcal{E}$-measurable then $\alpha^w \in \mathrm{M}(\mathcal{E})$ and hence the kernel π^w is $\mathcal{E}$-measurable. If π is a strict $\mathcal{E}$-measurable kernel then so is π^w, since if $g \in \mathrm{M}(\mathcal{E})$, $f \in \mathrm{M}(\mathcal{F})$ then

$$\pi^w(gf) = \alpha^w\pi(wgf) = \alpha^w g\pi(wf) = g\alpha^w\pi(wf) = g\pi^w(f) .$$

In this case the mapping $\alpha^w \in \mathrm{M}(\mathcal{E})$ can be taken inside the kernel, and thus

$$\pi^w(f) = \pi(\alpha^w wf) = \pi(\alpha^w j^w wf)$$

for all $f \in \mathrm{M}(\mathcal{F})$. In many applications this final form has the advantage that the mapping $\alpha^w j^w w$ does not take on the value ∞.

The above construction applied to a strict $\mathcal{E}$-measurable kernel π is the standard method of obtaining a strict $\mathcal{E}$-measurable quasi-probability kernel. Let us state the result of this construction as a proposition:

Proposition 3.1 *If $\pi \in \mathrm{K}(\mathcal{F})$ is a strict $\mathcal{E}$-measurable kernel then the mapping $\pi^w : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ defined by putting*

$$\pi^w(f) = (\pi(w))^{-1}\pi(wf)$$

for each $f \in \mathrm{M}(\mathcal{F})$ is a strict $\mathcal{E}$-measurable quasi-probability kernel. Moreover

$$\pi^w(f) = \pi(\alpha^w wf) = \pi(\alpha^w j^w wf)$$

for all $f \in \mathrm{M}(\mathcal{F})$, where the mappings $\alpha^w \in \mathrm{M}(\mathcal{E})$ and $j^w \in \mathrm{M}(\mathcal{F})$ are defined above.

Proof This was shown above. □

Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$. For the remainder of the chapter we look at the structure of the convex set of probability measures which are invariant under a given strict $\mathcal{E}$-measurable quasi-probability kernel. This is the prototype of a result to be found in Chapter 6 describing the structure of the convex set of Gibbs states given by a specification (which is a family of strict quasi-probability kernels). We start with two simple but useful concepts.

Let $\mu \in \mathrm{P}(\mathcal{F})$. Then $f \in \mathrm{M}(\mathcal{F})$ is said to be *μ-a.e. equal to a constant* if $f = \mu(f)$ μ-a.e. (noting that if $f = c$ μ-a.e. for some $c \in \mathbb{R}^+_\infty$ then c has to be equal to $\mu(f)$).

Lemma 3.4 *Let $\mu \in \mathrm{P}(\mathcal{F})$ and $f \in \mathrm{M}(\mathcal{E})$. Then the following are equivalent:*

(1) f is μ-a.e. equal to a constant.

(2) $\mu(gf) = \mu(g)\mu(f)$ for all $g \in \mathrm{M}(\mathcal{E})$.

(3) $\mu(gf) = \mu(g)\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F})$.

Proof (2) ⇒ (1): Since $\mu(gf) = \mu(g)\mu(f) = \mu(g\mu(f))$ for all $g \in \mathrm{M}(\mathcal{E})$ it follows from Lemma M.10.4 (2) that $f = \mu(f)$ μ-a.e. (Note that $\mu(g)\mu(f) = \mu(g\mu(f))$ still holds when $\mu(f) = \infty$.)

(1) ⇒ (3): Again using Lemma M.10.4 (2) $\mu(gf) = \mu(g\mu(f)) = \mu(g)\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F})$.

(3) ⇒ (2) is clear. □

A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is said to be *trivial* on $\mathcal{E}$ if $\mu(E) \in \{0, 1\}$ for all $E \in \mathcal{E}$. This can also be formulated in terms of mappings:

Lemma 3.5 *A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is trivial on $\mathcal{E}$ if and only if each $f \in \mathrm{M}(\mathcal{E})$ is μ-a.e. equal to a constant.*

Proof Suppose first that the measure μ is trivial on $\mathcal{E}$ and let $f \in \mathrm{M}(\mathcal{F})$; then $N = \{g \in \mathrm{M}(\mathcal{E}) : \mu(gf) = \mu(g)\mu(f)\}$ is a complete subspace of $\mathrm{M}(\mathcal{E})$. Let $E \in \mathcal{E}$; then $\mu(I_E f) = \mu(f) = \mu(I_E)\mu(f)$ if $\mu(E) = 1$; on the other hand, if $\mu(E) = 0$ then $\mu(I_E f) = 0 = \mu(I_E)\mu(f)$. Thus $I_E \in N$ for all $E \in \mathcal{E}$ and therefore by Proposition M.9.2 $N = \mathrm{M}(\mathcal{E})$. Hence by Lemma 3.4 f is μ-a.e. equal to a constant. Suppose conversely that f is μ-a.e. equal to a constant for each $f \in \mathrm{M}(\mathcal{E})$. Let $E \in \mathcal{E}$; then $I_E = \mu(I_E) = \mu(E)$ μ-a.e., and thus $\mu(E)$ is either 0 or 1. This shows that μ is trivial on $\mathcal{E}$. □

A subset C of $\mathrm{P}(\mathcal{F})$ is *convex* if $a\mu_1 + (1-a)\mu_2 \in C$ for all $\mu_1, \mu_2 \in C$ and all $0 \le a \le 1$. If C is convex then $\mu \in C$ is an *extreme point* of C, if whenever $\mu_1, \mu_2 \in C$ and $0 < a < 1$ with $\mu = a\mu_1 + (1-a)\mu_2$ then $\mu = \mu_1 = \mu_2$. The set of extreme points will be denoted by $\operatorname{ext} C$.

If π is a quasi-probability kernel then put

$$\mathcal{I}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\pi\} ;$$

moreover, if in addition π is $\mathcal{E}$-measurable then put

$$\mathcal{J}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu(gf) = \mu(g\pi(f)) \text{ for all } g \in \mathrm{M}(\mathcal{E}),\ f \in \mathrm{M}(\mathcal{F})\} .$$

Thus $\mathcal{J}(\pi) \subset \mathcal{I}(\pi)$, with equality when π is a strict $\mathcal{E}$-measurable kernel, since here if $\mu \in \mathcal{I}(\pi)$ then $\mu(gf) = (\mu\pi)(gf) = \mu(\pi(gf)) = \mu(g\pi(f))$ holds for all $g \in \mathrm{M}(\mathcal{E})$, $f \in \mathrm{M}(\mathcal{F})$. The sets $\mathcal{I}(\pi)$ and $\mathcal{J}(\pi)$ are both convex subsets of $\mathrm{P}(\mathcal{F})$.

In what follows let π be a strict $\mathcal{E}$-measurable quasi-probability kernel.

Proposition 3.2 *(1) An element of $\mathcal{J}(\pi)$ is extreme if and only if it is trivial on the σ-algebra $\mathcal{E}$.*

(2) The elements of $\mathcal{J}(\pi)$ are determined on $\mathcal{E}$, in that if $\mu_1, \mu_2 \in \mathcal{J}(\pi)$ with $\mu_1(E) = \mu_2(E)$ for all $F \in \mathcal{E}$ then $\mu_1 = \mu_2$.

(3) If $\mu_1, \mu_2 \in \operatorname{ext}\mathcal{J}(\pi)$ with $\mu_1 \ne \mu_2$ then there exists $E \in \mathcal{E}$ with $\mu_1(E) = 1$ and $\mu_2(E) = 0$. This says that distinct elements of $\operatorname{ext}\mathcal{J}(\pi)$ are mutually singular, even considered as measures on $\mathcal{E}$.

Proof We first need a lemma. Note if $\mu \in \mathrm{P}(\mathcal{F})$ and $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) = 1$ then the measure $\mu{\cdot}f$ (given by $\mu{\cdot}f\,(g) = \mu(fg)$ for each $g \in \mathrm{M}(\mathcal{F})$) is also a probability measure.

Lemma 3.6 *Let $\mu \in \mathcal{J}(\pi)$ and $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) = 1$. Then $\mu{\cdot}f \in \mathcal{J}(\pi)$ if and only if there exists $f' \in \mathrm{M}(\mathcal{E})$ with $f' = f$ μ-a.e. Moreover, in this case $\pi(f) = f$ μ-a.e.*

Proof Suppose there exists $f' \in \mathrm{M}(\mathcal{E})$ with $f' = f$ μ-a.e., so by Lemma M.10.4 (2) $\mu(fg) = \mu(f'g)$ for all $g \in \mathrm{M}(\mathcal{F})$. Then

$$\begin{aligned}((\mu{\cdot}f)\pi)(g) &= (\mu{\cdot}f)(\pi(g)) = \mu(f\pi(g)) = \mu(f'\pi(g))\\ &= \mu(\pi(f'g)) = (\mu\pi)(f'g) = \mu(f'g) = \mu(fg) = (\mu{\cdot}f)(g)\end{aligned}$$

for all $g \in \mathrm{M}(\mathcal{F})$ and hence $\mu{\cdot}f \in \mathcal{J}(\pi)$.

Suppose conversely that $\mu{\cdot}f \in \mathcal{J}(\pi)$; then for all $g \in \mathrm{M}(\mathcal{F})$

$$\mu(fg) = (\mu{\cdot}f)(g) = (\mu{\cdot}f)(\pi(g)) = \mu(f\pi(g)) = \mu(\pi(f)\pi(g)) = \mu(\pi(f)g)$$

and therefore by Lemma M.10.4 (2) $\pi(f) = f$ μ-a.e. Moreover, $\pi(f) \in \mathrm{M}(\mathcal{E})$. □

We now look at the various parts of Proposition 3.2:

(1) If $\mu \in \mathcal{J}(\pi)$ is not extreme then there exist μ_1, $\mu_2 \in \mathcal{J}(\pi)$ with $\mu_1 \neq \mu_2$ and $0 < a < 1$ such that $\mu = a\mu_1 + (1-a)\mu_2$. Then $\mu_1 \ll \mu$ and so by Theorem M.12.1 there exists $f \in \mathrm{M}(\mathcal{F})$ with $\mu_1 = \mu{\cdot}f$, and $\mu(f) = \mu_1(1) = 1$. Therefore by Lemma 3.6 there exists $f' \in \mathrm{M}(\mathcal{E})$ with $f' = f$ μ-a.e., and in particular this implies that $\mu(f') = \mu(f) = 1$. Now $\mu \neq \mu_1$ and thus let $g \in \mathrm{M}(\mathcal{F})$ with $\mu(g) \neq \mu_1(g) = \mu(fg)$. Then $\mu(f'g) = \mu(fg) \neq \mu(g) = \mu(f')\mu(g)$ and hence by Lemmas 3.4 and 3.5 μ is not trivial on $\mathcal{E}$.

Conversely, suppose μ is not trivial on $\mathcal{E}$, there therefore exists $E \in \mathcal{E}$ with $0 < \mu(E) < 1$. Put $a = \mu(E)$ and let $\mu_1 = \mu{\cdot}g_1$ and $\mu_2 = \mu{\cdot}g_2$ with $g_1 = a^{-1}I_E$ and $g_2 = (1-a)^{-1}I_{X\setminus E}$. Then $g_j \in \mathrm{M}(\mathcal{E})$ and $\mu(g_j) = 1$ and so by Lemma 3.6 $\mu_j \in \mathcal{J}(\pi)$ for $j = 1, 2$. But $\mu = a\mu_1 + (1-a)\mu_2$ and clearly $\mu_1 \neq \mu_2$; therefore $\mu \notin \operatorname{ext}\mathcal{J}(\pi)$.

(2) Let μ_1, $\mu_2 \in \mathcal{J}(\pi)$ with $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{E}$. Then $\mu_1(g) = \mu_2(g)$ for all $g \in \mathrm{M}(\mathcal{E})$. But $\pi(f) \in \mathrm{M}(\mathcal{E})$ for each $f \in \mathrm{M}(\mathcal{F})$ and so

$$\mu_1(f) = (\mu_1\pi)(f) = \mu_1(\pi(f)) = \mu_2(\pi(f)) = (\mu_2\pi)(f) = \mu_2(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, i.e., $\mu_1 = \mu_2$.

(3) Let μ_1, μ_2 be distinct elements of $\operatorname{ext}\mathcal{J}(\pi)$. By (2) there exists $E \in \mathcal{E}$ with $\mu_1(E) \neq \mu_2(E)$, and by (1) $\mu_1(E)$ and $\mu_2(E)$ can only have the values 0 and 1. Hence μ_1 and μ_2 are mutually singular on $\mathcal{E}$.

This completes the proof of Proposition 3.2. □

Let $P_\pi = \{x \in X : \pi(x, X) = 1\}$ (and so $P_\pi \in \mathcal{E}$). Note that $\mu(P_\pi) = 1$ for each $\mu \in \mathcal{J}(\pi)$, since if $\mu = \mu\pi$ then $\mu(P_\pi) = \mu(\pi(1)) = (\mu\pi)(X) = \mu(X) = 1$.

Proposition 3.3 *(1)* $\mathcal{J}(\pi) = \mathcal{I}(\pi)$.

(2) $\pi\pi = \pi$ *and hence the mapping* $\mu \mapsto \mu\pi$ *maps* $\{\mu \in \mathrm{P}(\mathcal{E}) : \mu(P_\pi) = 1\}$ *onto* $\mathcal{J}(\pi)$.

(3) $\pi(x, \cdot) \in \operatorname{ext} \mathcal{J}(\pi)$ *for each* $x \in P_\pi$.

(4) If $\mu \in \operatorname{ext} \mathcal{J}(\pi)$ *then* $\mu(\{x \in X : \pi(x, F) = \mu(F)\}) = 1$ *for each* $F \in \mathcal{F}$.

Proof (1) It has already been observed that this holds, and it is only included here because it corresponds to a (non-trivial) statement in Theorem 6.2.

(2) Let $f \in \mathrm{M}(\mathcal{F})$; then by Lemma 3.3 (2) $(\pi\pi)(f) = \pi(f)\pi(1)$, and from this it follows that $(\pi\pi)(f)(x) = \pi(f)(x)$ if $\pi(1)(x) = 1$. But $(\pi\pi)(f)(x) = \pi(\pi(f))(x)$ and $\pi(f)(x)$ are both 0 if $\pi(1)(x) = 0$ and so $(\pi\pi)(f)(x) = \pi(f)(x)$ again holds. Therefore $(\pi\pi)(f) = \pi(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, i.e., $\pi\pi = \pi$.

Now if $\mu \in \mathrm{P}(\mathcal{E})$ with $\mu(P_\pi) = 1$ then $(\mu\pi)(X) = \mu(\pi(\cdot, X)) = \mu(P_\pi) = 1$ and so $\mu\pi \in \mathrm{P}(\mathcal{F})$; thus $\mu\pi \in \mathcal{J}(\pi)$, since $(\mu\pi)\pi = \mu(\pi\pi) = \mu\pi$ and $\mathcal{I}(\pi) = \mathcal{J}(\pi)$. Conversely, let $\mu \in \mathcal{J}(\pi)$ and let $\nu \in \mathrm{P}(\mathcal{E})$ be the restriction of the measure μ to $\mathcal{E}$. Then $\nu\pi = \mu\pi = \mu$ and $\nu(P_\pi) = (\nu\pi)(X) = \mu(X) = 1$.

(3) Here is a basic observation:

Lemma 3.7 *Let* ϱ_1, $\varrho_2 \in \mathrm{K}(\mathcal{F})$ *be kernels and* $x \in X$*; then* $\varrho_1(x, \cdot) \in \mathcal{I}(\varrho_2)$ *if and only if* $\varrho_1(x, X) = 1$ *and* $\varrho_1(x, \cdot) = (\varrho_1\varrho_2)(x, \cdot)$.

Proof By Lemma M.14.2 $\varrho_1(x, \cdot)\varrho_2 = (\varrho_1\varrho_2)(x, \cdot)$ and it therefore follows that $\varrho_1(x, \cdot)\varrho_2 = \varrho_1(x, \cdot)$ if and only if $\varrho_1(x, \cdot) = (\varrho_1\varrho_2)(x, \cdot)$. □

By (2) $\pi\pi = \pi$, and thus by Lemma 3.7 $\pi(x, \cdot) \in \mathcal{I}(\pi) = \mathcal{J}(\pi)$ for each $x \in P_\pi$. But $\pi(x, E) = I_E\pi(x, X) \in \{0, 1\}$ for all $x \in X$, $E \in \mathcal{E}$ and hence if $x \in P_\pi$ then by Proposition 3.2 (1) $\pi(x, \cdot) \in \operatorname{ext} \mathcal{J}(\pi)$.

(4) Let $\mu \in \operatorname{ext} \mathcal{J}(\pi)$ and $F \in \mathcal{F}$. If $g \in \mathrm{M}(\mathcal{E})$ then by Lemmas 3.4 and 3.5 and Proposition 3.2 (1)

$$\mu(g\pi(\cdot, F)) = \mu(g\pi(I_F)) = \mu(gI_F) = \mu(g)\mu(I_F) = \mu(g\mu(F))$$

and so by Lemma M.10.4 (2) $\mu(\{x \in X : \pi(x, F) = \mu(F)\}) = 1$.

This completes the proof of Proposition 3.3. □

The σ-algebra $\mathcal{F}$ is said to *countably generated* if $\mathcal{F} = \sigma(\mathcal{T})$ for some countable subset $\mathcal{T}$ of $\mathcal{F}$. In this case Proposition 3.3 (4) can be improved on with the help of the following fact:

Lemma 3.8 *Suppose that* $\mathcal{F}$ *is countably generated and let* $\mu \in \mathrm{P}(\mathcal{F})$. *Then* $E = \{x \in X : \pi(x, \cdot) = \mu\} \in \mathcal{E}$, *and* $\mu(\{x \in X : \pi(x, F) = \mu(F)\}) = 1$ *for all* $F \in \mathcal{F}$ *if and only if* $\mu(E) = 1$.

Proof Let $\mathcal{T}$ be a countable subset of $\mathcal{F}$ with $\mathcal{F} = \sigma(\mathcal{T})$ and let $\mathcal{S}$ be the set of all finite intersections of elements from $\mathcal{T}$. Then $\mathcal{S}$ is still countable and is closed under finite intersections, and of course $\mathcal{F} = \sigma(\mathcal{S})$. Now by Proposition M.3.3 $E = \bigcap_{F \in \mathcal{S}} E_F$, where $E_F = \{x \in X : \pi(x, F) = \mu(F)\}$. But $E_F \in \mathcal{E}$ for each F and so $E \in \mathcal{E}$. Moreover, by Lemma M.3.3 $\mu(E) = 1$ if and only if $\mu(E_F) = 1$ for all $F \in \mathcal{S}$, and hence if and only if $\mu(E_F) = 1$ for all $F \in \mathcal{F}$. (If $\mathcal{F}$ is countably generated then there is in fact a countable algebra $\mathcal{A}$ with $\mathcal{F} = \sigma(\mathcal{A})$, see Proposition M.16.2.) □

If $\mathcal{F}$ is countably generated then Proposition 3.3 (4) can thus be replaced by:

(4′) If $\mu \in \operatorname{ext} \mathcal{J}(\pi)$ then $\mu(\{x \in X : \pi(x, \cdot) = \mu\}) = 1$.

In particular, this implies that each element of $\operatorname{ext} \mathcal{J}(\pi)$ has the form $\pi(x, \cdot)$ for some $x \in P_\pi$.

4 Specifications

In this chapter we define a specification and its Gibbs states; these are the basic objects to be studied. A specification is based on a decreasing family $\mathbb{F} = \{\mathcal{F}_A\}_{A \in J}$ of σ-algebras and is a family of kernels $\{\pi_A\}_{A \in J}$, where π_A should be considered as specifying the conditional probabilities with respect to $\mathcal{F}_A$. The corresponding Gibbs states are then the probability measures whose conditional probabilities agree with those provided by the kernels.

To be a bit more precise about the family of σ-algebras, it will be assumed that the following objects are given: a measurable space $(X, \mathcal{F})$, a non-empty index set J equipped with a partial order $\preceq$, and a decreasing family $\mathbb{F} = \{\mathcal{F}_A\}_{A \in J}$ of sub-σ-algebras of $\mathcal{F}$, i.e., such that $\mathcal{F}_B \subset \mathcal{F}_A$ whenever $A, B \in J$ with $A \preceq B$. It will always be assumed that the partial order $\preceq$ is *directed* (i.e., for all $A_1, A_2 \in J$ there exists $A \in J$ with $A_1 \preceq A$ and $A_2 \preceq A$) and *countably generated* (i.e., there exists a countable subset J_0 of J such that for each $A \in J$ there is an element $A_0 \in J_0$ with $A \preceq A_0$). In this case there exists an *order generating sequence* from J, i.e., an increasing sequence $\{A_n\}_{n \geq 1}$ from J such that for each $A \in J$ there is an index $n \geq 1$ with $A \preceq A_n$.

What will be called an $\mathbb{F}$-specification is defined in terms of such a family $\mathbb{F}$. Before coming to this, however, we first look at how these families $\mathbb{F}$ typically occur. Consider a situation in which the following ingredients are present:

1. A pair $(S, \mathcal{N})$ consisting of a non-empty set S and a ring $\mathcal{N}$ of subsets of S. The set S should be thought of as a set of *sites*, and the elements of $\mathcal{N}$ as a suitable collection of *bounded* subsets of S.

(A *ring* $\mathcal{N}$ is a non-empty subset of $\mathcal{P}(S)$ for which both $\Lambda \cup \Delta$ and $\Lambda \setminus \Delta$ are elements of $\mathcal{N}$ for all $\Lambda, \Delta \in \mathcal{N}$. It then follows that $\varnothing \in \mathcal{N}$ and $\Lambda \cap \Delta \in \mathcal{N}$ for all $\Lambda, \Delta \in \mathcal{N}$.) The inclusion order $\subset$ on $\mathcal{N}$ is directed; it will be countably generated if and only if there exists a countable subset $\mathcal{N}_c$ of $\mathcal{N}$ such that each element of $\mathcal{N}$ is a subset of some element of $\mathcal{N}_c$. In particular, this is the case if the ring $\mathcal{N}$ is itself countable.

2. A non-empty set X and for each $\Lambda \in \mathcal{N}$ a σ-algebra $\mathcal{F}^\Lambda \subset \mathcal{P}(X)$ such that

$$\mathcal{F}^{\Lambda \cup \Delta} = \mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$$

for all $\Lambda, \Delta \in \mathcal{N}$, where $\mathcal{F}^\Lambda \vee \mathcal{F}^\Delta$ denotes the smallest σ-algebra containing both $\mathcal{F}^\Lambda$ and $\mathcal{F}^\Delta$. The set X should be thought of as a basic set of *configurations*, and for each $\Lambda \in \mathcal{N}$ the σ-algebra $\mathcal{F}^\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on in Λ.

In particular, it follows that $\mathcal{F}^\Lambda \subset \mathcal{F}^\Delta$ whenever $\Lambda, \Delta \in \mathcal{N}$ with $\Lambda \subset \Delta$, since $\mathcal{F}^\Delta = \mathcal{F}^\Lambda \vee \mathcal{F}^{\Delta \setminus \Lambda}$. It will be assumed that $\mathcal{F}^\varnothing$ is the trivial σ-algebra $\{\varnothing, X\}$.

A collection $(S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ with $(S, \mathcal{N})$, X and $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ as above and with the inclusion order $\subset$ on $\mathcal{N}$ countably generated will be called a *spatial system.*

The lattice and particle models considered in Chapters 1 and 2 are both spatial systems.

Let $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ be a spatial system. The algebra $\mathcal{A} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{F}^\Lambda$ will then be called the *algebra of local observables.* More generally, for each $\Lambda \in \mathcal{N}$ the algebra $\mathcal{A}_\Lambda = \bigcup_{\Delta\subset S\setminus\Lambda} \mathcal{F}^\Delta$ will be referred to as *the algebra of local observables associated with the complement of* Λ.

Put $\mathcal{F} = \sigma(\mathcal{A})$ and for each $\Lambda \in \mathcal{N}_0 = \{\Lambda \in \mathcal{N} : \Lambda \neq \varnothing\}$, let $\mathcal{F}_\Lambda = \sigma(\mathcal{A}_\Lambda)$. We consider the σ-algebra $\mathcal{F}_\Lambda$ as consisting of those subsets of configurations which can be determined by what is going on outside of Λ.

This gives us a measurable space $(X, \mathcal{F})$, a non-empty index set $\mathcal{N}_0$ for which the inclusion order $\subset$ is directed and countably generated, and a decreasing family $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ of sub-σ-algebras of $\mathcal{F}$ which is called the *family of σ-algebras associated with the spatial system* Σ. Note that $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda = \mathcal{F}$ for each $\Lambda \in \mathcal{N}_0$ since $\mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda \supset \mathcal{F}^{\Delta\setminus\Lambda} \vee \mathcal{F}^\Lambda = \mathcal{F}^{\Lambda\cup\Delta} \supset \mathcal{F}^\Delta$ for each $\Delta \in \mathcal{N}$.

The only reason for using $\mathcal{N}_0$ instead of $\mathcal{N}$ as the basic index set is to avoid some trivial difficulties (for example in the particle model). As far as the definition of a specification is concerned it does not make any difference whether $\mathcal{N}_0$ or $\mathcal{N}$ is taken as the index set.

In order to get anywhere with the above set-up it is usually necessary to make a further assumption. Let $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ be a spatial system. For each $\Lambda \in \mathcal{N}$ let $\mu_\Lambda \in \mathrm{P}(\mathcal{F}^\Lambda)$. Then the family of probability measures $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ is said to be *consistent* if $\mu_\Lambda(F) = \mu_\Delta(F)$ for all $F \in \mathcal{F}^\Lambda$ whenever $\Lambda \subset \Delta$. We say that the family of σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ has the *Kolmogorov extension property* if for each consistent family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ there exists a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}$. If μ exists then by Proposition M.3.3 it is unique, since it is determined by the family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ on the algebra $\mathcal{A}$ and $\sigma(\mathcal{A}) = \mathcal{F}$. Finally, we say that the spatial system Σ has the *Kolmogorov extension property* if the family of σ-algebras $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ does.

Most non-trivial results about spatial systems require the Kolmogorov extension property to hold. Proposition 1.3 shows that this is the case for the lattice model if each of the measurable spaces $(X_s, \mathcal{F}_s)$, $s \in S$, is substandard Borel. (Substandard Borel spaces are our substitute for standard Borel spaces and are introduced in Chapter M.18. In particular, by Proposition M.18.1 a standard Borel space is substandard Borel.) By Proposition 2.3 the Kolmogorov extension property holds for the particle model if $(S, \mathcal{S})$ is substandard Borel. A general framework which ensures its validity is when there exists an order-generating sequence $\{\Lambda_n\}_{n\geq 0}$ from $\mathcal{N}$ such that $(X, \mathcal{F}^{\Lambda_n})$ is a substandard Borel space for each $n \geq 0$ and $\mathcal{F}$ is the inverse limit of the sequence $\{\mathcal{F}^{\Lambda_n}\}_{n\geq 0}$ (this meaning that $\bigcap_{n\geq 0} F_n \neq \varnothing$ whenever $\{F_n\}_{n\geq 0}$ is a decreasing sequence with F_n an atom of $\mathcal{F}^{\Lambda_n}$ for each $n \geq 0$). This result is presented in Chapter M.19.

Suppose Σ has the Kolmogorov extension property and let $\mathcal{N}_1$ be a subset of $\mathcal{N}$ which is also directed and countably generated. Then any consistent family of probability measures $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_1}$ extends to a probability measure $\mu \in \mathrm{P}(\mathcal{F})$. More precisely, for each $\Lambda \in \mathcal{N}_1$ let $\mu_\Lambda \in \mathrm{P}(\mathcal{F}^\Lambda)$ and suppose $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_1}$ is consistent in that $\mu_\Lambda(F) = \mu_\Delta(F)$ for all $F \in \mathcal{F}^\Lambda$ whenever $\Lambda,\, \Delta \in \mathcal{N}_1$ with $\Lambda \subset \Delta$. Then there exists a unique probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \mu_\Lambda(F)$ for all $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}_1$. This follows because the family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_1}$ can clearly be extended to a consistent family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$. This observation can also be applied to kernels:

Lemma 4.1 *Suppose Σ has the Kolmogorov extension property and let $\mathcal{N}_1$ be a subset of $\mathcal{N}$ which is directed and countably generated. For each $\Lambda \in \mathcal{N}_1$ let $\pi^\Lambda \in \mathrm{K}(\mathcal{F}^\Lambda)$ be a probability kernel such that $\pi^\Delta(f) = \pi^\Lambda(f)$ holds for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$ whenever $\Lambda,\, \Delta \in \mathcal{N}_1$ with $\Lambda \subset \Delta$. Then there exists a unique probability kernel $\pi \in \mathrm{K}(\mathcal{F})$ such that $\pi(f) = \pi^\Lambda(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $\Lambda \in \mathcal{N}_1$.*

Proof Let $x \in X$; then the probability measures $\pi^\Lambda(x,\cdot) \in \mathrm{P}(\mathcal{F}^\Lambda)$, $\Lambda \in \mathcal{N}_1$, are consistent and so there exists a unique probability measure $\pi(x,\cdot) \in \mathrm{P}(\mathcal{F})$ such that $\pi(x,\cdot)(f) = \pi^\Lambda(x,\cdot)(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $\Lambda \in \mathcal{N}_1$. This implies there is a unique pre-kernel $\pi : X \times \mathcal{F} \to \mathbb{R}^+_\infty$ such that $\pi(f) = \pi^\Lambda(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $\Lambda \in \mathcal{N}_1$, and π is finite, since $\pi(1) = 1$. Now if $A \in \mathcal{A}$ then $I_A \in \mathrm{M}(\mathcal{F}^\Lambda)$ for some $\Lambda \in \mathcal{N}_1$ and therefore $\pi(I_A) = \pi^\Lambda(I_A) \in \mathrm{M}(\mathcal{F}^\Lambda)$. In particular $\pi(I_A) \in \mathrm{M}(\mathcal{F})$ for all $A \in \mathcal{A}$, and hence by Proposition M.14.5 π is a kernel (and thus a probability kernel). □

Now to the definition of a specification. Again consider the general set-up, thus what we have is a measurable space $(X, \mathcal{F})$, a non-empty index set J equipped with a directed, countably generated partial order $\preceq$, and a decreasing family $\mathbb{F} = \{\mathcal{F}_A\}_{A\in J}$ of sub-σ-algebras of $\mathcal{F}$.

Suppose for each $A \in J$ we have a strict $\mathcal{F}_A$-measurable quasi-probability kernel $\pi_A \in \mathrm{K}(\mathcal{F})$. Then the family $\mathcal{V} = \{\pi_A\}_{A\in J}$ will be called an $\mathbb{F}$-*specification* if $\pi_B = \pi_B\pi_A$ whenever $A,\, B \in J$ with $A \preceq B$. The $\mathbb{F}$-specification $\mathcal{V} = \{\pi_A\}_{A\in J}$ is said to be *simple* if all the π_A, $A \in J$, are probability kernels.

Let $\mathcal{V} = \{\pi_A\}_{A\in J}$ be an $\mathbb{F}$-specification; then a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is called a *Gibbs state with specification* $\mathcal{V}$ if its conditional probabilities agree with those given by the kernels $\{\pi_A\}_{A\in J}$ in the sense that $\pi_A(I_F)$ should be a version of the conditional expectation $\mathrm{E}_\mu(I_F|\mathcal{F}_A)$ of I_F with respect to $\mathcal{F}_A$ for all $F \in \mathcal{F}$, $A \in J$ (see Chapter M.12 for the definition of conditional expectation). This can be of course be expressed without making any reference to conditional expectations (which is useful for those who don't know what such things are) as follows: A measure $\mu \in \mathrm{P}(\mathcal{F})$ is a Gibbs state if and only if

$$\mu(I_{F'} I_F) = \mu(I_{F'} \pi_A(I_F))$$

for all $F \in \mathcal{F}$, $F' \in \mathcal{F}_A$, $A \in J$. But

$$\mu(I_{F'} \pi_A(I_F)) = \mu(\pi_A(I_{F'} I_F)) = (\mu\pi_A)(I_{F'} I_F)$$

and thus μ is a Gibbs state if and only if $\mu = \mu\pi_A$ for each $A \in J$.

The set of Gibbs states with specification $\mathcal{V}$ will be denoted by $\mathcal{G}(\mathcal{V})$, hence

$$\mathcal{G}(\mathcal{V}) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\pi_A \text{ for all } A \in J\} \, .$$

This definition of Gibbs states originates from Dobrushin [1], [2], [3], [4], and Lanford and Ruelle [12], [15], and the equations which define the elements of $\mathcal{G}(\mathcal{V})$ (i.e., $\mu = \mu\pi_A$ for all $A \in J$, or these written in some other form) are thus often called the *DLR-equations.*

For the rest of this chapter we look at how specifications typically occur. The basic properties of Gibbs states themselves are dealt with in Chapters 5 and 6.

The simplest kind of specifications are obtained within the framework of spatial systems and involve what we call independent measures. Thus in the following let $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ be a spatial system and let $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be the associated family of σ-algebras.

Lemma 4.2 *Let $\mu \in \mathrm{P}(\mathcal{F})$ and $\Lambda \in \mathcal{N}_0$. Then there exists at most one kernel $\mu_\Lambda \in \mathrm{K}(\mathcal{F})$ such that $\mu_\Lambda(gf) = g\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$. Moreover, if μ_Λ exists then it is a strict $\mathcal{F}_\Lambda$-measurable probability kernel.*

Proof Let $\mathcal{D}_\Lambda$ denote the set of all elements of $\mathcal{F}$ having the form $F' \cap F$ with $F' \in \mathcal{F}_\Lambda$ and $F \in \mathcal{F}^\Lambda$. Then $X \in \mathcal{D}_\Lambda$, $\mathcal{D}_\Lambda$ is closed under finite intersections and $\mathcal{F} = \sigma(\mathcal{D}_\Lambda)$. Now if μ_Λ exists then $\mu_\Lambda(F' \cap F) = I_{F'}\mu(F)$ for all $F' \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$, which determines μ_Λ on $\mathcal{D}_\Lambda$. Hence by Proposition M.14.4 there is at most one kernel with this property. Thus suppose μ_Λ exists. Then the requirement $\mu_\Lambda(F' \cap F) = I_{F'}\mu(F)$ for all $F' \in \mathcal{F}_\Lambda$, $F \in \mathcal{F}^\Lambda$, implies that $\mu_\Lambda(I_D) \in \mathcal{F}_\Lambda$ for all $D \in \mathcal{D}_\Lambda$ and therefore by Proposition M.14.5 μ_Λ is $\mathcal{F}_\Lambda$-measurable. Finally, if g, $f_1 \in \mathrm{M}(\mathcal{F}_\Lambda)$ and $f_2 \in \mathrm{M}(\mathcal{F}^\Lambda)$ then

$$\mu_\Lambda(g(f_1 f_2)) = \mu_\Lambda((g f_1) f_2) = g f_1 \mu(f_2) = g \mu_\Lambda(f_1 f_2)$$

and so in particular $\mu_\Lambda(x, F \cap D) = I_F(x)\mu_\Lambda(x, D)$ for all $F \in \mathcal{F}_\Lambda$, $D \in \mathcal{D}_\Lambda$, $x \in X$, and therefore by Lemma 3.2 is a strict $\mathcal{F}_\Lambda$-measurable kernel. □

If the kernel μ_Λ exists for each $\Lambda \in \Lambda_0$ then we say that the family of kernels $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is *associated with* μ. If this family exists then it is a possible candidate for a specification and there is a simple characterisation of when this is the case involving independent measures. A measure $\lambda \in \mathrm{P}(\mathcal{F})$ is called *independent* if whenever Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda\cap\Delta = \varnothing$ then $\lambda(E\cap F) = \lambda(E)\lambda(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$. If $\lambda \in \mathrm{P}(\mathcal{F})$ is independent then $\lambda(fg) = \lambda(f)\lambda(g)$ for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^\Delta)$ whenever Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$.

Proposition 4.1 *Let $\mu \in \mathrm{P}(\mathcal{F})$ and suppose that the associated family of kernels $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ exists. Then $\mathcal{V} = \{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is an $\mathbb{F}$-specification if and only if μ is independent. Moreover, in this case $\mathcal{G}(\mathcal{V}) = \{\mu\}$.*

Proof Let Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$. By Proposition 3.1 (4) and (5) $\mu_\Delta\mu_\Lambda = \mu_\Lambda$ holds if and only if $(\mu_\Delta\mu_\Lambda)(f) = \mu_\Delta(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Delta)$, and this in turn holds by Proposition M.14.4 if and only if $(\mu_\Delta\mu_\Lambda)(fg) = \mu_\Delta(fg)$ for all $f \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ and all $g \in \mathrm{M}(\mathcal{F}^\Lambda)$. But

$$(\mu_\Delta\mu_\Lambda)(fg) = \mu_\Delta(\mu_\Lambda(fg)) = \mu_\Delta(f\mu(g)) = \mu_\Delta(f)\mu(g) = \mu(f)\mu(g)$$

and $\mu_\Delta(fg) = \mu(fg)$ for all $f \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ and all $g \in \mathrm{M}(\mathcal{F}^\Lambda)$. Therefore $\mathcal{V}$ is an $\mathbb{F}$-specification if and only if μ is independent. Finally, suppose that $\mathcal{V}$ is an $\mathbb{F}$-specification (and so μ is independent). If $\nu \in \mathcal{G}(\mathcal{V})$ then

$$\nu(g) = (\nu\mu_\Lambda)(g) = \nu(\mu_\Lambda(g)) = \nu(\mu(g)) = \mu(g)$$

for all $g \in \mathrm{M}(\mathcal{F}^\Lambda)$, $\Lambda \in \mathcal{N}_0$, thus $\nu(I_A) = \mu(I_A)$ for all $A \in \mathcal{A}$ and hence $\nu = \mu$. Conversely, if Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$ then

$$(\mu\mu_\Lambda)(fg) = \mu(\mu_\Lambda(fg)) = \mu(f\mu(g)) = \mu(f)\mu(g) = \mu(fg)$$

for all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^\Delta)$ and this implies that $\mu\mu_\Lambda = \mu$ for all $\Lambda \in \mathcal{N}_0$, i.e., $\mu \in \mathcal{G}(\mathcal{V})$. Therefore $\mathcal{G}(\mathcal{V}) = \{\mu\}$. □

Proposition 1.4 shows that any product of probability measures is independent in the lattice model. In the particle model Proposition 2.5 implies that any Poisson point process is independent.

Of course, Proposition 4.1 is not much use without knowing that the family of kernels associated with a measure exists, and so we now look at conditions which ensure that this is the case.

We need to make an assumption about the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$. Let us say that sub-$\sigma$-algebras $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{F}$ are *weakly independent* if $E_1 \cap E_2$ is non-empty whenever $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$ are both non-empty. We call the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ *weakly independent* if $\mathcal{F}^\Lambda$ and $\mathcal{F}^\Delta$ are weakly independent whenever Λ, $\Delta \in \mathcal{N}$ with $\Lambda \cap \Delta = \varnothing$. In the both the lattice and the particle model this is the case, as is shown by Propositions 1.2 and 2.2.

Proposition 4.2 *Suppose that the family* $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ *is weakly independent and that the spatial system* Σ *has the Kolmogorov extension property. Then for each* $\mu \in \mathrm{P}(\mathcal{F})$ *the associated family of kernels exists.*

Proof Fix $\Lambda \in \mathcal{N}_0$ and let $\Delta \in \mathcal{N}$ with $\Lambda \subset \Delta$. Then the σ-algebras $\mathcal{F}^\Lambda$ and $\mathcal{F}^{\Delta\setminus\Lambda}$ are weakly independent and so by Theorem M.15.4 there exists a unique $\mathcal{F}^{\Delta\setminus\Lambda}$-measurable probability kernel $\mu_\Lambda^\Delta \in \mathrm{K}(\mathcal{F}^\Delta)$ such that $\mu_\Lambda^\Delta(gf) = g\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$. If $\Lambda \subset \Delta \subset \Delta'$ then $\mu_\Lambda^{\Delta'}(gf) = g\mu(f) = \mu_\Lambda^\Delta(gf)$ for all $g \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, and so by Proposition M.14.4 $\mu_\Lambda^{\Delta'}(f) = \mu_\Lambda^\Delta(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Delta)$. Thus by Lemma 4.1 (with $\mathcal{N}_0 = \{\Delta \in \mathcal{N} : \Delta \supset \Lambda\}$) there exists a unique probability kernel $\mu_\Lambda \in \mathrm{K}(\mathcal{F})$ with $\mu_\Lambda(f) = \mu_\Lambda^\Delta(f)$ for all $f \in \mathrm{M}(\mathcal{F}^\Delta)$, $\Delta \supset \Lambda$. (and here we needed the Kolmogorov extension property).

Let $f \in \mathrm{M}(\mathcal{F}^\Lambda)$; if $\Delta \in \mathcal{N}_0$ with $\Delta \cap \Lambda = \varnothing$ then $\mu_\Lambda(gf) = \mu_\Lambda^{\Lambda\cup\Delta}(gf) = g\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F}^\Delta)$, and hence $\mu_\Lambda(I_A f) = I_A\mu(f)$ for all $A \in \mathcal{A}_\Lambda$; therefore by Proposition M.14.4 $\mu_\Lambda(gf) = g\mu(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, since $\mathcal{F}_\Lambda = \sigma(\mathcal{A}_\Lambda)$. □

Let $\lambda \in \mathrm{P}(\mathcal{F})$ be independent and suppose that the associated family of kernels $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ exists; thus by Proposition 4.1 $\mathcal{U}$ is a (simple) $\mathbb{F}$-specification. Here are a couple of useful properties of $\mathcal{U}$:

Lemma 4.3 *Let* $\Lambda, \Delta \in \mathcal{N}_0$ *with* $\Lambda \cap \Delta = \varnothing$*; then* $\lambda_{\Lambda\cup\Delta} = \lambda_\Delta\lambda_\Lambda$.

Proof Let $f_1 \in \mathrm{M}(\mathcal{F}^\Lambda)$, $f_2 \in \mathrm{M}(\mathcal{F}^\Delta)$. Then

$$\begin{aligned}(\lambda_\Delta\lambda_\Lambda)(f_1f_2) = \lambda_\Delta(\lambda_\Lambda(f_1f_2)) &= \lambda_\Delta(\lambda(f_1)f_2)\\ &= \lambda(f_1)\lambda(f_2) = \lambda(f_1f_2) = \lambda_{\Lambda\cup\Delta}(f_1f_2)\end{aligned}$$

and so by Proposition 3.1 (2) $\lambda_{\Lambda\cup\Delta}(f) = (\lambda_\Delta\lambda_\Lambda)(f)$ for all $f \in \mathrm{M}(\mathcal{F}^{\Lambda\cup\Delta})$. Thus if $f \in \mathrm{M}(\mathcal{F}^{\Lambda\cup\Delta})$, $g \in \mathrm{M}(\mathcal{F}_{\Lambda\cup\Delta})$ then

$$(\lambda_\Delta\lambda_\Lambda)(gf) = \lambda_\Delta(g\lambda_\Lambda(f)) = g(\lambda_\Delta\lambda_\Lambda)(f) = g\lambda_{\Lambda\cup\Delta}(f) = \lambda_{\Lambda\cup\Delta}(gf)$$

and so again applying Proposition 3.1 (2) $\lambda_{\Lambda\cup\Delta} = \lambda_\Delta\lambda_\Lambda$. □

Lemma 4.4 *Let* $\Lambda, \Delta \in \mathcal{N}_0$*; then* $\lambda_\Lambda(f) \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ *for all* $f \in \mathrm{M}(\mathcal{F}^\Delta)$.

Proof If $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, $g \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ then $\lambda_\Lambda(gf) = g\lambda(f) \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ and thus Proposition M.14.5 implies that $\lambda_\Lambda(f) \in \mathrm{M}(\mathcal{F}^{\Delta\setminus\Lambda})$ for all $f \in \mathrm{M}(\mathcal{F}^\Delta)$, since $\mathcal{F}^\Lambda \vee \mathcal{F}^{\Delta\setminus\Lambda} = \mathcal{F}^\Delta$. □

Now that we have a method of getting started using independent measures, the next problem is to construct new specifications from a given one, and this will be treated in the general situation. We are thus given a non-empty set J equipped with a directed and countably generated partial order $\preceq$, $(X, \mathcal{F})$ is an arbitrary measurable space and $\mathbb{F} = \{\mathcal{F}_A\}_{A \in J}$ is a decreasing family of sub-σ-algebras of $\mathcal{F}$. Let $\mathcal{V} = \{\pi_A\}_{A\in J}$ be an $\mathbb{F}$-specification; then there is really only one reasonable way to try to construct new specifications from $\mathcal{V}$: This involves considering a family $w = \{w_A\}_{A\in J}$, where $w_A \in \mathrm{M}(\mathcal{F})$ for each $A \in J$. Let $A \in J$; then by Proposition 3.1 the mapping $\pi_A^w : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ defined by

$$\pi_A^w(f) = (\pi_A(w_A))^{-1}\pi_A(w_A f)$$

for all $f \in \mathrm{M}(\mathcal{F})$ is a strict $\mathcal{F}_A$-measurable quasi-probability kernel. Moreover,

$$\pi_A^w(f) = \alpha_A^w \pi_A(w_A f) = \pi_A(\alpha_A^w w_A f) = \pi_A(\alpha_A^w j_A^w w_A f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, where $\alpha_A^w \in \mathrm{M}(\mathcal{F}_A)$ and $j_A^w \in \mathrm{M}(\mathcal{F})$ are given by

$$\alpha_A^w(x) = \begin{cases} (\pi_A(w_A)(x))^{-1} & \text{if } 0 < \pi_A(w_A)(x) < \infty\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$j_A^w(x) = \begin{cases} 1 & \text{if } 0 < w_A(x) < \infty\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This means that $\pi_A^w(f) = \pi_A(v_A f)$ for all $f \in \mathrm{M}(\mathcal{F})$, where v_A can be chosen to be either $\alpha_A^w w_A$ or $\alpha_A^w j_A^w w_A$ (and the latter expression has the advantage of not taking on the value ∞).

Since π_A^w is a strict $\mathcal{F}_A$-measurable quasi-probability kernel for each $A \in J$ it follows that $\mathcal{V}_w = \{\pi_A^w\}_{A\in J}$ will be an $\mathbb{F}$-specification provided $\pi_B^w = \pi_B^w \pi_A^w$ for all A, $B \in J$ with $A \preceq B$, and it turns out there is a natural condition under which this holds. The family $w = \{w_A\}_{A\in J}$ is said to be $\mathbb{F}$-*multiplicative* if for each A, $B \in J$ with $A \preceq B$ there exists $w_{B,A} \in \mathrm{M}(\mathcal{F}_A)$ such that $w_B = w_{B,A} w_A$.

Theorem 4.1 *If w is $\mathbb{F}$-multiplicative then $\{\pi_A^w\}_{A\in J}$ is an $\mathbb{F}$-specification.*

Proof This is a corollary of Proposition 4.3 below and the proof is given after that of Proposition 4.3. □

Note that the most obvious way to obtain a $\mathbb{F}$-multiplicative family $\{w_A\}_{A\in J}$ is just to let $w_A = w_0$ for all $A \in J$ for some $w_0 \in \mathrm{M}(\mathcal{F})$. The special case of this in which $w_0 = I_E$ for some $E \in \mathcal{F}$ then results in the $\mathbb{F}$-specification $\{\pi_A^E\}_{A\in J}$, where $\pi_A^E \in \mathrm{K}(\mathcal{F})$ is given by

$$\pi_A^E(f)(x) = \begin{cases} \pi_A(I_E f)(x)/\pi_A(I_E)(x) & \text{if } \pi_A(I_E)(x) > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Multiplicative families are usually obtained by exponentiating additive families. In Chapter 1 we introduced the extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$, denoted by $\mathbb{R}^\Diamond$, together with the natural σ-algebra $\mathcal{B}^\Diamond$ on $\mathbb{R}^\Diamond$. If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then the set of all measurable mappings from $(X, \mathcal{E})$ to $(\mathbb{R}^\Diamond, \mathcal{B}^\Diamond)$ is denoted by $\mathrm{M}^\Diamond(\mathcal{E})$, and if $v \in \mathrm{M}^\Diamond(X)$ then $\exp(-v) \in \mathrm{M}(\mathcal{E})$ if and only if $v \in \mathrm{M}^\Diamond(\mathcal{E})$.

Now for each $A \in J$ let $e_A \in \mathrm{M}^\Diamond(\mathcal{F})$; then the family $e = \{e_A\}_{A \in J}$ is said to be $\mathbb{F}$-*additive* if for each A, $B \in J$ with $A \preceq B$ there is a mapping $e_{B,A} \in \mathrm{M}^\Diamond(\mathcal{F}_A)$ such that $e_B = e_{B,A} + e_A$. If $e = \{e_A\}_{A \in J}$ is $\mathbb{F}$-additive and $w_A = \exp(-e_A)$ for each $A \in J$ then $w = \{w_A\}_{A \in J}$ is clearly $\mathbb{F}$-multiplicative.

In the lattice and particle models additive families were obtained via potentials (Propositions 1.7 and 2.8).

We now turn to Proposition 4.3, and so we are back in the general set-up with an $\mathbb{F}$-specification $\{\pi_A\}_{A \in J}$ and a family $w = \{w_A\}_{A \in J}$ from $\mathrm{M}(\mathcal{F})$. For each $A \in J$ let $i_A^w \in \mathrm{M}(\mathcal{F}_A)$ be the mapping defined by

$$i_A^w(x) = \begin{cases} 1 & \text{if } 0 < \pi_A(w_A)(x) < \infty \,, \\ 0 & \text{otherwise} \,; \end{cases}$$

thus $i_A^w = \pi_A^w(1) = \alpha_A \pi_A(w_A)$, where as above $\alpha_A^w \in \mathrm{M}(\mathcal{F}_A)$ is given by

$$\alpha_A^w(x) = \begin{cases} (\pi_A(w_A)(x))^{-1} & \text{if } 0 < \pi_A(w_A)(x) < \infty \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Proposition 4.3 *Let A, $B \in J$ with $A \preceq B$. Then the following statements are equivalent:*

(1) $\pi_B^w \pi_A^w = \pi_B^w$,

(2) conditions ($\diamond$) *and* ($\sharp$) *hold,*

(3) conditions ($\diamond$) *and* ($\flat$) *hold,*

where

($\diamond$) $i_B^w \pi_B(w_B) = i_B^w \pi_B(i_A^w w_B)$,

($\sharp$) $\pi_B(\pi_A(w_A f)\pi_A(w_B g)) = \pi_B(\pi_A(w_B f)\pi_A(w_A g))$ *for all* f, $g \in \mathrm{M}(\mathcal{F})$,

($\flat$) $\pi_B(\pi_A(w_A f)\pi_A(w_B)) = \pi_B(\pi_A(w_B f)\pi_A(w_A))$ *for all* $f \in \mathrm{M}(\mathcal{F})$.

Before starting with the proof let us point out that it is very much a matter of taste how the terms in ($\sharp$) and ($\flat$) should be expressed. Each side of the two equations has more-or-less the same form, to be definite consider the left-hand side $\pi_B(\pi_A(w_A f)\pi_A(w_B g))$ of the equation in ($\sharp$). Then, since $\pi_A(w_A f) \in \mathrm{M}(\mathcal{F}_A)$,

$$\begin{aligned} \pi_B(\pi_A(w_A f)\pi_A(w_B g)) &= \pi_B(\pi_A(\pi_A(w_A f) w_B g)) \\ &= (\pi_B \pi_A)(\pi_A(w_A f) w_B g) = \pi_B(\pi_A(w_A f) w_B g) \end{aligned}$$

and in the same way $\pi_B(\pi_A(w_A f)\pi_A(w_B g)) = \pi_B(w_A f \pi_A(w_B g))$. This means

$$\pi_B(\pi_A(w_A f)\pi_A(w_B g)) = \pi_B(\pi_A(w_A f)w_B g) = \pi_B(w_A f \pi_A(w_B g)) ,$$

which gives us two alternative expressions for $\pi_B(\pi_A(w_A f)\pi_A(w_B g))$. (In fact, the manipulation performed here also plays a role in the proof below.)

Proof of Proposition 4.3: Note first the following simple fact:

Lemma 4.5 *Condition (◇) holds if and only if* $\pi^w_B(1) = \pi^w_B(\pi^w_A(1))$.

Proof This is clear, since $\pi^w_B(1) = \alpha^w_B \pi_B(w_B)$ and

$$\pi^w_B(\pi^w_A(1)) = \alpha^w_B \pi_B(w_B \alpha^w_A \pi_A(w_A)) = \alpha^w_B \pi_B(w_B i^w_A) = \alpha^w_B \pi_B(i^w_A w_B) . \quad \square$$

The following will be needed in the proof that (3) ⇒ (1):

Lemma 4.6 *If (◇) holds then for all* $f \in \mathrm{M}(\mathcal{F})$

$$i^w_B \pi_B(w_B f) = i^w_B \pi_B(i^w_A w_B f) .$$

Proof Since $0 \le 1 - i^w_A \le 1$, $i^w_B \pi_B(w_B) < \infty$ and

$$i^w_B \pi_B(w_B) = i^w_B \pi_B((1 - i^w_A)w_B) + i^w_B \pi_B(i^w_A w_B) = i^w_B \pi_B((1 - i^w_A)w_B) + i^w_B \pi_B(w_B)$$

it follows that $i^w_B \pi_B((1 - i^w_A)w_B) = 0$. Therefore by Proposition M.14.3 (1) we have $i^w_B \pi_B((1 - i^w_A)w_B f) = 0$ and this means that $i^w_B \pi_B(w_B f) = i^w_B \pi_B(i^w_A w_B f)$ for all $f \in \mathrm{M}(\mathcal{F})$. □

(1) ⇒ (2): Let $g \in \mathrm{M}(\mathcal{F})$, $h \in \mathrm{M}(\mathcal{F}_A)$; then

$$\pi^w_B(hg) = \alpha^w_B \pi_B(w_B hg) = \alpha^w_B \pi_B(\pi_A(w_B hg)) = \alpha^w_B \pi_B(h \pi_A(w_B g)) ,$$

and also

$$\begin{aligned} \pi^w_B(\pi^w_A(hg)) &= \alpha^w_B \pi_B(w_B \alpha^w_A \pi_A(w_A hg)) = \alpha^w_B \pi_B(w_B \alpha^w_A h \pi_A(w_A g)) \\ &= \alpha^w_B \pi_B(\pi_A(w_B \alpha^w_A h \pi_A(w_A g))) = \alpha^w_B \pi_B(\alpha^w_A h \pi_A(w_B \pi_A(w_A g))) \\ &= \alpha^w_B \pi_B(\alpha^w_A h \pi_A(w_A g) \pi_A(w_B)) , \end{aligned}$$

and since $\pi^w_B \pi^w_A = \pi^w_B$ this implies that

$$\alpha^w_B \pi_B(h \pi_A(w_B g)) = \alpha^w_B \pi_B(\alpha^w_A h \pi_A(w_A g) \pi_A(w_B)) .$$

Let $f \in \mathrm{M}(\mathcal{F})$; then applying this with $h = \pi_A(w_A f)$ gives

$$\alpha^w_B \pi_B(\pi_A(w_A f)\pi_A(w_B g)) = \alpha^w_B \pi_B(\alpha^w_A \pi_A(w_A f)\pi_A(w_A g)\pi_A(w_B)) \ .$$

But the right-hand side has the same value if f and g are interchanged, and so

$$\begin{aligned} \alpha^w_B \pi_B(\pi_A(w_A f)\pi_A(w_B g)) &= \alpha^w_B \pi_B(\pi_A(w_A g)\pi_A(w_B f)) \\ &= \alpha^w_B \pi_B(\pi_A(w_B f)\pi_A(w_A g)) \end{aligned}$$

for all f, $g \in \mathrm{M}(\mathcal{F})$. Therefore

$$i^w_B \pi_B(\pi_A(w_A f)\pi_A(w_B g)) = i^w_B \pi_B(\pi_A(w_B f)\pi_A(w_A g))$$

for all f, $g \in \mathrm{M}(\mathcal{F})$, which is ($\sharp$). Lemma 4.5 implies that ($\diamond$) holds.

(2) $\Rightarrow$ (3): This is clear, since ($\flat$) is just the special case of ($\sharp$) with $g = 1$.

(3) $\Rightarrow$ (1): Let $f \in \mathrm{M}(\mathcal{F})$; then, since $\alpha^w_A \pi_A(w_B f) \in \mathrm{M}(\mathcal{F}_A)$,

$$\begin{aligned} \pi^w_B(\pi^w_A(f)) &= \alpha^w_B \pi_B(w_B \alpha^w_A \pi_A(w_A f)) = \alpha^w_B \pi_B(\pi_A(w_B \alpha^w_A \pi_A(w_A f))) \\ &= \alpha^w_B \pi_B(\alpha^w_A \pi_A(w_A f)\pi_A(w_B)) = \alpha^w_B \pi_B(\pi_A(w_A \alpha^w_A f)\pi_A(w_B)) \ ; \end{aligned}$$

on the other hand, by Lemma 4.6 and the fact that $\pi_A(w_B) \in \mathrm{M}(\mathcal{F}_A)$,

$$\begin{aligned} \pi^w_B(f) &= \alpha^w_B \pi_B(w_B f) = \alpha^w_B \pi_B(i^w_A w_B f) = \alpha^w_B \pi_B(\alpha^w_A \pi_A(w_A) w_B f) \\ &= \alpha^w_B \pi_B(\pi_A(\alpha^w_A \pi_A(w_A) w_B f)) = \alpha^w_B \pi_B(\alpha^w_A \pi_A(w_A)\pi_A(w_B f)) \\ &= \alpha^w_B \pi_B(\alpha^w_A \pi_A(w_B f)\pi_A(w_A)) = \alpha^w_B \pi_B(\pi_A(w_B \alpha^w_A f)\pi_A(w_A)) \ . \end{aligned}$$

Thus by ($\flat$) $(\pi^w_B \pi^w_A)(f) = \pi^w_B(\pi^w_A(f)) = \pi^w_B(f)$ for all $f \in \mathrm{M}(\mathcal{F})$ and therefore $\pi^w_B \pi^w_A = \pi^w_B$. □

Proof of Theorem 4.1: Let A, $B \in J$ with $A \preceq B$; we show that conditions ($\diamond$) and ($\sharp$) in Proposition 4.3 hold. By the definition of being $\mathbb{F}$-multiplicative there exists $w_{B,A} \in \mathrm{M}(\mathcal{F}_A)$ such that $w_B = w_{B,A} w_A$. Then

$$\pi_B(i^w_A w_B) = \pi_B(\pi_A(i^w_A w_B)) = \pi_B(\pi_A(i^w_A w_{B,A} w_A)) = \pi_B(i^w_A w_{B,A} \pi_A(w_A))$$

and in exactly the same way it follows that $\pi_B(w_B) = \pi_B(w_{B,A}\pi_A(w_A))$. Let $x \in X$ with $\pi_B(w_B)(x) < \infty$; then $\pi_B(w_{B,A}\pi_A(w_A))(x) < \infty$ and therefore by Proposition M.14.3 (2) $\pi_B(w_{B,A}\pi_A(w_A))(x) = \pi_B(i^w_A w_{B,A}\pi_A(w_A))(x)$ holds, since $\{y \in X : 0 < w_{B,A}\pi_A(w_A)(y) < \infty\} \subset \{y \in X : 0 < \pi_A(w_A)(y) < \infty\}$. Hence $i^w_B \pi_B(w_A) = i^w_B \pi(i^w_A w_B)$, which shows ($\diamond$) holds. Now for all f, $g \in \mathrm{M}(\mathcal{F})$

$$\begin{aligned} \pi_A(w_A f)\pi_A(w_B g) &= \pi_A(w_A f)\pi_A(w_{B,A} w_A g) \\ &= \pi_A(w_A f) w_{B,A} \pi_A(w_A g) = w_{B,A}\pi_A(w_A f)\pi_A(w_A g) \\ &= \pi_A(w_{B,A} w_A f)\pi_A(w_A g) = \pi_A(w_B f)\pi_A(w_A g) \end{aligned}$$

and so in particular $\pi_B(\pi_A(w_A f)\pi_A(w_B g)) = \pi_B(\pi_A(w_B f)\pi_A(w_A g))$. This shows that ($\sharp$) holds. Thus by Proposition 4.3 $\pi^w_B = \pi^w_B \pi^w_A$. □

5 Existence of Gibbs states

The aim in this chapter is to find reasonable sufficient conditions on a specification $\mathcal{V}$ to ensure that the set of Gibbs states $\mathcal{G}(\mathcal{V})$ is non-empty. The methods which will be used to this end also give sufficient conditions for a subset Q of $\mathcal{G}(\mathcal{V})$ to be, in a certain sense, sequentially compact.

It is possible for $\mathcal{G}(\mathcal{V})$ to be empty, even when $\mathcal{V}$ is defined in a 'reasonable' way. Let $Q : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$ be the transition matrix of a random walk (i.e., $Q(s,t)$ depends only on $s - t$). If $Q(s,t) > 0$ for all $s, t \in \mathbb{Z}$ then the natural conditional probabilities arising from Q define a specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$, where $\mathcal{N}_0$ consists of all intervals of the form $\{k \in \mathbb{Z} : n \leq k \leq m\}$. However, it was shown by Spitzer [16] that $\mathcal{G}(\mathcal{V})$ is empty.

We only consider specifications arising from spatial systems, although most of the results could be reformulated to deal with more general situations. Thus in what follows let $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda \in \mathcal{N}})$ be a spatial system (as defined in Chapters 4) and let $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ be the associated family of σ-algebras. Thus the basic measurable space is $(X, \mathcal{F})$ with $\mathcal{F} = \sigma(\mathcal{A})$, where $\mathcal{A} = \bigcup_{\Lambda \in \mathcal{N}} \mathcal{F}^\Lambda$ is the algebra of local observables.

We assume that Σ has the Kolmogorov extension property, and emphasise that it is this property which ensures that everything works.

Before going any further, let us state the main result of this chapter in the form which can be applied to most lattice models. Let $\lambda \in \mathrm{P}(\mathcal{F})$ be independent and assume that the associated family of kernels $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ exists. Thus λ_Λ is the unique kernel with $\lambda_\Lambda(gf) = g\lambda(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$ and by Proposition 4.1 $\mathcal{U}$ is a simple $\mathbb{F}$-specification. Now let $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ be an $\mathbb{F}$-additive family (with $e_\Lambda \in \mathrm{M}^\Diamond(\mathcal{F})$ for each $\Lambda \in \Lambda_0$) and for each $\Lambda \in \mathcal{N}_0$ put $w_\Lambda = \exp(-e_\Lambda)$, so $w = \{w_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is an $\mathbb{F}$-multiplicative family. By Theorem 4.1 there is then the $\mathbb{F}$-specification $\mathcal{U}_w = \{\lambda^w_\Lambda\}_{\Lambda \in \mathcal{N}_0}$, where

$$\lambda^w_\Lambda(f) = (\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $\Lambda \in \mathcal{N}_0$. A mapping $u \in \mathrm{M}^\Diamond(\mathcal{F})$ will be called *quasi-local* if for each $\varepsilon > 0$ there exists $\Lambda \in \mathcal{N}$ and a bounded mapping $v \in \mathrm{M}^\Diamond(\mathcal{F}^\Lambda)$ such that $|u - v| \leq \varepsilon$ (and so in particular u is also bounded).

Proposition 5.1 *Suppose the mapping e_Λ is quasi-local for each $\Lambda \in \mathcal{N}_0$. Then the set of Gibbs states $\mathcal{G}(\mathcal{U}_w)$ is non-empty. Moreover, for each sequence $\{\mu_n\}_{n \geq 1}$ from $\mathcal{G}(\mathcal{U}_w)$ there exists a subsequence $\{n_j\}_{j \geq 1}$ and a Gibbs state $\mu \in \mathcal{G}(\mathcal{U}_w)$ such that $\lim_j \mu_{n_j}(A) = \mu(A)$ for all $A \in \mathcal{A}$.*

Proof This will follow from Theorems 5.1 and 5.2 below. Note that, since e_Λ is bounded for each $\Lambda \in \mathcal{N}_0$, it easily follows that the specification $\mathcal{U}_w$ occurring here is simple. □

For the lattice model Proposition 1.8 gives a condition on a potential Φ which ensures that the conditional energy E^{Φ}_{Λ} is quasi-local for each $\Lambda \in \mathcal{N}_0$.

We now start to develop a general framework for establishing the existence of Gibbs states and first need to introduce the quasi-local mappings in $\mathrm{M}(\mathcal{F})$. Recall that if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then $\mathrm{M}_{\mathsf{B}}(\mathcal{E})$ denotes the set of bounded mappings in $\mathrm{M}(\mathcal{E})$ and that by Lemma M.9.5 (1) $\mathrm{M}_{\mathsf{B}}(\mathcal{E})$ is a subspace of $\mathrm{M}(\mathcal{E})$ and thus also of $\mathrm{M}(\mathcal{F})$. Denote by $\mathrm{M}_{\mathsf{B}}(\mathcal{A})$ the union of the sets $\mathrm{M}_{\mathsf{B}}(\mathcal{F}^{\Lambda})$, $\Lambda \in \mathcal{N}$. Also let $\mathrm{M}_{\mathsf{E}}(\mathcal{A})$ be the union of the sets $\mathrm{M}_{\mathsf{E}}(\mathcal{F}^{\Lambda})$, $\Lambda \in \mathcal{N}$, thus $\mathrm{M}_{\mathsf{E}}(\mathcal{A})$ consists of those mappings $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ for which $f(X)$ is a finite subset of $\mathbb{R}^+$.

Lemma 5.1 *(1)* $\mathrm{M}_{\mathsf{B}}(\mathcal{A})$ *is a normal subspace of* $\mathrm{M}_{\mathsf{B}}(\mathcal{F})$.

(2) If $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ *then for each* $\varepsilon > 0$ *there exists a mapping* $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$ *such that* $g \le f \le g + \varepsilon$.

Proof These all follow from Lemma M.9.5 and the fact that the order $\subset$ on $\mathcal{N}$ is directed. □

A mapping $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ is said to be *quasi-local* if for each $\varepsilon > 0$ there exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ such that $g \le f + \varepsilon$ and $f \le g + \varepsilon$ (i.e., such that $|f - g| \le \varepsilon$). The set of all such quasi-local mappings will be denoted by $\mathrm{M}_{\diamondsuit}(\mathcal{A})$. In particular $\mathrm{M}_{\mathsf{B}}(\mathcal{A}) \subset \mathrm{M}_{\diamondsuit}(\mathcal{A})$.

Lemma 5.2 *(1)* $\mathrm{M}_{\diamondsuit}(\mathcal{A})$ *is a subspace of* $\mathrm{M}_{\mathsf{B}}(\mathcal{F})$.

(2) A mapping $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ *is quasi-local if and only if for each* $\varepsilon > 0$ *there exists* $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ *with* $g \le f \le g + \varepsilon$.

(3) If $f \in \mathrm{M}_{\diamondsuit}(\mathcal{A})$ *then for each* $\varepsilon > 0$ *there exists a mapping* $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$ *such that* $g \le f \le g + \varepsilon$.

(4) Let $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ *and suppose that for each* $\varepsilon > 0$ *there exists* $g \in \mathrm{M}_{\diamondsuit}(\mathcal{A})$ *with* $g \le f \le g + \varepsilon$. *Then* $f \in \mathrm{M}_{\diamondsuit}(\mathcal{A})$.

Proof (2) Let $f \in \mathrm{M}_{\diamondsuit}(\mathcal{A})$, let $\varepsilon > 0$ and put $\eta = \varepsilon/2$; there therefore exists $h \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $h \le f + \eta$ and $f \le h + \eta$. Now $\eta \le h \vee \eta$ and so by Lemma 5.1 (1) the mapping $g = h \vee \eta - \eta$ is in $\mathrm{M}_{\mathsf{B}}(\mathcal{A})$. Let $x \in X$; if $h(x) \le \eta$ then $g(x) = 0$ and hence $g(x) \le f(x) \le h(x) + \eta \le \eta + \eta = \varepsilon = g(x) + \varepsilon$. On the other hand, if $h(x) > \eta$ then $g(x) = h(x) - \eta$ and here

$$g(x) = h(x) - \eta \le f(x) + \eta - \eta = f(x) \le h(x) + \eta = g(x) + \eta + \eta = g(x) + \varepsilon \,.$$

Thus $g(x) \le f(x) \le g(x) + \varepsilon$ for all $x \in X$, i.e., $g \le f \le g + \varepsilon$. The converse is clear.

(1) Let f_1, $f_2 \in \mathrm{M}_\diamondsuit(\mathcal{A})$ and a_1, $a_2 \in \mathbb{R}^+$. Let $\varepsilon > 0$; by (2) there then exist g_1, $g_2 \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ such that $g_1 \le f_1 \le g_1 + \eta$ and $g_2 \le f_2 \le g_2 + \eta$, where $\eta = \varepsilon/(a_1 + a_2 + 1)$; then by Lemma 5.1 (1) $a_1 g_1 + a_2 g_2 \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ and

$$\begin{aligned} a_1 g_1 + a_2 g_2 \le a_1 f_1 + a_2 f_2 &\le a_1(g_1 + \eta) + a_2(g_2 + \eta) \\ &= a_1 g_1 + a_2 g_2 + (a_1 + a_2)\eta \le a_1 g_1 + a_2 g_2 + \varepsilon \ . \end{aligned}$$

Thus by (2) $a_1 f_1 + a_2 f_2 \in \mathrm{M}_\diamondsuit(\mathcal{A})$ and this shows $\mathrm{M}_\diamondsuit(\mathcal{A})$ is a subspace of $\mathrm{M}_{\mathsf{B}}(\mathcal{F})$, since also $0 \in \mathrm{M}_\diamondsuit(\mathcal{A})$.

(3) Let $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$ and $\varepsilon > 0$; then by (2) there exists a mapping $h \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $h \le f \le h + \varepsilon/2$ and by Lemma 5.1 (2) there exists $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$ with $g \le h \le g + \varepsilon/2$. Thus $g \le h \le f \le h + \varepsilon/2 \le g + \varepsilon/2 + \varepsilon/2 = g + \varepsilon$, i.e., $g \le f \le g + \varepsilon$.

(4) Let $\varepsilon > 0$; then there exists $h \in \mathrm{M}_\diamondsuit(\mathcal{A})$ with $h \le f \le h + \varepsilon/2$ and by (2) there there then exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $g \le h \le g + \varepsilon/2$. Thus (as in (3)) $g \le f \le g + \varepsilon$, and this shows that $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$. □

A sequence $\{\mu_n\}_{n\ge 1}$ from $\mathrm{P}(\mathcal{F})$ will be said to *converge locally* to $\mu \in \mathrm{P}(\mathcal{F})$ if $\mu(A) = \lim_n \mu_n(A)$ for all $A \in \mathcal{A}$. (The second part of Proposition 5.1 is thus a statement about local convergence.) If $\{\mu_n\}_{n\ge 1}$ converges locally to both μ and μ' then by Proposition M.3.3 $\mu = \mu'$.

Lemma 5.3 *A sequence $\{\mu_n\}_{n\ge 1}$ from $\mathrm{P}(\mathcal{F})$ converges locally to $\mu \in \mathrm{P}(\mathcal{F})$ if and only if $\mu(f) = \lim_n \mu_n(f)$ for all $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$.*

Proof Suppose $\{\mu_n\}_{n\ge 0}$ converges locally to μ. Then $\mu(g) = \lim_n \mu_n(g)$ clearly holds for each $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$. Let $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$; then for each $\varepsilon > 0$ there exists by Lemma 5.2 (2) $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$ such that $g \le f \le g + \varepsilon$. Thus

$$\limsup_{n\to\infty} \mu_n(f) \le \limsup_{n\to\infty} \mu_n(g + \varepsilon) = \lim_{n\to\infty} \mu_n(g) + \varepsilon = \mu(g) + \varepsilon \le \mu(f) + \varepsilon$$

and in the same way

$$\mu(f) \le \mu(g) + \varepsilon = \lim_{n\to\infty} \mu_n(g) + \varepsilon \le \liminf_{n\to\infty} \mu_n(f) + \varepsilon \ .$$

From this it follows that $\mu(f) = \lim_n \mu_n(f)$. The converse is clear. □

Let $\mathcal{E}$ be a σ-algebra of subsets of X; a decreasing sequence $\{E_k\}_{k\ge 1}$ from $\mathcal{E}$ with $\bigcap_{k\ge 1} E_k = \varnothing$ will be called a *null sequence from* $\mathcal{E}$. A subset Q of $\mathrm{P}(\mathcal{E})$ is said to be *equicontinuous* if for each each null sequence $\{E_k\}_{k\ge 1}$ from $\mathcal{E}$ and each $\varepsilon > 0$ there exists $p \ge 1$ such that $\mu(E_p) < \varepsilon$ for all $\mu \in Q$.

Proposition 5.2 *Let Q be an equicontinuous subset of* $\mathrm{P}(\mathcal{E})$. *Then for each sequence* $\{\mu_n\}_{n\geq 1}$ *from Q there exists a subsequence* $\{n_j\}_{j\geq 1}$ *and* $\mu \in \mathrm{P}(\mathcal{E})$ *such that* $\mu(E) = \lim_j \mu_{n_j}(E)$ *for all* $E \in \mathcal{E}$.

Proof This is the elementary half of the Dunford-Pettis theorem; a proof can be found in Chapter M.17. □

The next result gives a simple criterion for showing that a subset Q of $\mathrm{P}(\mathcal{E})$ is equicontinuous.

Lemma 5.4 *Let Q be a subset of* $\mathrm{P}(\mathcal{E})$ *and suppose for each* $\varepsilon > 0$ *there exists a finite measure* ω *on* $\mathcal{E}$ *and* $\delta > 0$ *such that* $\mu(E) < \varepsilon$ *for all* $\mu \in Q$ *whenever* $E \in \mathcal{E}$ *with* $\omega(E) < \delta$. *Then Q is equicontinuous.*

Proof This is clear, since if ω is a finite measure on $\mathcal{E}$ and $\{E_k\}_{k\geq 1}$ is a null sequence from $\mathcal{E}$ then by Lemma M.3.2 $\lim_n \omega(E_n) = 0$. □

The converse of Lemma 5.4 is also true (see Dunford and Schwartz [7], IV.9).

Now a subset Q of $\mathrm{P}(\mathcal{F})$ will be called *locally equicontinuous* if for each $\Lambda \in \mathcal{N}$ the restrictions of the measures in Q to $\mathcal{F}^\Lambda$ are equicontinuous, i.e., if for each $\Lambda \in \mathcal{N}$, each null sequence $\{F_k\}_{k\geq 1}$ from $\mathcal{F}^\Lambda$ and each $\varepsilon > 0$ there exists $p \geq 0$ such that $\mu(F_p) < \varepsilon$ for all $\mu \in Q$.

Proposition 5.3 *Let* $Q \subset \mathrm{P}(\mathcal{F})$ *be locally equicontinuous and let* $\{\mu_n\}_{n\geq 1}$ *be a sequence from Q. Then there exists a subsequence* $\{n_j\}_{j\geq 1}$ *such that* $\{\mu_{n_j}\}_{j\geq 1}$ *converges locally to some element* $\mu \in \mathrm{P}(\mathcal{F})$.

Proof This is essentially the same as the proof of Theorem M.19.2. Let $\{\Lambda_k\}_{k\geq 0}$ be an increasing order generating sequence from $\mathcal{N}$. Applying Proposition 5.2 to the restrictions of the measures in Q to $\mathcal{F}^{\Lambda_k}$ for each $k \geq 0$, and employing the usual diagonal argument (Theorem M.23.1), shows there exists a subsequence $\{n_j\}_{j\geq 1}$ and for each $k \geq 0$ a measure $\nu_k \in \mathrm{P}(\mathcal{F}^{\Lambda_k})$ such that $\nu_k(F) = \lim_j \mu_{n_j}(F)$ for all $F \in \mathcal{F}^{\Lambda_k}$, $k \geq 0$. But the sequence $\{\nu_k\}_{n\geq 0}$ is then clearly consistent (in that $\nu_k(F) = \nu_{k+1}(F)$ for all $F \in \mathcal{F}^{\Lambda_k}$, $k \geq 0$) and hence, since $\{\Lambda_k\}_{k\geq 0}$ is order generating and Σ has the Kolmogorov extension property, there exists a unique probability measure $\mu \in \mathrm{P}(\mathcal{F})$ such that $\mu(F) = \nu_{\Lambda_k}(F)$ for all $F \in \mathcal{F}^{\Lambda_k}$, $k \geq 0$. Let $A \in \mathcal{A}$; then $A \in \mathcal{F}^{\Lambda_k}$ for some $k \geq 0$ and so $\mu(A) = \nu_k(A) = \lim_j \mu_{n_j}(A)$; this implies that $\mu(A) = \lim_j \mu_{n_j}(A)$ for all $A \in \mathcal{A}$. □

It is also useful to have local equicontinuity for families indexed by $\mathcal{N}_0$: A family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ of elements from $\mathrm{P}(\mathcal{F})$ will be called *locally equicontinuous* if for each $\Delta \in \mathcal{N}$, each null sequence $\{F_k\}_{k\geq 1}$ from $\mathcal{F}^\Delta$ and each $\varepsilon > 0$ there exists $\Lambda' \in \mathcal{N}_0$ and $p \geq 1$ so that $\mu_\Lambda(F_p) < \varepsilon$ for all $\Lambda \in \mathcal{N}$ with $\Lambda' \subset \Lambda$.

Lemma 5.5 *Let $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be a locally equicontinuous family from* $\mathrm{P}(\mathcal{F})$*. Then there exists an increasing order-generating sequence $\{\Lambda_n\}_{n\ge1}$ from $\mathcal{N}_0$ such that $\{\mu_{\Lambda_n}\}_{n\ge1}$ converges locally to some element $\mu\in\mathrm{P}(\mathcal{F})$.*

Proof Let $\{\Lambda_n\}_{n\ge1}$ be any increasing order-generating sequence from $\mathcal{N}_0$ and put $Q=\{\mu_{\Lambda_n}: n\ge1\}$. Then Q is a locally equicontinuous subset of $\mathrm{P}(\mathcal{F})$: Let $\Delta\in\mathcal{N}$, let $\{F_k\}_{k\ge1}$ be a null sequence from $\mathcal{F}^\Delta$ and let $\varepsilon>0$. There thus exists $\Gamma\in\mathcal{N}_0$ and $p\ge1$ so that $\mu_\Lambda(F_p)<\varepsilon$ for all $\Lambda\in\mathcal{N}_0$ with $\Gamma\subset\Lambda$. But (since $\{\Lambda_n\}_{n\ge1}$ is order-generating) there then exists $m\ge1$ such that $\Gamma\subset\Lambda_n$ for all $n\ge m$. Moreover, $\lim_k\mu_{\Lambda_n}(F_k)=0$ for each $n\ge1$, and hence there exists $q\ge p$ such that $\mu_{\Lambda_n}(F_q)<\varepsilon$ for $q=1,\ldots,m-1$, which means $\mu_{\Lambda_n}(F_q)<\varepsilon$ for all $n\ge1$. Now by Proposition 5.3 there exists a subsequence $\{n_j\}_{j\ge1}$ such that if $\Lambda'_j=\Lambda_{n_j}$ then $\{\mu_{\Lambda'_j}\}_{j\ge1}$ converges locally to some $\mu\in\mathrm{P}(\mathcal{F})$. This completes the proof, since the sequence $\{\Lambda'_n\}_{n\ge1}$ is also order-generating. □

Let us say that a kernel $\pi\in\mathrm{K}(\mathcal{F})$ *bounded* if $\pi(1)\in\mathrm{M}_{\mathsf{B}}(\mathcal{F})$, i.e., if there exists $c\in\mathbb{R}^+$ such that $\pi(x,X)\le c$ for all $x\in X$. If this case $\pi(f)\in\mathrm{M}_{\mathsf{B}}(\mathcal{F})$ for all $f\in\mathrm{M}_{\mathsf{B}}(\mathcal{F})$, since if $\pi(1)\le c$ and $f\le b$ then $\pi(f)\le bc$. A bounded kernel π will be called *quasi-local* if $\pi(\cdot,F)\in\mathrm{M}_\diamondsuit(\mathcal{A})$ for all $F\in\mathcal{A}$.

Lemma 5.6 *A bounded kernel π is quasi-local if and only if $\pi(f)\in\mathrm{M}_\diamondsuit(\mathcal{A})$ for all $f\in\mathrm{M}_\diamondsuit(\mathcal{A})$.*

Proof Suppose π is quasi-local, let $f\in\mathrm{M}_\diamondsuit(\mathcal{A})$ and $\varepsilon>0$. Choose $c>0$ with $\pi(1)\le c$. By Lemma 5.2 (3) there exists $g\in\mathrm{M}_{\mathsf{E}}(\mathcal{A})$ with $g\le f\le g+\varepsilon/c$, and therefore $\pi(g)\le\pi(f)\le\pi(g)+\varepsilon$, since $\pi(\varepsilon/c)\le\varepsilon$. Now g has the form $c_1I_{A_1}+\cdots+c_mI_{A_m}$ with $c_1,\ldots,c_m\in\mathbb{R}^+$ and $A_1,\ldots,A_m\in\mathcal{A}$ and thus

$$\pi(g)=c_1\pi(I_{A_1})+\cdots+c_m\pi(I_{A_m})=c_1\pi(\cdot,A_1)+\cdots+c_m\pi(\cdot,A_m)\,.$$

Therefore by Lemma 5.2 (1) $\pi(g)\in\mathrm{M}_\diamondsuit(\mathcal{A})$ and, since this holds for all $\varepsilon>0$, Lemma 5.2 (4) implies that $\pi(f)\in\mathrm{M}_\diamondsuit(\mathcal{A})$. The converse is clear. □

Now in what follows let $\mathcal{V}=\{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be an $\mathbb{F}$-specification (and so a specification finally enters the picture). Let us say that $\mathcal{V}$ is *quasi-local* if each of the kernels π_Λ has this property.

Lemma 5.7 *Let $\mathcal{V}$ be quasi-local and let $\{\mu_n\}_{n\ge1}$ be a sequence from $\mathrm{P}(\mathcal{F})$ which converges locally to $\mu\in\mathrm{P}(\mathcal{F})$. Suppose that for each $\Lambda\in\mathcal{N}_0$ there exists $m\ge1$ such that $\mu_n\pi_\Lambda=\mu_n$ for all $n\ge m$. Then $\mu\in\mathcal{G}(\mathcal{V})$.*

Proof Let $\Lambda \in \mathcal{N}$; for each $A \in \mathcal{A}$ it follows from Lemma 5.3 that

$$\begin{aligned}(\mu\pi_\Lambda)(I_A) = \mu(\pi_\Lambda(I_A)) = \lim_{n\to\infty}\mu_n(\pi_\Lambda(I_A)) &= \lim_{n\to\infty}(\mu_n\pi_\Lambda)(I_A) \\ &= \lim_{n\to\infty}\mu_n(I_A) = \mu(I_A)\end{aligned}$$

since $\mu_n\pi_\Lambda = \mu_n$ for all large enough n. Thus $\mu\pi_\Lambda = \mu$. □

The next result states that if $\mathcal{V}$ is quasi-local then the set of Gibbs states $\mathcal{G}(\mathcal{V})$ is closed under local convergence.

Proposition 5.4 *Let $\mathcal{V}$ be quasi-local and let $\{\mu_n\}_{n\ge1}$ be a sequence from $\mathcal{G}(\mathcal{V})$ which converges locally to $\mu \in \mathrm{P}(\mathcal{F})$. Then $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof This follows immediately from Lemma 5.7. □

Here is the first existence result for Gibbs states:

Theorem 5.1 *Suppose that $\mathcal{V}$ is quasi-local and there exists a family $\{x^\Lambda\}_{\Lambda\in\mathcal{N}_0}$ of elements of X such that $\{\pi_\Lambda(x^\Lambda,\cdot)\}_{\Lambda\in\mathcal{N}_0}$ is a locally equicontinuous family from $\mathrm{P}(\mathcal{F})$. Then $\mathcal{G}(\mathcal{V})$ is non-empty.*

Proof For each $\Lambda \in \mathcal{N}_0$ put $\mu_\Lambda = \pi_\Lambda(x^\Lambda,\cdot)$. By Lemma 5.5 there is an increasing order-generating sequence $\{\Lambda_n\}_{n\ge1}$ from $\mathcal{N}_0$ and $\mu \in \mathrm{P}(\mathcal{F})$ such that $\{\mu_{\Lambda_n}\}_{n\ge1}$ converges locally to μ. Let $\Lambda \in \mathcal{N}_0$; then then exists $m \ge 1$ with $\Lambda \subset \Lambda_m$ and thus by Lemma M.14.2

$$\mu_n\pi_\Lambda = \pi_{\Lambda_n}(x^{\Lambda_n},\cdot)\pi_\Lambda = (\pi_{\Lambda_n}\pi_\Lambda)(x^{\Lambda_n},\cdot) = \pi_{\Lambda_n}(x^{\Lambda_n},\cdot) = \mu_n$$

for all $n \ge m$. Therefore by Lemma 5.7 $\mu \in \mathcal{G}(\mathcal{V})$. □

Theorem 5.2 *Suppose that $\mathcal{V}$ is quasi-local and Q is a locally equicontinuous subset of $\mathcal{G}(\mathcal{V})$. Then for each sequence $\{\mu_n\}_{n\ge1}$ from Q there exists a subsequence $\{n_j\}_{j\ge1}$ and $\mu \in \mathcal{G}(\mathcal{V})$ such that $\{\mu_{n_j}\}_{j\ge1}$ converges locally to μ.*

Proof This follows immediately from Proposition 5.3 and Lemma 5.7. □

Proof of Proposition 5.1: Let $\Lambda \in \mathcal{N}_0$, $F \in \mathcal{A}$ and $\varepsilon > 0$. Since e_Λ is quasi-local there exists $\Delta \in \mathcal{N}$ and a bounded mapping $u \in \mathrm{M}^\diamond(\mathcal{F}^\Delta)$ with $|e_\Lambda - u| \le \varepsilon$. Put $v = \exp(-u)$ and let $f = (\lambda_\Lambda(v))^{-1}\lambda_\Lambda(vI_F)$. Then $v \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}^\Delta)$ and hence by Lemma 4.4 $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}^{\Delta\setminus\Lambda}) \subset \mathrm{M}_{\mathsf{B}}(\mathcal{A})$. But

$$\begin{aligned}f = (\lambda_\Lambda(v))^{-1}\lambda_\Lambda(vI_F) &\le \exp(2\varepsilon)(\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda I_F) \\ &= \exp(2\varepsilon)\lambda^w_\Lambda(\cdot,F) = \lambda^w_\Lambda(\cdot,F) + (\exp(2\varepsilon)-1)\lambda^w_\Lambda(\cdot,F) \\ &\le \lambda^w_\Lambda(\cdot,F) + (\exp(2\varepsilon)-1)\end{aligned}$$

and in the same way $\lambda^w_\Lambda(\cdot, F) \le f + (\exp(2\varepsilon) - 1)$. This shows that $\pi_\Lambda(\cdot, F)$ is quasi-local for all $F \in \mathcal{A}$, $\Lambda \in \mathcal{N}_0$, and therefore the specification $\mathcal{U}_w$ is quasi-local.

For $\Lambda \in \mathcal{N}_0$ let $a_\Lambda = \exp(\sup\{|e_\Lambda(x)| : x \in X\})$; then $1/a_\Lambda \le w_\Lambda \le a_\Lambda$, and thus for all $\Lambda \in \mathcal{N}_0$, $F \in \mathcal{F}^\Lambda$

$$\begin{aligned}\lambda^w_\Lambda(I_F) &= (\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda I_F)\\ &\le (\lambda_\Lambda(1/a_\Lambda))^{-1}\lambda_\Lambda(a_\Lambda I_F) = a^2_\Lambda \lambda_\Lambda(I_F) = a^2_\Lambda \lambda(F)\ .\end{aligned}$$

Now let Λ, $\Delta \in \mathcal{N}_0$ with $\Lambda \subset \Delta$. Then

$$\lambda^w_\Delta(I_F) = \lambda^w_\Delta(\lambda^w_\Lambda(I_F)) \le \lambda^w_\Delta(a^2_\Lambda \lambda(F)) = a^2_\Lambda \lambda(F)$$

for all $F \in \mathcal{F}^\Lambda$, i.e., $\lambda^w_\Delta(x, F) = \lambda^w_\Delta(I_F)(x) \le a^2_\Lambda \lambda(F)$ for all $x \in X$, $F \in \mathcal{F}^\Lambda$. Hence by Lemma 5.4 the family $\{\pi_\Lambda(x^\Lambda, \cdot)\}_{\Lambda \in \mathcal{N}_0}$ is locally equicontinuous for any choice of the family $\{x^\Lambda\}_{\Lambda \in \mathcal{N}_0}$, and so by Theorem 5.1 $\mathcal{G}(\mathcal{U}_w)$ is non-empty.

But the same estimate also shows that $\mu(F) \le a^2_\Lambda \lambda(F)$ for all $\mu \in \mathcal{G}(\mathcal{U}_w)$, $F \in \mathcal{F}^\Lambda$, $\Lambda \in \mathcal{N}_0$ (since $\mu(F) = \mu(\lambda^w_\Lambda(I_F)))$, and this implies that $\mathcal{G}(\mathcal{U}_w)$ is itself a locally equicontinuous subset of $\mathrm{P}(\mathcal{F})$. The second statement in Proposition 5.1 thus follows from Theorem 5.2. □

For typical particle models the specifications which arise are not quasi-local, and so Theorems 5.1 and 5.2 cannot be applied directly. Since the specifications are less well-behaved it is necessary to work with a restricted class of probability measures (which in the applications will correspond to the so-called tempered measures).

Denote by $\mathcal{A}_\delta$ the set of all countable intersections of elements from $\mathcal{A}$ and note that by Lemma M.3.2 $\mu(B) = \inf\{\mu(A) : A \in \mathcal{A} \text{ with } A \supset B\}$ for each $B \in \mathcal{A}_\delta$.

Now fix a non-empty subset $\mathcal{C}$ of $\mathcal{A}_\delta$ for which the inclusion order is directed and countably generated. A mapping $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ will be called *$\mathcal{C}$-local* if for each $U \in \mathcal{C}$ and each $\varepsilon > 0$ there exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ such that $g(x) \le f(x) + \varepsilon$ and $f(x) \le g(x) + \varepsilon$ for all $x \in U$ (i.e., such that $|f - g| I_U \le \varepsilon$). The set of all $\mathcal{C}$-local mappings will be denoted by $\mathrm{M}_{\mathcal{C}}(\mathcal{A})$.

The simplest possibility here is with $\mathcal{C} = \{X\}$, and then $\mathrm{M}_{\mathcal{C}}(\mathcal{A}) = \mathrm{M}_\diamondsuit(\mathcal{A})$.

Lemma 5.8 *(1) $\mathrm{M}_{\mathcal{C}}(\mathcal{A})$ is a subspace of $\mathrm{M}_{\mathsf{B}}(\mathcal{F})$ which contains $\mathrm{M}_\diamondsuit(\mathcal{A})$.*

(2) A mapping $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ is $\mathcal{C}$-local if and only if for each $\varepsilon > 0$ and for each $U \in \mathcal{C}$ there exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ such that $g(x) \le f(x) \le g(x) + \varepsilon$ for all $x \in U$.

(3) Let $f \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$, $U \in \mathcal{C}$ and $\varepsilon > 0$; then for each $b \in \mathbb{R}^+$ with $f \le b$ there exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $g \le b$ such that $g(x) \le f(x) \le g(x) + \varepsilon$ for all $x \in U$.

(4) Let $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ and suppose for each $\varepsilon > 0$ there exists $g \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$ such that $g \le f \le g + \varepsilon$. Then $f \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$.

Proof (2), (1) and (4) follow exactly as in the proofs of Lemma 5.2 (2), (1) and (4).

(3) Let $f \in \mathrm{M}_{\mathcal{C}}(\mathcal{F})$, $U \in \mathcal{C}$ and $\varepsilon > 0$; there thus exists $h \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $h(x) \le f(x) \le h(x) + \varepsilon$ for all $x \in U$. But then by Lemma 5.1 (1) $g = h \wedge b$ is an element of $\mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $g \le b$ which still satisfies $g(x) \le f(x) \le g(x) + \varepsilon$ for all $x \in U$. □

An element $\mu \in \mathrm{P}(\mathcal{F})$ will be called $\mathcal{C}$*-regular* if for each $\varepsilon > 0$ there exists $U \in \mathcal{C}$ with $\mu(U) \ge 1 - \varepsilon$, and the set of all $\mathcal{C}$-regular probability measures will be denoted by $\mathrm{P}_{\mathcal{C}}(\mathcal{F})$. If $\mathcal{C} = \{X\}$ then of course $\mathrm{P}_{\mathcal{C}}(\mathcal{F}) = \mathrm{P}(\mathcal{F})$.

A sequence $\{\mu_n\}_{n\ge1}$ from $\mathrm{P}(\mathcal{F})$ will be said to *converge* $\mathcal{C}$*-locally* to an element $\mu \in \mathrm{P}(\mathcal{F})$ if $\mu(f) = \lim_n \mu_n(f)$ for all $f \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$.

If a sequence converges $\mathcal{C}$-locally then by Lemmas 5.3 and 5.8 (1) it also converges locally. The next result gives a condition under which the converse holds.

Lemma 5.9 *Let $\{\mu_n\}_{n\ge1}$ be a sequence from $\mathrm{P}(\mathcal{F})$ which converges locally to an element $\mu \in \mathrm{P}(\mathcal{F})$. Suppose for each $\varepsilon > 0$ there exists $U \in \mathcal{C}$ and $n_0 \ge 0$ such that $\mu_n(U) \ge 1-\varepsilon$ for all $n \ge n_0$. Then in fact $\mu \in \mathrm{P}_{\mathcal{C}}(\mathcal{F})$ and $\{\mu_n\}_{n\ge1}$ converges $\mathcal{C}$-locally to μ.*

Proof Let $\varepsilon > 0$; then there exists $U \in \mathcal{C}$ and $n_0 \ge 0$ such that $\mu_n(U) \ge 1 - \varepsilon$ for all $n \ge n_0$. Thus if $A \in \mathcal{A}$ with $A \supset U$ then $\mu_n(A) \ge 1 - \varepsilon$ for all $n \ge n_0$ and hence $\mu(A) = \lim_n \mu_n(A) \ge 1 - \varepsilon$. Therefore $\mu(U) \ge 1 - \varepsilon$, since $U \in \mathcal{A}_\delta$, which shows that $\mu \in \mathrm{P}_{\mathcal{C}}(\mathcal{F})$.

Let $f \in \mathrm{M}_{\mathcal{C}}(\mathcal{F})$ and $\varepsilon > 0$; let $b > 0$ be such that $f \le b$. Now choose $U \in \mathcal{C}$ and $n_0 \ge 1$ such that $\mu_n(U) \ge 1 - \varepsilon/(4b)$ for all $n \ge n_0$; as above it then follows that $\mu(U) \ge 1 - \varepsilon/(4b)$. By Lemma 5.8 (3) there exists $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{A})$ with $g \le b$ such that $g(x) \le f(x) \le g(x) + \varepsilon/2$ for all $x \in U$. By Lemma 5.3 $\lim_n \mu_n(g) = \mu(g)$ and thus

$$\begin{aligned}
\limsup_{n\to\infty} \mu_n(f) &\le \limsup_{n\to\infty} \mu_n(bI_{X\setminus U} + g + \varepsilon/2) \\
&\le \limsup_{n\to\infty} \bigl(b(1 - \mu_n(U)) + \varepsilon/2 + \mu_n(g)\bigr) \\
&\le \limsup_{n\to\infty} \bigl(3\varepsilon/4 + \mu_n(g)\bigr) = \varepsilon/4 + \mu(g) \\
&\le 3\varepsilon/4 + \mu(bI_{X\setminus U} + f) = 3\varepsilon/4 + b(1 - \mu(U)) + \mu(f) \\
&\le \varepsilon + \mu(f) \,.
\end{aligned}$$

In the same way

$$\begin{aligned}
\mu(f) &\le \mu(bI_{X\setminus U} + g + \varepsilon/2) = b(1 - \mu(U)) + \varepsilon/2 + \mu(g) \\
&\le 3\varepsilon/4 + \mu(g) = 3\varepsilon/4 + \lim_{n\to\infty} \mu_n(g) \\
&\le 3\varepsilon/4 + \liminf_{n\to\infty} \mu_n(bI_{X\setminus U} + f) \le \varepsilon + \liminf_{n\to\infty} \mu_n(f) \,.
\end{aligned}$$

This implies that $\mu(f) = \lim_n \mu_n(f)$. □

A subset Q of $\mathrm{P}(\mathcal{F})$ will be called *$\mathcal{C}$-tight* if for each $\varepsilon > 0$ there exists $U \in \mathcal{C}$ such that $\mu(U) \geq 1 - \varepsilon$ for all $\mu \in Q$. Of course, in this case $Q \subset \mathrm{P}_{\mathcal{C}}(\mathcal{F})$.

Proposition 5.5 *Let Q be a $\mathcal{C}$-tight subset of $\mathrm{P}(\mathcal{F})$ and $\{\mu_n\}_{n\geq 1}$ be a sequence from Q which converges locally to an element $\mu \in \mathrm{P}(\mathcal{F})$. Then $\mu \in \mathrm{P}_{\mathcal{C}}(\mathcal{F})$ and $\{\mu_n\}_{n\geq 1}$ converges $\mathcal{C}$-locally to μ.*

Proof This follows immediately from Lemma 5.9. □

Proposition 5.6 *Let Q be a locally equicontinuous $\mathcal{C}$-tight subset of $\mathrm{P}(\mathcal{F})$. Then for each sequence $\{\mu_n\}_{n\geq 1}$ from Q there exists a subsequence $\{n_j\}_{j\geq 1}$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges $\mathcal{C}$-locally to some element $\mu \in \mathrm{P}_{\mathcal{C}}(\mathcal{F})$.*

Proof This follows immediately from Propositions 5.3 and 5.5. □

A family $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ of elements from $\mathrm{P}(\mathcal{F})$ will be called *$\mathcal{C}$-tight* if for each $\varepsilon > 0$ there exists $U \in \mathcal{C}$ and $\Lambda' \in \mathcal{N}_0$ such that $\mu_\Lambda(U) > 1 - \varepsilon$ for all $\Lambda \in \mathcal{N}_0$ with $\Lambda' \subset \Lambda$.

Lemma 5.10 *If $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is a locally equicontinuous $\mathcal{C}$-tight family in $\mathrm{P}(\mathcal{F})$ then there exists an increasing order-generating sequence $\{\Lambda_n\}_{n\geq 1}$ from $\mathcal{N}_0$ such that $\{\mu_{\Lambda_n}\}_{n\geq 1}$ converges $\mathcal{C}$-locally to some element $\mu \in \mathrm{P}_{\mathcal{C}}(\mathcal{F})$.*

Proof This follows immediately from Lemmas 5.5 and 5.9. □

A bounded kernel π will be called *$\mathcal{C}$-local* if the mapping $\pi(\cdot, F)$ is $\mathcal{C}$-local for all $F \in \mathcal{A}$.

Lemma 5.11 *A bounded kernel π is $\mathcal{C}$-local if and only if $\pi(f)$ is $\mathcal{C}$-local for all $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$.*

Proof Suppose π is $\mathcal{C}$-local, let $f \in \mathrm{M}_\diamondsuit(\mathcal{A})$ and $\varepsilon > 0$. Choose $c > 0$ with $\pi(x, X) \leq c$ for all $x \in X$. By Lemma 5.2 (2) there then exists $g \in \mathrm{M}_{\mathsf{E}}(\mathcal{A})$ with $g \leq f \leq g + \varepsilon/c$, and as in Lemma 5.6 it follows that $\pi(g) \leq \pi(f) \leq \pi(g) + \varepsilon$ and $\pi(g) \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$ (since by Lemma 5.8 (1) $\mathrm{M}_{\mathcal{C}}(\mathcal{A})$ is a subspace of $\mathrm{M}_{\mathsf{B}}(\mathcal{F})$). For each $\varepsilon > 0$ there thus exists $h \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$ with $h \leq \pi(f) \leq h + \varepsilon$ and hence by Lemma 5.8 (4) $\pi(f) \in \mathrm{M}_{\mathcal{C}}(\mathcal{A})$. The converse is clear. □

The $\mathbb{F}$-specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ will be called *$\mathcal{C}$-local* if each of the kernels π_Λ is $\mathcal{C}$-local. Put $\mathcal{G}_{\mathcal{C}}(\mathcal{V}) = \mathcal{G}(\mathcal{V}) \cap \mathrm{P}_{\mathcal{C}}(\mathcal{F})$.

Lemma 5.12 *Let $\mathcal{V}$ be $\mathcal{C}$-local and let $\{\mu_n\}_{n\geq 1}$ be a sequence from $\mathrm{P}(\mathcal{F})$ which converges $\mathcal{C}$-locally to $\mu \in \mathrm{P}(\mathcal{F})$. Suppose that for each $\Lambda \in \mathcal{N}_0$ there exists $m \geq 1$ such that $\mu_n \pi_\Lambda = \mu_n$ for all $n \geq m$. Then $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof Let $\Lambda \in \mathcal{N}_0$; for each $F \in \mathcal{A}$ it follows that

$$\begin{aligned}(\mu\pi_\Lambda)(F) = \mu(\pi_\Lambda(\cdot, F)) = \lim_{n\to\infty} \mu_n(\pi_\Lambda(\cdot, F)) &= \lim_{n\to\infty} (\mu_n\pi_\Lambda)(F) \\ &= \lim_{n\to\infty} \mu_n(F) = \mu(F)\end{aligned}$$

since $\mu_n \pi_\Lambda = \mu_n$ for all large enough n. Thus $\mu\pi_\Lambda = \mu$. □

The next result states that if $\mathcal{V}$ is $\mathcal{C}$-local then the set of Gibbs states $\mathcal{G}(\mathcal{V})$ is closed under $\mathcal{C}$-local convergence.

Proposition 5.7 *Let $\mathcal{V}$ be $\mathcal{C}$-local and let $\{\mu_n\}_{n\geq 1}$ be a sequence from $\mathcal{G}(\mathcal{V})$ which converges $\mathcal{C}$-locally to $\mu \in \mathrm{P}(\mathcal{F})$. Then $\mu \in \mathcal{G}_\mathcal{C}(\mathcal{V})$.*

Proof This follows immediately from Lemma 5.12. □

Theorem 5.3 *Suppose that $\mathcal{V}$ is $\mathcal{C}$-local and there exists a family $\{x^\Lambda\}_{\Lambda\in\mathcal{N}_0}$ of elements of X such that $\{\pi_\Lambda(x^\Lambda, \cdot)\}_{\Lambda\in\mathcal{N}_0}$ is a locally equicontinuous $\mathcal{C}$-tight family from $\mathrm{P}(\mathcal{F})$. Then $\mathcal{G}_\mathcal{C}(\mathcal{V})$ is non-empty.*

Proof For each $\Lambda \in \mathcal{N}_0$ put $\mu_\Lambda = \pi_\Lambda(x^\Lambda, \cdot)$. By Lemma 5.10 there is an increasing order-generating sequence $\{\Lambda_n\}_{n\geq 1}$ from $\mathcal{N}_0$ and $\mu \in \mathrm{P}_\mathcal{C}(\mathcal{F})$ such that $\{\mu_{\Lambda_n}\}_{n\geq 1}$ converges $\mathcal{C}$-locally to μ. Let $\Lambda \in \mathcal{N}_0$; then then exists $m \geq 1$ with $\Lambda \subset \Lambda_m$ and thus by Lemma M.14.2

$$\mu_n\pi_\Lambda = \pi_{\Lambda_n}(x^{\Lambda_n}, \cdot)\pi_\Lambda = (\pi_{\Lambda_n}\pi_\Lambda)(x^{\Lambda_n}, \cdot) = \pi_{\Lambda_n}(x^{\Lambda_n}, \cdot) = \mu_n$$

for all $n \geq m$. Therefore by Lemma 5.12 $\mu \in \mathcal{G}_\mathcal{C}(\mathcal{V})$. □

Theorem 5.4 *Suppose that $\mathcal{V}$ is $\mathcal{C}$-local and that Q is a locally equicontinuous $\mathcal{C}$-tight subset of $\mathcal{G}_\mathcal{C}(\mathcal{V})$. Then for each sequence $\{\mu_n\}_{n\geq 1}$ from Q there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathcal{G}_\mathcal{C}(\mathcal{V})$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges $\mathcal{C}$-locally to μ.*

Proof This follows immediately from Proposition 5.6 and Lemma 5.12. □

6 Representation of Gibbs states

This chapter continues the study of $\mathcal{G}(\mathcal{V})$, the set of Gibbs states corresponding to an $\mathbb{F}$-specification $\mathcal{V} = \{\pi_A\}_{A\in J}$ (as defined in Chapter 4). Since

$$\mathcal{G}(\mathcal{V}) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\pi_A \text{ for all } A \in J\}$$

it follows that $\mathcal{G}(\mathcal{V})$ is a convex subset of $\mathrm{P}(\mathcal{F})$. (A subset C of $\mathrm{P}(\mathcal{F})$ is convex if $a\mu_1 + (1-a)\mu_2 \in C$ for all μ_1, $\mu_2 \in C$ and all $0 \le a \le 1$.) It is thus natural to look at $\operatorname{ext}\mathcal{G}(\mathcal{V})$, the set of extreme points of $\mathcal{G}(\mathcal{V})$. (If C is a convex subset of $\mathrm{P}(\mathcal{F})$ then $\mu \in C$ is in $\operatorname{ext} C$, i.e., μ is an extreme point of C, if whenever μ_1, $\mu_2 \in C$ and $0 < a < 1$ with $\mu = a\mu_1 + (1-a)\mu_2$ then $\mu = \mu_1 = \mu_2$.)

In Theorem 6.1 $\operatorname{ext}\mathcal{G}(\mathcal{V})$ is characterised as the set of elements of $\mathcal{G}(\mathcal{V})$ which are trivial on the σ-algebra $\mathcal{F}_\infty = \bigcap_{A\in J}\mathcal{F}_A$, the so-called *tail field* (for the family $\mathbb{F}$), where if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then $\mu \in \mathrm{P}(\mathcal{F})$ is said to be *trivial* on $\mathcal{E}$ if $\mu(E)$ is either 0 or 1 for each $F \in \mathcal{E}$. In general it is possible to have $\mathcal{G}(\mathcal{V}) \neq \varnothing$ but $\operatorname{ext}\mathcal{G}(\mathcal{V}) = \varnothing$. However, if it is assumed that $(X, \mathcal{F})$ is substandard Borel then this cannot happen. This follows from Theorem 6.2, where it is also shown that if $(X, \mathcal{F})$ is substandard Borel then each element of $\mathcal{G}(\mathcal{V})$ has a integral representation as a convex mixture of elements from $\operatorname{ext}\mathcal{G}(\mathcal{V})$. Theorem 6.2 is a version due to Föllmer [10] of a construction of Dynkin [8] for the entrance boundary of a stochastic process.

Substandard Borel spaces are our substitute for standard Borel spaces and the results we need about them are to be found in Chapter M.18. (In particular, Proposition M.18.1 implies that a standard Borel space is substandard Borel.) Proposition 1.3 shows that $(X, \mathcal{F})$ is substandard Borel in the lattice model if each of the measurable spaces $(X_s, \mathcal{F}_s)$, $s \in S$, is. In the particle model Proposition 2.3 implies that $(X, \mathcal{F})$ is substandard Borel whenever $(S, \mathcal{S})$ is.

If $\pi \in \mathrm{K}(\mathcal{F})$ is a quasi-probability kernel then as in Chapter 3 put

$$\mathcal{I}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\pi\} ;$$

moreover, if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ and π is $\mathcal{E}$-measurable then put

$$\mathcal{J}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu(gf) = \mu(g\pi(f)) \text{ for all } g \in \mathrm{M}(\mathcal{E}),\ f \in \mathrm{M}(\mathcal{F})\} .$$

Thus $\mathcal{J}(\pi) \subset \mathcal{I}(\pi)$, with equality when π is a strict $\mathcal{E}$-measurable kernel, since in this case if $\mu \in \mathcal{I}(\pi)$ then $\mu(gf) = (\mu\pi)(gf) = \mu(\pi(gf)) = \mu(g\pi(f))$ holds for all $g \in \mathrm{M}(\mathcal{E})$, $f \in \mathrm{M}(\mathcal{F})$.

In particular, with this notation we have $\mathcal{G}(\mathcal{V}) = \bigcap_{A\in J}\mathcal{I}(\pi_A) = \bigcap_{A\in J}\mathcal{J}(\pi_A)$.

Before going further the reader should perhaps look again at Propositions 3.2 and 3.3, which considered the set $\mathcal{I}(\pi) = \mathcal{J}(\pi)$ for a single strict $\mathcal{E}$-measurable quasi-probability kernel π. These results are the prototypes for Theorems 6.1 and 6.2 and will be used in their proofs. Recall that if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then $\mu \in \mathrm{P}(\mathcal{F})$ is said to be trivial on $\mathcal{E}$ if $\mu(E) \in \{0,1\}$ for all $E \in \mathcal{E}$, and that Lemmas 3.4 and 3.5 gives various conditions which are equivalent to being trivial.

Theorem 6.1 *(1) A Gibbs state $\mu \in \mathcal{G}(\mathcal{V})$ is extreme if and only if it is trivial on $\mathcal{F}_\infty$.*

(2) The elements of $\mathcal{G}(\mathcal{V})$ are determined on $\mathcal{F}_\infty$, in that if μ_1, $\mu_2 \in \mathcal{G}(\mathcal{V})$ with $\mu_1(F) = \mu_2(F)$ for all $F \in \mathcal{F}_\infty$ then $\mu_1 = \mu_2$.

(3) If μ_1, $\mu_2 \in \operatorname{ext}\mathcal{G}(\mathcal{V})$ with $\mu_1 \neq \mu_2$ then there exists $F \in \mathcal{F}_\infty$ with $\mu_1(F) = 1$ and $\mu_2(F) = 0$. Thus distinct elements of $\operatorname{ext}\mathcal{G}(\mathcal{V})$ are mutually singular, even considered as measures on $\mathcal{F}_\infty$.

Proof The following is the analogue of Lemma 3.6:

Lemma 6.1 *Let $\mu \in \mathcal{G}(\mathcal{V})$ and $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) = 1$. Then $\mu{\cdot}f \in \mathcal{G}(\mathcal{V})$ if and only if there exists $f' \in \mathrm{M}(\mathcal{F}_\infty)$ with $f' = f$ μ-a.e.*

Proof Suppose there exists $f' \in \mathrm{M}(\mathcal{F}_\infty)$ with $f' = f$ μ-a.e.; thus $\mu(fg) = \mu(f'g)$ for all $g \in \mathrm{M}(\mathcal{F})$. Then

$$\begin{aligned}((\mu{\cdot}f)\pi_A)(g) &= (\mu{\cdot}f)(\pi_A(g)) = \mu(f\pi_A(g)) = \mu(f'\pi_A(g)) \\ &= \mu(\pi_A(f'g)) = (\mu\pi_A)(f'g) = \mu(f'g) = \mu(fg) = (\mu{\cdot}f)(g)\end{aligned}$$

for all $A \in J$, $g \in \mathrm{M}(\mathcal{F})$ and hence $\mu{\cdot}f \in \mathcal{G}(\mathcal{V})$. (This proof is identical with the proof of of the corresponding half of Lemma 3.6.) Suppose conversely $\mu{\cdot}f \in \mathcal{G}(\mathcal{V})$; then for each $A \in J$ there exists by Lemma 3.6 $f'_A \in \mathrm{M}(\mathcal{F}_A)$ with $\mu(E_A) = 1$, where $E_A = \{x \in X : f'_A(x) = f(x)\}$. Choose an increasing sequence $\{A_n\}_{n\geq 1}$ generating the order on J, so $\mathcal{F}_\infty = \bigcap_{n\geq 1} \mathcal{F}_{A_n}$, and let $f' = \liminf_n f'_{A_n}$. Then $f' \in \mathrm{M}(\mathcal{F}_\infty)$ and if $E = \{x \in X : f'(x) = f(x)\}$ then $\bigcap_{n\geq 1} E_{A_n} \subset E$, which implies that $\mu(E) = 1$. □

The proof of Theorem 6.1 is now identical with the proof of Proposition 3.2, replacing $\mathcal{E}$ with $\mathcal{F}_\infty$ and using Lemma 6.1 instead of Lemma 3.6. □

We next give a standard characterisation of the elements in $\mathrm{P}(\mathcal{F})$ which are trivial on $\mathcal{F}_\infty$. A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is said to have *short range correlations* (with respect to $\mathbb{F}$) if given any bounded $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ and any $\varepsilon > 0$ there exists $A \in J$ such that $|\mu(gf) - \mu(g)\mu(f)| < \varepsilon$ for all $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}_A)$ with $g \leq 1$.

Proposition 6.1 *A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is trivial on $\mathcal{F}_\infty$ if and only if it is has short range correlations.*

Proof If μ has short range correlations and $F \in \mathcal{F}_\infty$ then

$$|\mu(F) - \mu(F)\mu(F)| = |\mu(I_F I_F) - \mu(I_F)\mu(I_F)| < \varepsilon$$

for any $\varepsilon > 0$. This implies $\mu(F)$ is either 0 or 1 and thus that μ is trivial on $\mathcal{F}_\infty$. Conversely, suppose the measure μ is trivial on $\mathcal{F}_\infty$ and let $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ with $f \le b$. Then by Lemma 3.4 $f = \mu(f)$ μ-a.e., which means that the constant mapping $\mu(f)$ is a version of the conditional expectation of f with respect to $\mathcal{F}_\infty$. Choose an increasing sequence $\{A_n\}_{n\ge 1}$ generating the order on J, so $\mathcal{F}_\infty = \bigcap_{n\ge 1} \mathcal{F}_{A_n}$, and for each let f_n be a version of the conditional expectation of f with respect to $\mathcal{F}_{A_n}$ with $f_n \le b$ (which exists by Theorem M.12.4). Then by Proposition M.20.1

$$\lim_{n\to\infty} \mu(|f_n - \mu(f)|) = 0 \ .$$

Let $\varepsilon > 0$, and thus there exists $n \ge 1$ with $\mu(|f_n - \mu(f)|) < \varepsilon$. Then for all $g \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}_{A_n})$ with $g \le 1$

$$\begin{aligned} |\mu(gf) - \mu(g)\mu(f)| &= |\mu(gf_n) - \mu(g)\mu(f)| = |\mu(g(f_n - \mu(f)))| \\ &\le \mu(|g(f_n - \mu(f))|) = \mu(|g||f_n - \mu(f)|) \le \mu(|f_n - \mu(f)|) < \varepsilon \ . \quad \square \end{aligned}$$

Note that Proposition 6.1 is in some sense the analogue of Lemmas 3.4 and 3.5.

It follows from Theorem 6.1 and Proposition 6.1 that a Gibbs state is extreme if and only if it has short range correlations. In examples originating from statistical physics the elements of $\mathrm{ext}\,\mathcal{G}(\mathcal{V})$ should represent 'pure phases', and as such should possess correlation properties of the type occurring here.

The main result of this chapter (Theorem 6.2) will now be presented, and for this we require $(X, \mathcal{F})$ to be a substandard Borel space. Let us recall the definition from Chapter M.18. Let $M = \{0,1\}^{\mathbb{N}}$, considered as a compact metric space with respect to the metric $d : M \times M \to \mathbb{R}^+$ given by

$$d(\{z_n\}_{n\ge 0}, \{z'_n\}_{n\ge 0}) = \sum_{n\ge 0} 2^{-n}|z_n - z'_n|$$

(or any equivalent metric), and let $\mathcal{B}$ be the σ-algebra of Borel subsets of M (the smallest σ-algebra containing the open subsets of M). Then a measurable space $(X, \mathcal{F})$ is defined to be a *substandard Borel space* if there exists a mapping $f : X \to M$ with $f^{-1}(\mathcal{B}) = \mathcal{F}$ such that $f(X) \in \mathcal{B}$.

Certainly some assumption on $(X, \mathcal{F})$ is needed for Theorem 6.2 to be true, because in particular it implies that $\mathrm{ext}\,\mathcal{G}(\mathcal{V})$ is non-empty whenever $\mathcal{G}(\mathcal{V})$ is non-empty, and when $(X, \mathcal{F})$ is sufficiently bad this need not be the case.

Let $(X, \mathcal{F})$ be a substandard Borel space; then Proposition M.16.2 implies that $\mathcal{F}$ is countably generated. It is worth pointing out however that, although $\mathcal{F}_\infty$ is a sub-σ-algebra of $\mathcal{F}$, this does not imply that $\mathcal{F}_\infty$ is countably generated. In fact, this is never the case in any non-trivial example. (Believing that $\mathcal{F}_\infty$ must be countably generated has led to many false results in this area, and the reader can check that the proof of Theorem 6.2 would be very straightforward under this mistaken assumption.)

If $\pi \in \mathrm{K}(\mathcal{F})$ is a quasi-probability kernel then put $P_\pi = \{x \in X : \pi(x, X) = 1\}$, thus $I_{P_\pi} = \pi(1)$. Note that $\mu(P_\pi) = 1$ for each $\mu \in \mathcal{I}(\pi)$, since if $\mu = \mu\pi$ then $\mu(P_\pi) = \mu(\pi(1)) = (\mu\pi)(X) = \mu(X) = 1$.

Theorem 6.2 *Let $(X, \mathcal{F})$ be a substandard Borel space. Then there exists an $\mathcal{F}_\infty$-measurable quasi-probability kernel $\pi \in \mathrm{K}(\mathcal{F})$ such that:*

(1) $\mathcal{G}(\mathcal{V}) = \mathcal{J}(\pi) = \mathcal{I}(\pi)$.

(2) $\pi\pi = \pi$, and hence the mapping $\mu \mapsto \mu\pi$ maps $\{\mu \in \mathrm{P}(\mathcal{F}_\infty) : \mu(P_\pi) = 1\}$ onto $\mathcal{G}(\mathcal{V})$.

(3) $\pi(x, \cdot) \in \operatorname{ext} \mathcal{G}(\mathcal{V})$ for each $x \in P_\pi$.

(4) If $\mu \in \operatorname{ext} \mathcal{G}(\mathcal{V})$ then $\mu(\{x \in X : \pi(x, \cdot) = \mu\}) = 1$, and so in particular there exists $x \in P_\pi$ such that $\mu = \pi(x, \cdot)$.

Proof Below. □

Of course, if $\mathcal{G}(\mathcal{V}) = \varnothing$ then Theorem 6.2 holds trivially with $\pi = 0$. Note that by Lemma 3.8 the set $\{x \in X : \pi(x, \cdot) = \mu\}$ in (4) is an element of $\mathcal{F}_\infty$, since $\mathcal{F}$ is countably generated.

Let us point out that, although the the kernel π occurring in Theorem 6.2 has the same properties as the kernel in Proposition 3.3, there is in general no hope that π here will be a strict $\mathcal{F}_\infty$-measurable kernel, and it is this which gives rise to the technical problems in the proof.

Now for the proof of Theorem 6.2. This separates into two parts: The first is to construct an $\mathcal{F}$-measurable quasi-probability kernel ϱ such that $\mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\varrho)$. The the second is to modify ϱ appropriately to obtain a kernel π with all the required properties. The first part is quite straightforward, but relies heavily on $(X, \mathcal{F})$ being substandard Borel; the second part is less straightforward, but only requires $\mathcal{F}$ to be countably generated.

We start with the second part.

Proposition 6.2 *Suppose that $\mathcal{F}$ is countably generated and that there exists an $\mathcal{F}_\infty$-measurable quasi-probability kernel $\varrho \in \mathrm{K}(\mathcal{F})$ such that $\mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\varrho)$. Then there exists an $\mathcal{F}_\infty$-measurable quasi-probability kernel $\pi \in \mathrm{K}(\mathcal{F})$ satisfying the conditions (1), (2), (3) and (4) in Theorem 6.2.*

Proof Let us fix an increasing sequence $\{A_n\}_{n\ge1}$ generating the order on J and for each $n \ge 1$ put $\mathcal{F}_n = \mathcal{F}_{A_n}$ and $\pi_n = \pi_{A_n}$. Then $\{\mathcal{F}_n\}_{n\ge1}$ is a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_\infty = \bigcap_{n\ge1} \mathcal{F}_n$, each π_n is a strict $\mathcal{F}_n$-measurable quasi-probability kernel and $\mathcal{G}(\mathcal{V}) = \bigcap_{n\ge1} \mathcal{J}(\pi_n) = \bigcap_{n\ge1} \mathcal{I}(\pi_n)$.

Let us also fix a countable set $\mathcal{S}$ closed under finite intersections with $\mathcal{F} = \sigma(\mathcal{S})$. (The proof of Lemma 3.8 shows that such a set $\mathcal{S}$ exists.)

Lemma 6.2 *Let $U_n = \{x \in X : \varrho(x,\cdot) \in \mathcal{J}(\pi_n)\}$; then $U_n \in \mathcal{F}_\infty$ and $\mu(U_n) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof By Proposition M.3.3 and Lemma 3.7 (and since $\mathcal{J}(\pi_n) = \mathcal{I}(\pi_n)$)

$$U_n = \{x \in X : \varrho(x, X) = 1\} \cap \bigcap_{A\in\mathcal{S}} U_n^A \ ,$$

where $U_n^A = \{x \in X : \varrho(x, A) = (\varrho\pi_n)(x, A)\}$. In particular this implies that $U_n \in \mathcal{F}_\infty$. Let $\mu \in \mathcal{G}(\mathcal{V})$; then $\mu \in \mathcal{J}(\pi_n) \cap \mathcal{J}(\varrho)$ and so for all $g \in \mathrm{M}(\mathcal{F}_\infty)$

$$\begin{aligned}\mu(g(\varrho\pi_n)(\cdot, A)) &= \mu(g(\varrho\pi_n)(I_A)) = \mu(g(\varrho(\pi_n(I_A))) = \mu(g\pi_n(I_A)) = \mu(\pi_n(gI_A)) \\ &= (\mu\pi_n)(gI_A) = \mu(gI_A) = \mu(g\varrho(I_A)) = \mu(g\varrho(\cdot, A)) \ ,\end{aligned}$$

and this shows that $\mu(U_n^A) = 1$. Finally,

$$\mu(\{x \in X : \varrho(x, X) = 1\}) = \mu(\varrho(\cdot, X)) = (\mu\varrho)(X) = \mu(X) = 1$$

and therefore by Lemma M.3.3 $\mu(U_n) = 1$. □

Lemma 6.3 *Let $U = \{x \in X : \varrho(x,\cdot) \in \mathcal{G}(\mathcal{V})\}$; then $U \in \mathcal{F}_\infty$ and $\mu(U) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof This follows immediately from Lemma 6.2, since $U = \bigcap_{n\ge1} U_n$. □

For each $x \in X$ let $\Delta_x = \{y \in X : \varrho(y,\cdot) = \varrho(x,\cdot)\}$; by Proposition M.3.3

$$\Delta_x = \{y \in X : \varrho(y, A) = \varrho(x, A) \text{ for all } A \in \mathcal{S}\}$$

and so $\Delta_x \in \mathcal{F}_\infty$. Put $D = \{x \in X : \varrho(x, \Delta_x) = 1\}$.

Lemma 6.4 *$D \cap U \in \mathcal{F}_\infty$ and $\mu(D \cap U) = 1$ for all $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof For $F \in \mathcal{F}$ put $\Delta_x^F = \{y \in X : \varrho(y,F) = \varrho(x,F)\}$; then $\Delta_x = \bigcap_{A\in\mathcal{S}} \Delta_x^A$ and so by Lemma M.3.3 $\varrho(x,\Delta_x) = 1$ if and only if $\varrho(x,\Delta_x^A) = 1$ for each $A \in \mathcal{S}$. This means that $D = \bigcap_{A\in\mathcal{S}} D_A$, where $D_F = \{x \in X : \varrho(x,\Delta_x^F) = 1\}$ for each $F \in \mathcal{F}$, and thus also $D \cap U = \bigcap_{A\in\mathcal{S}} D_A \cap U$.

Now let $x \in U$ (and so $\varrho(x,X) = 1$) and let $F \in \mathcal{F}$; put $c = \varrho(x,F)$. Then $\varrho(y,F) = c$ if and only if $(c - \varrho(y,F))^2 = 0$; thus $\varrho(x,\Delta_x^F) = 1$ if and only if $\varrho((c-\varrho(\cdot,F))^2)(x) = \varrho(x,\cdot)((c-\varrho(\cdot,F))^2) = 0$. But $\varrho(x,\cdot) \in \mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\varrho) \subset \mathcal{I}(\varrho)$ and so by Lemma 3.7 $\varrho(\varrho(\cdot,F))(x) = (\varrho\varrho)(x,F) = \varrho(x,F) = c$. Therefore

$$\begin{aligned}
\varrho((c-\varrho(\cdot,F))^2)(x) + (\varrho(x,F))^2 &= \varrho((c-\varrho(\cdot,F))^2)(x) + 2c^2 - c^2 \\
&= \varrho((c-\varrho(\cdot,F))^2)(x) + 2c\varrho(\varrho(\cdot,F))(x) - c^2 \\
&= \varrho((c-\varrho(\cdot,F))^2 + 2c\varrho(\cdot,F))(x) - c^2 \\
&= \varrho(c^2 + (\varrho(\cdot,F))^2)(x) - c^2 \\
&= \varrho((\varrho(\cdot,F))^2)(x)\ .
\end{aligned}$$

Hence $\varrho((\varrho(\cdot,F))^2)(x) \geq (\varrho(x,F))^2$ for all $x \in U$ and all $F \in \mathcal{F}$, and $D_F \cap U$ consists exactly of those elements $x \in U$ for which

$$\varrho((\varrho(\cdot,F))^2)(x) = (\varrho(x,F))^2\ .$$

In particular $D_F \cap U \in \mathcal{F}_\infty$. Now let $\mu \in \mathcal{G}(\mathcal{V})$; then $\mu \in \mathcal{I}(\varrho)$ and so

$$\mu(\varrho((\varrho(\cdot,F))^2)) = (\mu\varrho)((\varrho(\cdot,F))^2) = \mu((\varrho(\cdot,F))^2)\ ;$$

hence $\mu(D_F \cap U) = 1$ (since $\varrho((\varrho(\cdot,F))^2)(x) \geq (\varrho(x,F))^2$ for all $x \in U$ and by Lemma 6.3 $\mu(U) = 1$). It then follows that both $D \cap U \in \mathcal{F}_\infty$ and $\mu(D \cap U) = 1$, since $\mathcal{S}$ is countable. □

Lemma 6.5 *If $x \in D \cap U$ then $\varrho(x,F) \in \{0,1\}$ for each $F \in \mathcal{F}_\infty$.*

Proof Let $x \in D \cap U$; then $\varrho(x,\cdot) \in \mathcal{J}(\varrho)$ and $\varrho(x,\Delta_x) = 1$. Thus

$$\begin{aligned}
\varrho(x,F) &= \varrho(x,\cdot)(I_F I_F) = \varrho(x,\cdot)(I_F\varrho(I_F)) = \varrho(x,\cdot)(I_{\Delta_x} I_F \varrho(I_F)) \\
&= \varrho(x,\cdot)(I_{\Delta_x} I_F \varrho(x,F)) = \varrho(x,\cdot)(I_F\varrho(x,F)) = (\varrho(x,F))^2
\end{aligned}$$

for each $F \in \mathcal{F}_\infty$ and hence $\varrho(x,F) \in \{0,1\}$. □

Now define an $\mathcal{F}_\infty$-measurable quasi-probability kernel $\pi \in \mathrm{K}(\mathcal{F})$ by

$$\pi(x,F) = I_{D\cap U}(x)\varrho(x,F)$$

for all $x \in X$, $F \in \mathcal{F}$; thus $\pi(f) = I_{D\cap U}\varrho(f)$ for all $f \in \mathrm{M}(\mathcal{F})$. If $\mu \in \mathcal{G}(\mathcal{V})$ then for all $g \in \mathrm{M}(\mathcal{F}_\infty)$, $f \in \mathrm{M}(\mathcal{F})$

$$\mu(gf) = \mu(g\varrho(f)) = \mu(gI_{D\cap U}\varrho(f)) = \mu(g\pi(f))\ ,$$

since by Lemma 6.4 $\mu(D \cap U) = 1$, which means that $\mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\pi)$.

Lemma 6.6 *$\pi = \pi\pi_n$ for each $n \geq 1$, and hence $\mathcal{I}(\pi) \subset \mathcal{G}(\mathcal{V})$.*

Proof If $x \in D \cap U$ then $\varrho(x,\cdot) \in \mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\pi_n) = \mathcal{I}(\pi_n)$ and so by Lemma 3.7

$$\pi(x,\cdot) = \varrho(x,\cdot) = (\varrho\pi_n)(x,\cdot) = (\pi\pi_n)(x,\cdot)\ .$$

On the other hand $\pi(x,\cdot) = 0 = (\pi\pi_n)(x,\cdot)$ whenever $x \in X \setminus D \cap U$, and thus $\pi = \pi\pi_n$. Finally, let $\mu \in \mathcal{I}(\pi)$; then $\mu\pi = \mu$ and hence

$$\mu\pi_n = (\mu\pi)\pi_n = \mu(\pi\pi_n) = \mu\pi = \mu$$

for all $n \geq 1$, i.e., $\mu \in \mathcal{G}(\mathcal{V})$. □

We now have $\mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\pi) \subset \mathcal{I}(\pi) \subset \mathcal{G}(\mathcal{V})$ and thus $\mathcal{G}(\mathcal{V}) = \mathcal{J}(\pi) = \mathcal{I}(\pi)$, i.e., π satisfies condition (0) in Theorem 6.2.

Let $x \in U \cap D$; then $\pi(x,\cdot) = \varrho(x,\cdot) \in \mathcal{G}(\mathcal{V}) = \mathcal{I}(\pi)$ and therefore by Lemma 3.7 $(\pi\pi)(x,\cdot) = \pi(x,\cdot)$. On the other hand, if $x \notin U \cap D$ then $\pi(x,\cdot) = 0$ and so $(\pi\pi)(x,\cdot) = 0 = \pi(x,\cdot)$. Thus $\pi\pi = \pi$ and from this it follows exactly as in the proof of Proposition 3.3 (2) that $\mu \mapsto \mu\pi$ maps $\{\mu \in \mathrm{P}(\mathcal{F}_\infty) : \mu(P_\pi) = 1\}$ onto $\mathcal{G}(\mathcal{V})$. Hence π satisfies condition (1) in Theorem 6.2.

Note that $P_\pi = U \cap D$; therefore if $x \in P_\pi$ then $\pi(x,\cdot) = \varrho(x,\cdot) \in \mathcal{G}(\mathcal{V})$ and by Lemma 6.5 $\pi(x,F) = \varrho(x,F) \in \{0,1\}$ for all $F \in \mathcal{F}_\infty$. Hence by Theorem 6.1 (1) $\pi(x,\cdot) \in \mathrm{ext}\,\mathcal{G}(\mathcal{V})$, i.e., π satisfies condition (2) in Theorem 6.2.

Finally, π satisfies condition (3) in Theorem 6.2: This follows exactly as in the proof of Proposition 3.3 (4) (together with Lemma 3.8).

This completes the proof of Proposition 6.2. □

Proposition 6.3 *Let $(X,\mathcal{F})$ be a substandard Borel space. Then there exists an $\mathcal{F}_\infty$-measurable quasi-probability kernel $\varrho \in \mathrm{K}(\mathcal{F})$ such that $\mathcal{G}(\mathcal{V}) \subset \mathcal{J}(\varrho)$.*

Proof This proof makes use of the notation and results from Chapter M.18. Again fix an increasing sequence $\{A_n\}_{n\geq 1}$ generating the order on J and for each $n \geq 1$ put $\mathcal{F}_n = \mathcal{F}_{A_n}$ and $\pi_n = \pi_{A_n}$. This means that $\{\mathcal{F}_n\}_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_\infty = \bigcap_{n\geq 1} \mathcal{F}_n$, each π_n is a strict $\mathcal{F}_n$-measurable quasi-probability kernel and $\mathcal{G}(\mathcal{V}) = \bigcap_{n\geq 1} \mathcal{J}(\pi_n) = \bigcap_{n\geq 1} \mathcal{I}(\pi_n)$.

Since $(X,\mathcal{F})$ is a substandard Borel space there exists a mapping $f : X \to M$ with $f^{-1}(\mathcal{B}) = \mathcal{F}$ and $f(X) \in \mathcal{B}$. Put

$$X_{\mathcal{C}} = \left\{x \in X : \lim_{n\to\infty} \pi_n(x, f^{-1}(C)) \text{ exists for all } C \in \mathcal{C}\right\}\ ,$$

where $\mathcal{C} \subset \mathcal{B}$ is the countable algebra of cylinder sets.

Lemma 6.7 *$X_{\mathcal{C}} \in \mathcal{F}_\infty$ and $\mu(X_{\mathcal{C}}) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$.*

Proof For each $C \in \mathcal{C}$ let X_C denote the set of those elements $x \in X$ for which the limit $\lim_n \pi_n(x, f^{-1}(C))$ exists, thus $X_{\mathcal{C}} = \bigcap_{C \in \mathcal{C}} X_C$ and therefore, since $\mathcal{C}$ is countable, it is enough by Lemma M.3.3 to show for each $C \in \mathcal{C}$ that $X_C \in \mathcal{F}_\infty$ and $\mu(X_C) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$. Now by Lemma M.9.7 (and the fact that X_C doesn't depend on the first m terms of the sequence $\{\pi_n(\cdot, f^{-1}(C))\}_{n \geq 1}$ for any $m \geq 1$) it follows that $X_C \in \mathcal{F}_\infty$. Let $\mu \in \mathcal{G}(\mathcal{V})$ and put $F = f^{-1}(C)$; then $\mu(h\pi_n(\cdot, F)) = \mu(hI_F)$ for all $h \in \mathrm{M}(\mathcal{F}_n)$, which means that $\pi_n(\cdot, F)$ is a version of the conditional expectation of I_F with respect to $\mathcal{F}_n$ (with the measure μ here fixed). Thus by Proposition M.20.1 $\mu(X_C) = 1$. □

Define a mapping $\tau : X \times \mathcal{C} \to \mathbb{R}^+_\infty$ by letting

$$\tau(x, C) = \begin{cases} \lim_{n \to \infty} \pi_n(x, f^{-1}(C)) & \text{if } x \in X_{\mathcal{C}} \,, \\ 0 & \text{if } x \in X \setminus X_{\mathcal{C}} \,. \end{cases}$$

Lemma 6.8 *Let $C \in \mathcal{C}$; then $\tau(\cdot, C) \in \mathrm{M}(\mathcal{F}_\infty)$ and*

$$\mu(G \cap f^{-1}(C)) = \mu(I_G \tau(\cdot, C))$$

for all $G \in \mathcal{F}_\infty$, $\mu \in \mathcal{G}(\mathcal{V})$.

Proof It is clear that $\tau(\cdot, C) \in \mathrm{M}(\mathcal{F}_\infty)$, since by Lemma 6.7 $X_{\mathcal{C}} \in \mathcal{F}_\infty$. If $\mu \in \mathcal{G}(\mathcal{V})$ and $G \in \mathcal{F}_\infty$ $\mu(G \cap f^{-1}(C)) = \mu(I_G \pi_n(\cdot, f^{-1}(C)))$ for each $n \geq 1$ and by Lemma 6.7 $\mu(X_{\mathcal{C}}) = 1$; thus by Theorem M.10.4

$$\mu(G \cap f^{-1}(C)) = \lim_{n \to \infty} \mu(I_G I_{X_{\mathcal{C}}} \pi_n(\cdot, f^{-1}(C))) = \mu(I_G \tau(\cdot, C)) \,. \quad \square$$

Lemma 6.9 *The mapping $\tau(x, \cdot) : \mathcal{C} \to \mathbb{R}^+_\infty$ is additive with $\tau(x, M) \in \{0, 1\}$ for each $x \in X$.*

Proof This is trivially true if $x \in X \setminus X_{\mathcal{C}}$ so let $x \in X_{\mathcal{C}}$. Clearly $\tau(x, M)$ must be either 0 or 1, since $\pi_n(x, X)$ takes on only these values, and if C_1, $C_2 \in \mathcal{C}$ with $C_1 \cap C_2 = \varnothing$ then $f^{-1}(C_1) \cap f^{-1}(C_2) = \varnothing$ and

$$\begin{aligned} \tau(x, C_1 \cup C_2) &= \lim_{n \to \infty} \pi_n(x, f^{-1}(C_1 \cup C_2)) = \lim_{n \to \infty} \pi_n(x, f^{-1}(C_1) \cup f^{-1}(C_2)) \\ &= \lim_{n \to \infty} (\pi_n(x, f^{-1}(C_1)) + \pi_n(x, f^{-1}(C_2))) = \tau(x, C_1) + \tau(x, C_2) \,. \quad \square \end{aligned}$$

Let $x \in X$; by Proposition M.16.6 the additive mapping $\tau(x, \cdot) : \mathcal{C} \to \mathbb{R}^+_\infty$ has a unique extension to a measure on $\mathcal{B}$ which will also be denoted by $\tau(x, \cdot)$, and of course $\tau(x, \cdot)$ must be either 0 or an element of $\mathrm{P}(M, \mathcal{B})$. This defines a finite pre-kernel $\tau : X \times \mathcal{B} \to \mathbb{R}^+_\infty$.

Lemma 6.10 *The mapping* $\tau : X \times \mathcal{B} \to \mathbb{R}^+_\infty$ *is a kernel with* $\tau(\cdot, B) \in \mathrm{M}(\mathcal{F}_\infty)$ *for all* $B \in \mathcal{B}$ *and* $\mu(G \cap f^{-1}(B)) = \mu(I_G \tau(\cdot, B))$ *for all* $G \in \mathcal{F}_\infty$, $\mu \in \mathcal{G}(\mathcal{V})$.

Proof This follows directly from Lemma 6.8 and Proposition M.14.5. □

Now for the first time the fact that $f(X) \in \mathcal{B}$ will be needed. Let

$$X_f = \{x \in X : \tau(x, f(X)) = 1\} .$$

Then $X_f \in \mathcal{F}_\infty$ and applying Lemma 6.10 with $B = f(X)$ and $G = X$ shows

$$\mu(\tau(\cdot, f(X))) = \mu(f^{-1}(f(X))) = \mu(X) = 1$$

and hence that $\mu(X_f) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$. Define a mapping $\eta : X \times \mathcal{B} \to \mathbb{R}^+_\infty$ by $\eta(x, B) = I_{X_f}(x)\tau(x, B)$. Then $\eta(x, \cdot)$ is either 0 or an element of $\mathrm{P}(M, \mathcal{B})$ with $\eta(x, f(X)) = 1$ for each $x \in X$, $\eta(\cdot, B) \in \mathrm{M}(\mathcal{F}_\infty)$ for each $B \in \mathcal{B}$ and $\mu(G \cap f^{-1}(B)) = \mu(I_G \eta(\cdot, B))$ for all $G \in \mathcal{F}_\infty$, $\mu \in \mathcal{G}(\mathcal{V})$, (since $\mu(X_f) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V})$).

By Theorem M.13.1 there now exists for each $x \in X$ a unique measure $\varrho(x, \cdot)$ (either 0 or an element of $\mathrm{P}(\mathcal{F})$) such that $\eta(x, \cdot) = f_* \varrho(x, \cdot)$. The resulting mapping $\varrho : X \times \mathcal{F} \to \mathbb{R}^+_\infty$ is then an $\mathcal{F}_\infty$-measurable quasi-probability kernel. If $F \in \mathcal{F}$ then $F = f^{-1}(B)$ for some $B \in \mathcal{B}$ and thus

$$\mu(G \cap F) = \mu(G \cap f^{-1}(B)) = \mu(I_G \eta(\cdot, B)) = \mu(I_G \varrho(\cdot, f^{-1}(B))) = \mu(I_G \varrho(\cdot, F))$$

for all $G \in \mathcal{F}_\infty$, $\mu \in \mathcal{G}(\mathcal{V})$. This completes the proof of Proposition 6.3. □

Theorem 6.2 follows immediately from Propositions 6.2 and 6.3.

7 Invariant specifications

Most of the examples involving the lattice model considered in Chapter 1 will have $S = \mathbb{Z}^d$ for some $d \geq 1$, and $(X_s, \mathcal{F}_s) = (X_0, \mathcal{F}_0)$ for each $s \in \mathbb{Z}^d$. In this situation the group $\mathbb{Z}^d$ induces a group of transformations $\{\psi_s\}_{s\in\mathbb{Z}^d}$ which act on the basic space $X = \prod_{s\in\mathbb{Z}^d} X_s$; the mapping $\psi_t : X \to X$ is defined by $(\psi_t(x))_s = x_{s-t}$ for each $x \in X$ (where x_s denotes the s-th component of $x \in X$). Each mapping ψ_t is clearly a bijection and $\psi_{s+t} = \psi_s \circ \psi_t$ for all s, $t \in \mathbb{Z}^d$.

Note that $\psi_t^{-1}(\prod_{s\in\mathbb{Z}^d} F_s) = \prod_{s\in\mathbb{Z}^d} F_{s+t}$. Thus if $R \in \mathcal{R}^\Lambda$ for some $\Lambda \in \mathcal{N}$ then $\psi_t^{-1}(R) \in \mathcal{R}^{\Lambda-t}$, where $\Lambda - t = \{s - t : s \in \Lambda\}$. Moreover, each rectangle $R' \in \mathcal{R}^{\Lambda-t}$ has the form $\psi_t^{-1}(R)$ with $R \in \mathcal{R}^\Lambda$, therefore $\psi_t^{-1}(\mathcal{R}^\Lambda) = \mathcal{R}^{\Lambda-t}$ and thus by Proposition M.2.4 $\psi_t^{-1}(\mathcal{F}^\Lambda) = \mathcal{F}^{\Lambda-t}$ for all $\Lambda \in \mathcal{N}$, $t \in \mathbb{Z}^d$. In particular this implies (again using Proposition M.2.4) that $\psi_t^{-1}(\mathcal{F}) = \mathcal{F}$ for all $t \in \mathbb{Z}^d$.

In this situation we can consider the set $\mathrm{P}_{\mathbb{Z}^d}(\mathcal{F})$ of probability measures on $\mathcal{F}$ invariant under translations, i.e., those $\mu \in \mathrm{P}(\mathcal{F})$ for which $\mu(\psi_t^{-1}(F)) = \mu(F)$ for all $t \in \mathbb{Z}^d$, $F \in \mathcal{F}$. In particular, if $\omega_0 \in \mathrm{P}(X_0, \mathcal{F}_0)$ and $\omega_s = \omega_0$ for all $s \in \mathbb{Z}^d$ then it is easy to see that the product of these measures is an element of $\mathrm{P}_{\mathbb{Z}^d}(\mathcal{F})$.

Suppose we also have a specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ which is translation invariant in that $\pi_{\Lambda+t}(x, F) = \pi_\Lambda(\psi^{-1}(x), \psi^{-1}(F))$ for all $x \in X$, $F \in \mathcal{F}$ and $t \in \mathbb{Z}^d$, where $\Lambda + t = \{s + t : s \in \Lambda\}$. It is then natural to look at the set of translation invariant Gibbs states

$$\mathcal{G}_{\mathbb{Z}^d}(\mathcal{V}) = \mathcal{G}(\mathcal{V}) \cap \mathrm{P}_{\mathbb{Z}^d}(\mathcal{F}) .$$

Even if the kernels in the specification are translation invariant it is still possible for $\mathcal{G}_{\mathbb{Z}^d}(\mathcal{V})$ to be strictly smaller than $\mathcal{G}(\mathcal{V})$, i.e., the conditional probabilities being translation invariant does not imply that all the Gibbs states are. This corresponds in models from statistical physics to the phenomenon of symmetry breakdown. A particular example where it occurs is in the Ising model with $d \geq 3$ and with the inverse temperature large enough, a result which was proved by Dobrushin [6].

In this chapter we look at some general (but not very deep) results which can be applied to models such as the one above. We will work with a spatial system $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ as defined in Chapter 4 (although most of the results could be reformulated to deal with other situations). Thus the basic measurable space is $(X, \mathcal{F})$ with $\mathcal{F} = \sigma(\mathcal{A})$, where $\mathcal{A} = \bigcup_{\Lambda\in\mathcal{N}} \mathcal{F}^\Lambda$ is the algebra of local observables. Let $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be the associated family of σ-algebras.

First consider a *measurable* mapping $\psi : (X, \mathcal{F}) \to (X, \mathcal{F})$. There is then the mapping $\psi^* : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ given by $\psi^* f = f \circ \psi$ for all $f \in \mathrm{M}(\mathcal{F})$, i.e., with $\psi^*(f)(x) = f(\psi(x))$ for all $x \in X$. Put $\mathcal{I}_\psi = \{F \in \mathcal{F} : \psi^{-1}(F) = F\}$; then $\mathcal{I}_\psi$ is clearly a sub-σ-algebra of $\mathcal{F}$. A mapping $f \in \mathrm{M}(\mathcal{F})$ is said to be *ψ-invariant* if $\psi^* f = f$. If $F \in \mathcal{F}$ then I_F is ψ-invariant if and only if $F \in \mathcal{I}_\psi$, since $\psi^* I_F = I_{\psi^{-1}(F)}$.

Lemma 7.1 *A mapping $f \in \mathrm{M}(\mathcal{F})$ is ψ-invariant if and only if $f \in \mathrm{M}(\mathcal{I}_\psi)$.*

Proof Let $f \in \mathrm{M}(\mathcal{F})$ and $B \in \mathcal{B}_\infty^+$; then

$$\psi^{-1}(\{x \in X : f(x) \in B\}) = \{x \in X : (\psi^* f)(x) \in B\} .$$

Thus if $f \in \mathrm{M}(\mathcal{I}_\psi)$ then $\{x \in X : f(x) \in B\}) = \{x \in X : (\psi^* f)(x) \in B\}$ for each $B \in \mathcal{B}_\infty^+$ and so $\psi^* f = f$. Conversely, if $\psi^* f = f$ then $\{x \in X : f(x) \in B\} \in \mathcal{I}_\psi$ for each $B \in \mathcal{B}_\infty^+$ and therefore $f \in \mathrm{M}(\mathcal{I}_\psi)$. □

A probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is called *ψ-invariant* if $\psi_* \mu = \mu$ (and hence μ is ψ-invariant if and only if $\mu(\psi^{-1}(F)) = \mu(F)$ for all $F \in \mathcal{F}$).

A *measurable bijection acting on* $(X, \mathcal{F})$ is a bijective mapping $\psi : X \to X$ with $\psi^{-1}(\mathcal{F}) = \mathcal{F}$. The inverse mapping $\psi^{-1} : X \to X$ is then also a measurable bijection. The mapping $\psi^* : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ is a bijection with inverse $(\psi^{-1})^*$.

Now fix a countable group H of measurable bijections acting on $(X, \mathcal{F})$, meaning that $\mathrm{id}_X \in H$, $\psi \circ \varphi \in H$ for all φ, $\psi \in H$ and $\psi^{-1} \in H$ for all $\psi \in H$. An object will be called *H-invariant* if it is ψ-invariant for each $\psi \in H$. The set of H-invariant elements of $\mathrm{P}(\mathcal{F})$ is denoted by $\mathrm{P}_H(\mathcal{F})$. Note that the set $\mathrm{P}_H(\mathcal{F})$ is convex. Put $\mathcal{I}_H = \{F \in \mathcal{F} : \psi^{-1}(F) = F \text{ for all } \psi \in H\}$; then $\mathcal{I}_H$ is clearly a sub-σ-algebra of $\mathcal{F}$. By Lemma 7.1 a mapping $f \in \mathrm{M}(\mathcal{F})$ is H-invariant if and only if $f \in \mathrm{M}(\mathcal{I}_H)$.

We assume that H is countable because it greatly simplifies things (and in fact it is essential in some of the proofs). This is no problem for the lattice models, since here we almost always have $H = \mathbb{Z}^d$ for some $d \geq 1$. In the particle models the group is typically $\mathbb{R}^d$, but the action of this group is usually determined by the action of the countable subgroup $\mathbb{Q}^d$ and so the results presented in this chapter can still be applied.

A measure $\mu \in \mathrm{P}_H(\mathcal{F})$ is said to be *H-ergodic* if it is trivial on $\mathcal{I}_H$ (but let us emphasise that this definition is only really correct (at least in the sense that Proposition 7.1 holds) because the group H is countable). The following is a standard result from ergodic theory.

Proposition 7.1 $\mathrm{ext}\,\mathrm{P}_H(\mathcal{F}) = \{\mu \in \mathrm{P}_H(\mathcal{F}) : \mu$ *is H-ergodic*$\}$.

Proof This is given below. □

Let $\mu \in \mathrm{P}(\mathcal{F})$ and $\psi \in H$. A mapping $f \in \mathrm{M}(\mathcal{F})$ is called *μ-almost ψ-invariant* if $\psi^* f = f$ holds μ-a.e.

Lemma 7.2 *Let $\psi \in H$, let $\mu \in \mathrm{P}(\mathcal{F})$ be ψ-invariant and let f, $g \in \mathrm{M}(\mathcal{F})$ with $f = g$ μ-a.e. Then $\psi^* f = \psi^* g$ μ-a.e.*

Proof Let $E = \{x \in X : f(x) = g(x)\}$; then

$$\mu(\{x \in X : (\psi^* f)(x) = (\psi^* g)(x)\}) = \mu(\psi^{-1}(E)) = \mu(E) = 1\ . \quad \square$$

Lemma 7.3 *Let $\psi \in H$, let $\mu \in \mathrm{P}(\mathcal{F})$ be ψ-invariant and let $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) = 1$. Then $\mu{\cdot}f$ is ψ-invariant if and only if f is μ-almost ψ-invariant.*

Proof Let $g \in \mathrm{M}(\mathcal{F})$; then $\mu((\psi^* f)(\psi^* g)) = \mu(\psi^*(fg)) = (\psi_*\mu)(fg) = \mu(fg)$. But $\mu{\cdot}f$ is ψ-invariant if and only if $\mu(fg) = (\mu{\cdot}f)(g) = \psi_*(\mu{\cdot}f)(g) = \mu(f\psi^* g)$ for all $g \in \mathrm{M}(\mathcal{F})$. Hence $\mu{\cdot}f$ is ψ-invariant if and only if $\mu((\psi^* f)(\psi^* g)) = \mu(f\psi^* g)$ and thus if and only if $\mu((\psi^* f)g) = \mu(fg)$ for all $g \in \mathrm{M}(\mathcal{F})$, since the mapping ψ^* is surjective. Therefore by Lemma M.10.4 (2) $\mu{\cdot}f$ is ψ-invariant if and only if $\psi^* f = f$ μ-a.e. $\square$

Lemma 7.4 *Let $\mu \in \mathrm{P}(\mathcal{F})$ and let $f \in \mathrm{M}(\mathcal{F})$ be μ-almost ψ-invariant for each $\psi \in H$. Then there exists $g \in \mathrm{M}(\mathcal{I}_H)$ with $g = f$ μ-a.e.*

Proof Put $g = \inf\{\psi^* f : \psi \in H\}$; then by Lemma M.9.4 $g \in \mathrm{M}(\mathcal{F})$, and $\psi^* g = g$ for each $\psi \in H$. Thus by Lemma 7.1 $g \in \mathrm{M}(\mathcal{I}_H)$. But if $E_\psi = \{x \in X : \psi^* f = f\}$ and $E = \bigcap_{\psi \in H} E_\psi$ then $\mu(E) = 1$, since $\mu(E_\psi) = 1$ for each $\psi \in H$ and H is countable, and $g(x) = f(x)$ for all $x \in E$. Hence $g = f$ μ-a.e. $\square$

Lemma 7.5 *Let $\mu \in \mathrm{P}_H(\mathcal{F})$ and $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) = 1$. Then $\mu{\cdot}f \in \mathrm{P}_H(\mathcal{F})$ if and only if there exists $g \in \mathrm{M}(\mathcal{I}_H)$ with $g = f$ μ-a.e.*

Proof If $\mu{\cdot}f \in \mathrm{P}_H(\mathcal{F})$ then by Lemma 7.3 f is μ-almost ψ-invariant for each $\psi \in H$, and therefore by Lemma 7.4 there exists $g \in \mathrm{M}(\mathcal{I}_H)$ with $g = f$ μ-a.e. Conversely, if there exists $g \in \mathrm{M}(\mathcal{I}_H)$ with $g = f$ μ-a.e. then by Lemma M.10.4 (2) $\mu{\cdot}f = \mu{\cdot}g$, and by Lemma 7.3 $\mu{\cdot}g \in \mathrm{P}_H(\mathcal{F})$. Thus $\mu{\cdot}f \in \mathrm{P}_H(\mathcal{F})$. $\square$

Proof of Proposition 7.1: This is similar to the the proof of Theorem 6.1 (1), but using Lemma 7.5 instead of Lemma 6.1.

If $\mu \in \mathrm{P}_H(\mathcal{F})$ is not extreme then there exist $\mu_1, \mu_2 \in \mathrm{P}_H(\mathcal{F})$ with $\mu_1 \neq \mu_2$ and $0 < a < 1$ such that $\mu = a\mu_1 + (1-a)\mu_2$. Then $\mu_1 \ll \mu$ and so by Theorem M.12.1 there exists $f \in \mathrm{M}(\mathcal{F})$ with $\mu_1 = \mu{\cdot}f$, and $\mu(f) = \mu_1(1) = 1$. Therefore by Lemma 7.5 there exists $g \in \mathrm{M}(\mathcal{I}_H)$ with $g = f$ μ-a.e. Now $\mu \neq \mu_1$ and thus let $h \in \mathrm{M}(\mathcal{F})$ with $\mu(h) \neq \mu_1(h) = \mu(fh)$. Then by Lemma M.10.4 (2) $\mu(gh) = \mu(fh) \neq \mu(h) = \mu(f)\mu(h) = \mu(g)\mu(h)$ and hence by Lemma 3.5 g is not μ-a.e. equal to a constant. Thus by Lemma 3.5 μ is not H-ergodic.

Conversely, suppose that μ is not H-ergodic, then there exists $E \in \mathcal{I}_H$ with $0 < \mu(E) < 1$. Put $a = \mu(E)$ and let $\mu_1 = \mu{\cdot}g_1$ and $\mu_2 = \mu{\cdot}g_2$ with $g_1 = a^{-1}I_E$ and $g_2 = (1-a)^{-1}I_{X\setminus E}$. Then $g_j \in \mathrm{M}(\mathcal{I}_H)$ and $\mu(g_j) = 1$ and so by Lemma 7.5 $\mu_j \in \mathrm{P}_H(\mathcal{F})$ for $j = 1, 2$. But $\mu = a\mu_1 + (1-a)\mu_2$ and clearly $\mu_1 \neq \mu_2$; therefore $\mu \notin \operatorname{ext} \mathrm{P}_H(\mathcal{F})$. □

Let us now add an assumption about the relationship between the group H and the set $\mathcal{N}$. We assume that there is a group, isomorphic to H and also denoted by H, of measurable bijections acting on $(S, \mathcal{S})$ such that $\psi^{-1}(\Lambda) \in \mathcal{N}$ and with $\psi^{-1}(\mathcal{F}^\Lambda) = \mathcal{F}^{\psi^{-1}(\Lambda)}$ for all $\Lambda \in \mathcal{N}$, $\psi \in H$. Thus in particular $\psi^{-1}(\mathcal{A}) \subset \mathcal{A}$ for each $\psi \in H$. In particular, this condition on H and $\mathcal{N}$ is satisfied by the lattice model considered at the beginning of the chapter. As before let $\mathcal{F}_\infty = \bigcap_{\Lambda \in \mathcal{N}_0} \mathcal{F}_\Lambda$ be the tail field.

Lemma 7.6 $\psi^{-1}(\mathcal{F}_\infty) \subset \mathcal{F}_\infty$ *for each* $\psi \in H$*.*

Proof Let $\psi \in H$ and $\Lambda \in \mathcal{N}$ and put $\Lambda' = \psi(\Lambda)$; thus $\Lambda' = (\psi^{-1})^{-1}(\Lambda) \in \mathcal{N}$. Now if $\Delta \in \mathcal{N}$ with $\Delta \cap \Lambda' = \varnothing$ then $\psi^{-1}(\Delta) \cap \Lambda = \varnothing$ and it therefore follows that $\psi^{-1}(\mathcal{F}^\Delta) = \mathcal{F}^{\psi^{-1}(\Delta)} \subset \mathcal{F}_\Lambda$. Hence by Proposition M.2.4

$$\psi^{-1}(\mathcal{F}_{\Lambda'}) = \psi^{-1}\Big(\sigma\Big(\bigcup_{\Delta \subset S\setminus\Lambda'} \mathcal{F}^\Delta\Big)\Big) = \sigma\Big(\bigcup_{\Delta \subset S\setminus\Lambda'} \psi^{-1}(\mathcal{F}^\Delta)\Big) \subset \mathcal{F}_\Lambda\ ,$$

thus $\psi^{-1}(\mathcal{F}_\infty) \subset \psi^{-1}(\mathcal{F}_{\Lambda'}) \subset \mathcal{F}_\Lambda$ for each $\Lambda \in \mathcal{N}$, i.e., $\psi^{-1}(\mathcal{F}_\infty) \subset \mathcal{F}_\infty$. □

Now also fix an $\mathbb{F}$-specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ and put $\mathcal{G}_H(\mathcal{V}) = \mathcal{G}(\mathcal{V}) \cap \mathrm{P}_H(\mathcal{F})$. In particular the set $\mathcal{G}_H(\mathcal{V})$ is convex.

Proposition 7.2 $\operatorname{ext} \mathcal{G}_H(\mathcal{V}) = \{\mu \in \mathcal{G}_H(\mathcal{V}) : \mu$ *is trivial on* $\mathcal{F}_\infty \cap \mathcal{I}_H\}$*. This implies in particular (together with Theorem 6.1 (1)) that any* H*-ergodic element of* $\mathcal{G}_H(\mathcal{V})$ *is extreme.*

Proof This follows exactly as in the proof of Proposition 7.1, but using Lemma 7.7 below instead of Lemma 7.5. □

Lemma 7.7 *Let* $\mu \in \mathcal{G}_H(\mathcal{V})$ *and let* $f \in \mathrm{M}(\mathcal{F})$ *with* $\mu(f) = 1$*. Then* $\mu{\cdot}f \in \mathcal{G}_H(\mathcal{V})$ *if and only if there exists* $g \in \mathrm{M}(\mathcal{F}_\infty \cap \mathcal{I}_H)$ *with* $g = f$ μ*-a.e.*

Proof If $\mu{\cdot}f \in \mathcal{G}_H(\mathcal{V})$ then by Lemma 7.5 there exists $h' \in \mathrm{M}(\mathcal{I}_H)$ with $h' = f$ μ-a.e. and by Lemma 6.1 there exists $h \in \mathrm{M}(\mathcal{F}_\infty)$ with $h = f$ μ-a.e. Thus $h = h'$ μ-a.e., and so $\psi^* h = h$ μ-a.e. (since $\psi^* h' = h'$ and by Lemma 7.2 $\psi^* h = \psi^* h'$ μ-a.e.); moreover, by Lemma 7.6 $\psi^* h \in \mathrm{M}(\mathcal{F}_\infty)$ for each $\psi \in H$. Now the proof of Lemma 7.4 shows that $g = \inf\{\psi^* h : \psi \in H\} \in \mathrm{M}(\mathcal{F}_\infty \cap \mathcal{I}_H)$ and that $g = h$ μ-a.e., and then also $g = f$ μ-a.e. Conversely, if there exists $g \in \mathrm{M}(\mathcal{F}_\infty \cap \mathcal{I}_H)$ with $g = f$ μ-a.e. then by Lemma M.10.4 (2) $\mu{\cdot}f = \mu{\cdot}g$, by Lemma 7.5 $\mu{\cdot}g \in \mathrm{P}_H(\mathcal{F})$ and by Lemma 6.1 $\mu{\cdot}g \in \mathcal{G}(\mathcal{V})$. Thus $\mu{\cdot}f \in \mathcal{G}_H(\mathcal{V})$. □

Usually Proposition 7.2 can be improved on because there is another relationship between $\mathcal{I}_H$ and $\mathcal{F}_\infty$, which involves the following definition: Let us say that the group H *separates* $\mathcal{N}$ if for all Λ, $\Delta \in \mathcal{N}$ there exists $\psi \in H$ such that $\psi^{-1}(\Lambda) \cap \Delta = \varnothing$. Again, this condition on H and $\mathcal{N}$ is certainly satisfied by the lattice model considered at the beginning of the chapter.

Theorem 7.1 *If H separates $\mathcal{N}$ then* $\mathrm{ext}\,\mathcal{G}_H(\mathcal{V})$ *consists exactly of the H-ergodic elements in $\mathcal{G}_H(\mathcal{V})$.*

Proof It was already noted that Proposition 7.2 and Theorem 6.1 (1) imply that any H-ergodic element of $\mathcal{G}_H(\mathcal{V})$ is extreme. For the converse the following two lemmas are needed:

Lemma 7.8 *Let $\mu \in \mathrm{P}(\mathcal{F})$ be ψ-invariant and $f \in \mathrm{M}(\mathcal{F})$ with $\mu(f) < \infty$ be μ-almost ψ-invariant. Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$ and $f' \in \mathrm{M}(\mathcal{E})$ be a version of the conditional expectation of f with respect to $\mathcal{E}$. Then $\psi^* f'$ is a version of the conditional expectation of f with respect to $\psi^{-1}(\mathcal{E})$.*

Proof Let $g \in \mathrm{M}(\psi^{-1}(\mathcal{E}))$; then $g = \psi^* g'$ for some $g' \in \mathrm{M}(\mathcal{E})$, since ψ^* is surjective. Hence, since $\mu = \psi_* \mu$ and $f = \psi^* f$ μ-a.e.,

$$\begin{aligned}\mu(g\psi^* f') &= \mu((\psi^* g')(\psi^* f')) = (\psi_*\mu)(g'f') = \mu(g'f') = \mu(g'f)\\ &= (\psi_*\mu)(g'f) = \mu(\psi^*(g'f)) = \mu((\psi^* g')(\psi^* f)) = \mu(g\psi^* f) = \mu(gf)\,,\end{aligned}$$

and this shows that $\psi^* f'$ is a version of the conditional expectation of f with respect to $\psi^{-1}(\mathcal{E})$. □

Lemma 7.9 *Suppose H separates $\mathcal{N}$, let $\mu \in \mathrm{P}_H(\mathcal{F})$ and $f \in \mathrm{M}_{\mathsf{B}}(\mathcal{F})$ be μ-almost H-invariant; let $f_\infty \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}_\infty)$ be a version of the conditional expectation of f with respect to $\mathcal{F}_\infty$. Then $f = f_\infty$ μ-a.e.*

Proof Let $b \in \mathbb{R}^+$ be such that $f \le b$ and let $\{\Lambda_n\}_{n\ge1}$ be an increasing sequence from $\mathcal{N}$ with $\bigcup_{n\ge1} \Lambda_n = S$ (and hence $\mathcal{F} = \sigma(\bigcup_{n\ge1} \mathcal{F}^{\Lambda_n})$). For each $n \ge 1$ let $f_n \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}^{\Lambda_n})$ with $f_n \le b$ be a version of the conditional expectation of f with respect to $\mathcal{F}^{\Lambda_n}$ and for each $\Delta \in \mathcal{N}_0$ let $f_\Delta \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}_\Delta)$ with $f_A \le b$ be a version of the conditional expectation of f with respect to $\mathcal{F}_\Delta$. Let $\varepsilon > 0$;

then by Proposition M.20.1 there exists $\Delta \in \mathcal{N}_0$ such that $\mu(|f_\Delta - f_\infty|) < \varepsilon/3$ and by Proposition M.20.2 there exists $n \geq 1$ such that $\mu(|f - f_n|) < \eta$, where both $\eta < \varepsilon/3$ and $b\eta < (\varepsilon/3)^2$. Choose $\psi \in H$ with $\psi^{-1}(\Lambda_n) \cap \Delta = \varnothing$; then $\psi^{-1}(\mathcal{F}^{\Lambda_n}) = \mathcal{F}^{\psi^{-1}(\Lambda_n)} \subset \mathcal{F}_\Delta$, by Lemma 7.7 $\psi^* f_n$ is a version of the conditional expectation of f with respect to $\psi^{-1}(\mathcal{F}^{\Lambda_n})$ and

$$\begin{aligned}\mu(|f - \psi^* f_n|) = \mu(|\psi^* f - \psi^* f_n|) &= \mu(\psi^* |f - f_n|) \\ &= (\psi_* \mu)(|f - f_n|) = \mu(|f - f_n|) \ .\end{aligned}$$

Moreover, by Lemma M.12.4 and Proposition M.10.7 (Cauchy-Schwarz)

$$\begin{aligned}\big(\mu(|f_\Delta - \psi^* f_n|)\big)^2 \leq \mu((f_\Delta - \psi^* f_n)^2) &\leq \mu((f - \psi^* f_n)^2) \\ &\leq b\mu(|f - \psi^* f_n|) = b\mu(|f - f_n|) \leq b\eta\end{aligned}$$

and so $\mu(|f_\Delta - \psi^* f_n|) < \varepsilon/3$. Therefore

$$\begin{aligned}\mu(|f - f_\infty|) &\leq \mu(|f - \psi^* f_n|) + \mu(|\psi^* f_n - f_\Delta|) + \mu(|f_\Delta - f_\infty|) \\ &= \mu(|f - f_n|) + \mu(|\psi^* f_n - f_\Delta|) + \mu(|f_\Delta - f_\infty|) < \varepsilon \ .\end{aligned}$$

Thus $\mu(|f - f_\infty|) = 0$, since $\varepsilon > 0$ was arbitrary, and then Proposition M.10.6 (1) implies that $\mu(\{x \in X : |f(x) - f_\infty(x)| > 0\}) = 0$, i.e., $f = f_\infty$ μ-a.e. □

Now to the rest of the proof of Theorem 7.1, i.e., to show that each $\mu \in \operatorname{ext} \mathcal{G}_H(\mathcal{V})$ is H-ergodic. Suppose $\mu \in \mathcal{G}_H(\mathcal{V})$ is not H-ergodic, so there exists $E \in \mathcal{I}_H$ with $0 < \mu(E) < 1$. Put $a = \mu(E)$ and let $\mu_1 = \mu{\cdot}g_1$ and $\mu_2 = \mu{\cdot}g_2$ with $g_1 = a^{-1} I_E$ and $g_2 = (1-a)^{-1} I_{X \setminus E}$. Then $g_j \in \mathrm{M}_{\mathsf{B}}(\mathcal{I}_H)$ and $\mu(g_j) = 1$ and so by Lemma 7.5 $\mu_j \in \mathrm{P}_H(\mathcal{F})$ for $j = 1, 2$. For $j = 1, 2$ let $h_j \in \mathrm{M}_{\mathsf{B}}(\mathcal{F}_\infty)$ be a version of the conditional expectation of g_j with respect to $\mathcal{F}_\infty$. Then by Lemma 7.9 $g_j = h_j$ μ-a.e., and so by Lemma 6.1 $\mu_j \in \mathcal{G}(\mathcal{V})$. This shows that $\mu_1, \mu_2 \in \mathcal{G}_H(\mathcal{V})$. But $\mu = a\mu_1 + (1-a)\mu_2$ and clearly $\mu_1 \neq \mu_2$; therefore $\mu \notin \operatorname{ext} \mathcal{G}_H(\mathcal{V})$. Hence each $\mu \in \operatorname{ext} \mathcal{G}_H(\mathcal{V})$ is H-ergodic, which completes the proof of Theorem 7.1. □

Remark: Theorem 7.1 says that if H separates $\mathcal{N}$ then

$$\operatorname{ext} \mathcal{G}_H(\mathcal{V}) = \mathcal{G}_H(\mathcal{V}) \cap \operatorname{ext} \mathrm{P}_H(\mathcal{F}) \ ,$$

and in the appropriate topological setting this would mean that $\mathcal{G}_H(\mathcal{V})$ is a 'face' of $\mathrm{P}_H(X, \mathcal{F})$.

Theorem 7.1 and Proposition 7.2 do not require that the specification $\mathcal{V}$ be in any way invariant under the group H. However, in the remaining discussion the invariance of $\mathcal{V}$ plays a decisive role, and we must first explain how this invariance is defined in the present set-up.

Let $\psi : X \to X$ be a measurable bijection acting on $(X, \mathcal{F})$. If $\pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ is a kernel then we define a kernel $\psi^\diamond \pi : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ by

$$\psi^\diamond \pi = (\psi^*)^{-1} \circ \pi \circ \psi^* \ .$$

Thus $(\psi^\Diamond\pi)(f) = (\psi^*)^{-1}(\pi(\psi^* f))$ for all $f \in \mathrm{M}(\mathcal{F})$ and hence

$$(\psi^\Diamond\pi)(f)(x) = \pi(\psi^* f)(\psi^{-1}(x))$$

for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$, since $(\psi^*)^{-1} = (\psi^{-1})^*$.

Lemma 7.10 *Let $\mu \in \mathrm{P}(\mathcal{F})$, $\pi \in \mathrm{K}(\mathcal{F})$ with $\mu\pi = \mu$. Then $(\psi_*\mu)(\psi^\Diamond\pi) = \psi_*\mu$.*

Proof Since $\mu\pi = \mu$ it follows that

$$\begin{aligned}(\psi_*\mu)(\psi^\Diamond\pi) = (\mu \circ \psi^*) \circ (\psi^\Diamond\pi) &= (\mu \circ \psi^*) \circ ((\psi^*)^{-1} \circ \pi \circ \psi^*) \\ &= \mu \circ \pi \circ \psi^* = (\mu\pi) \circ \psi^* = \mu \circ \psi^* = \psi_*\mu \ . \ \square\end{aligned}$$

The $\mathbb{F}$-specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is said to be *H-invariant* if $\psi^\Diamond\pi_\Lambda = \pi_{\psi(\Lambda)}$ for all $\Lambda \in \mathcal{N}_0$, $\psi \in H$. The main fact we need about such specifications is given by the following result.

Proposition 7.3 *Let $\mathcal{V}$ be an H-invariant $\mathbb{F}$-specification. Then $\psi_*\mu \in \mathcal{G}(\mathcal{V})$ for all $\mu \in \mathcal{G}(\mathcal{V})$, $\psi \in H$.*

Proof Let $\mu \in \mathcal{G}(\mathcal{V})$, $\psi \in H$ and $\Lambda \in \mathcal{N}_0$; put $\Delta = \psi^{-1}(\Lambda)$. Then $\psi(\Delta) = \Lambda$ and thus $\psi^\Diamond\pi_\Delta = \pi_\Lambda$. Hence by Lemma 7.10 $(\psi_*\mu)\pi_\Lambda = (\psi_*\mu)(\psi^\Diamond\pi_\Delta) = \psi_*\mu$, since $\mu\pi_\Delta = \mu$, and this implies that $\psi_*\mu \in \mathcal{G}(\mathcal{V})$. $\square$

In order to get an idea of what H-invariance means for a specification consider an independent measure $\lambda \in \mathrm{P}(\mathcal{F})$ and suppose the associated family of kernels $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ exists. Thus $\lambda_\Lambda(gf) = g\lambda(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$ and by Proposition 4.1 $\mathcal{U}$ is an $\mathbb{F}$-specification.

Proposition 7.4 *If λ is H-invariant then the specification $\mathcal{U}$ is H-invariant.*

Proof Let $\psi \in H$ and let $\Lambda, \Delta \in \mathcal{N}_0$ with $\Delta \cap \Lambda = \varnothing$ and put $\varphi = \psi^{-1}$; then $\varphi^{-1}(\mathcal{F}^\Lambda) = \mathcal{F}^{\psi(\Lambda)}$ and so φ^* maps $\mathrm{M}(\mathcal{F}^\Lambda)$ bijectively onto $\mathrm{M}(\mathcal{F}^{\psi(\Lambda)})$ and in the same way $\mathrm{M}(\mathcal{F}^\Delta)$ bijectively onto $\mathrm{M}(\mathcal{F}^{\psi(\Delta)})$. Let $g \in \mathrm{M}(\mathcal{F}^\Delta)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$; then $\varphi^* g \in \mathrm{M}(\mathcal{F}^{\psi(\Delta)}) \subset \mathcal{F}_{\psi(\Lambda)}$, $\varphi^* f \in \mathrm{M}(\mathcal{F}^{\psi(\Lambda)})$ and thus

$$\begin{aligned}(\lambda_{\psi(\Lambda)} \circ \varphi^*)(gf) &= \lambda_{\psi(\Lambda)}(\varphi^*(gf)) = \lambda_{\psi(\Lambda)}((\varphi^* g)(\varphi^* f)) = (\varphi^* g)\lambda(\varphi^* f) \\ &= (\varphi^* g)(\varphi_*\lambda)(f) = (\varphi^* g)\lambda(f) = \varphi^*(g\lambda(f)) \\ &= \varphi^*(\lambda_\Lambda(gf)) = (\varphi^* \circ \lambda_\Lambda)(gf) \ .\end{aligned}$$

Therefore $(\lambda_{\psi(\Lambda)} \circ \varphi^*)(gf) = (\varphi^* \circ \lambda_\Lambda)(gf)$ for all $g \in \mathrm{M}(\mathcal{F}^\Delta)$, $\Delta \subset S \setminus \Lambda$ and all $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, hence by Proposition M.14.4 $\lambda_{\psi(\Lambda)} \circ \varphi^* = \varphi^* \circ \lambda_\Lambda$, and so

$$\lambda_{\psi(\Lambda)} = \varphi^* \circ \lambda_\Lambda \circ (\varphi^*)^{-1} = (\psi^*)^{-1} \circ \lambda_\Lambda \circ \psi^* = \psi^\Diamond\lambda_\Lambda \ . \ \square$$

Now let $w = \{w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be an $\mathbb{F}$-multiplicative family; then by Theorem 4.1 there is the specification $\mathcal{V}_w = \{\pi^w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$, where

$$\pi^w_\Lambda(f) = (\pi_\Lambda(w_\Lambda))^{-1}\pi_\Lambda(w_\Lambda f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $\Lambda \in \mathcal{N}_0$. Let us say that w is *H-invariant* if $w_\Lambda = \psi^* w_{\psi(\Lambda)}$ for all $\Lambda \in \mathcal{N}_0$, $\psi \in H$, i.e., if $w_\Lambda(x) = w_{\psi(\Lambda)}(\psi(x))$ for all $x \in X$, $\Lambda \in \mathcal{N}_0$, $\psi \in H$.

Proposition 7.5 *If $\mathcal{V}$ and w are both H-invariant then the specification $\mathcal{V}_w$ is also H-invariant.*

Proof Let $\Lambda \in \mathcal{N}_0$ and $\psi \in H$; then for all $f \in \mathrm{M}(\mathcal{F})$, $x \in X$

$$\begin{aligned}\pi_{\psi(\Lambda)}(w_{\psi(\Lambda)}f)(x) &= (\psi^\diamond\pi_\Lambda)(w_{\psi(\Lambda)}f)(x) = \pi_\Lambda(\psi^*(w_{\psi(\Lambda)}f))(\psi^{-1}(x))\\ &= \pi_\Lambda((\psi^* w_{\psi(\Lambda)})(\psi^* f))(\psi^{-1}(x)) = \pi_\Lambda(w_\Lambda(\psi^* f))(\psi^{-1}(x))\end{aligned}$$

and it therefore follows that

$$\begin{aligned}\pi^w_{\psi(\Lambda)}(f)(x) &= (\pi_{\psi(\Lambda)}(w_{\psi(\Lambda)})(x))^{-1}\pi_{\psi(\Lambda)}(w_{\psi(\Lambda)}f)(x)\\ &= (\pi_\Lambda(w_\Lambda)(\psi^{-1}(x)))^{-1}\pi_\Lambda(w_\Lambda(\psi^* f))(\psi^{-1}(x))\\ &= \pi^w_\Lambda(\psi^* f)(\psi^{-1}(x)) = (\psi^\diamond\pi^w_\Lambda)(f)(x)\ .\end{aligned}$$

Thus $\pi^w_{\psi(\Lambda)} = \psi^\diamond\pi^w_\Lambda$ for all $\Lambda \in \mathcal{N}_0$, $\psi \in H$, i.e., $\mathcal{V}_w$ is H-invariant. □

We next consider conditions which ensure that $\mathcal{G}_H(\mathcal{V})$ is non-empty, and we will work with the set-up considered in Chapter 5. In particular, we assume that the spatial system Σ has the Kolmogorov extension property.

Lemma 7.11 *Let $\{\mu_n\}_{n\ge1}$ be a sequence from $\mathrm{P}(\mathcal{F})$ which converges locally to $\mu \in \mathrm{P}(\mathcal{F})$. Then $\{\psi_*\mu_n\}_{n\ge1}$ converges locally to $\psi_*\mu$ for each $\psi \in H$.*

Proof Let $\psi \in H$; then, since $\psi^{-1}(\mathcal{A}) \subset \mathcal{A}$, it follows that

$$(\psi_*\mu)(A) = \mu(\psi^{-1}(A)) = \lim_{n\to\infty}\mu_n(\psi^{-1}(A)) = \lim_{n\to\infty}(\psi_*\mu_n)(A)$$

for all $A \in \mathcal{A}$. □

Let us say that a subset Q of $\mathrm{P}(\mathcal{F})$ is *closed under local convergence* if whenever $\{\mu_n\}_{n\ge1}$ is a sequence from Q which converges locally to $\mu \in \mathrm{P}(\mathcal{F})$ then $\mu \in Q$. In particular, if the specification $\mathcal{V}$ is quasi-local then by Proposition 5.4 $\mathcal{G}(\mathcal{V})$ is closed under local convergence.

We say further that Q is *H-invariant* if $\psi_*\mu \in Q$ for all $\mu \in Q$, $\psi \in H$. Thus if $\mathcal{V}$ is H-invariant then by Proposition 7.3 $\mathcal{G}(\mathcal{V})$ is H-invariant.

Until further notice suppose that H is abelian: This assumption is needed in Propositions 7.6, 7.7 and 7.8 and Theorem 7.2 (since they all essentially depend on something like the Markov-Kakutani fixed point theorem).

Proposition 7.6 *Suppose Q is a non-empty convex locally equicontinuous subset of* $\mathrm{P}(\mathcal{F})$ *which is both closed under local convergence and is H-invariant. Then*

$$\mathrm{Fix}_H(Q) = \{\mu \in Q : \psi_*\mu = \mu \text{ for all } \psi \in H\}$$

is non-empty. Moreover, if $\{\mu_n\}_{n\geq 1}$ is any sequence of elements from $\mathrm{Fix}_H(Q)$ *then there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathrm{Fix}_H(Q)$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ.*

Proof The proof is just that of the Markov-Kakutani fixed point theorem (see, for example, Dunford and Schwartz [7], Theorem V.10.6), but exploiting the fact that H is countable and that we have some kind of sequential compactness. Let $\mathcal{D}$ denote the set of non-empty convex subsets of Q which are both closed under local convergence and are H-invariant. Thus in particular $Q \in \mathcal{D}$. If $D \in \mathcal{D}$ and $\psi \in H$ then let $\mathrm{Fix}_\psi(D) = \{\mu \in D : \psi_*\mu = \mu\}$.

Lemma 7.12 $\mathrm{Fix}_\psi(D) \in \mathcal{D}$ *for all $D \in \mathcal{D}$, $\psi \in D$.*

Proof It is clear that $\mathrm{Fix}_\psi(D)$ is convex and it is H-invariant because H is abelian. (If $\mu \in \mathrm{Fix}_\psi(D)$ and $\varphi \in H$ then $\psi_*(\varphi_*\mu) = \varphi_*(\psi_*\mu) = \varphi_*\mu$ and so $\varphi_*\mu \in \mathrm{Fix}_\psi(D)$.) Moreover, if $\{\mu_n\}_{n\geq 1}$ is a sequence from $\mathrm{Fix}_\psi(D)$ which converges locally to $\mu \in \mathrm{P}(\mathcal{F})$ then by Lemma 7.11 the sequence $\{\psi_*\mu_n\}_{n\geq 1}$ converges locally to $\psi_*\mu$, and so $\psi_*\mu = \mu$, since $\psi_*\mu_n = \mu_n$ for each n, i.e., $\mu \in \mathrm{Fix}_\psi(D)$. Hence $\mathrm{Fix}_\psi(D)$ is also H-invariant.

It thus remains to show that $\mathrm{Fix}_\psi(D)$ is non-empty. Let $\mu \in D$ and for each $n \geq 1$ let $\mu_n = (1/n)(\mu + \psi_*\mu + \cdots + \psi_*^{n-1}\mu)$, where $\psi^1 = \psi$ and $\psi^{k+1} = \psi \circ \psi^k$. Then $\mu_n \in D$, since D is convex and H-invariant, and by Proposition 5.3 there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu' \in \mathrm{P}(\mathcal{F})$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ' (since Q is locally equicontinuous). Hence $\mu' \in D$, since D is closed under local convergence. Moreover, by Lemma 7.11 $\{\psi_*\mu_{n_j}\}_{j\geq 1}$ converges locally to $\psi_*\mu'$. But $k^{-1}\mu + \psi_*\mu_k = \mu_k + k^{-1}\psi_*^k\mu$ and thus $|(\psi_*\mu_k)(A) - \mu_k(A)| \leq 2/k$ for all $A \in \mathcal{A}$, $k \geq 1$. Therefore $(\psi_*\mu')(A) = \mu'(A)$ for all $A \in \mathcal{A}$ and so by Proposition M.3.3 $\psi_*\mu' = \mu'$. This shows that $\mu' \in \mathrm{Fix}_\psi(D)$, i.e., $\mathrm{Fix}_\psi(D)$ is non-empty. □

Now let $\{\psi_n\}_{n\geq 1}$ be some enumeration of the elements of H and using Lemma 7.12 define a decreasing sequence $\{D_n\}_{n\geq 1}$ from $\mathcal{D}$ by putting $D_0 = Q$ and letting $D_n = \mathrm{Fix}_{\psi_n}(D_{n-1})$ for each $n \geq 1$. Then $D = \bigcap_{n\geq 1} D_n \neq \varnothing$: For each $n \geq 1$ let $\mu \in D_n$; then by Proposition 5.3 there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathrm{P}(\mathcal{F})$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ (since Q is locally equicontinuous). However, for each $n \geq 1$ there exists $p \geq 1$ such that $\{\mu_{n_j}\}_{j\geq p}$ is a sequence in D_n (which converges locally to μ) and therefore $\mu \in D_n$, since D_n is closed under local convergence. Thus $\mu \in D$. But clearly $D \subset \mathrm{Fix}_H(Q)$ and so $\mathrm{Fix}_H(Q)$ is non-empty.

Finally, if $\{\mu_n\}_{n\geq 1}$ is any sequence from $\mathrm{Fix}_H(Q)$ then by Proposition 5.3 there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathrm{P}(\mathcal{F})$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ, and then by Lemma 7.11 $\mu \in \mathrm{Fix}_H(Q)$, since $\mu_n \in \mathrm{Fix}_\psi(Q)$ for all $n \geq 1$, $\psi \in H$. This completes the proof of Proposition 7.6. □

Theorem 7.2 *Suppose $\mathcal{V}$ is an H-invariant quasi-local specification for which $\mathcal{G}(\mathcal{V})$ is a non-empty locally equicontinuous subset of $\mathrm{P}(\mathcal{F})$. Then $\mathcal{G}_H(\mathcal{V})$ is non-empty, and if $\{\mu_n\}_{n\geq 1}$ is any sequence from $\mathcal{G}_H(\mathcal{V})$ then there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathcal{G}_H(\mathcal{V})$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ.*

Proof This is a special case of Proposition 7.6 (with $Q = \mathcal{G}(\mathcal{V})$) since, as was already noted, Proposition 5.4 implies $\mathcal{G}(\mathcal{V})$ is closed under local convergence and by Proposition 7.3 $\mathcal{G}(\mathcal{V})$ is H-invariant. □

Let us apply Theorem 7.2 to a more explicit result which can be applied to most lattice models. Again assume for the moment that the family $\{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}$ is weakly independent and let $\lambda \in \mathrm{P}(\mathcal{F})$ be independent. Then by Proposition 4.2 there exists a unique simple $\mathbb{F}$-specification $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ such that $\lambda_\Lambda(gf) = g\lambda(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$ and all $\Lambda \in \mathcal{N}_0$. Now let $e = \{e_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be an $\mathbb{F}$-additive family (with $e_\Lambda \in \mathrm{M}^\Diamond(\mathcal{F})$ for each $\Lambda \in \Lambda_0$) and for each $\Lambda \in \mathcal{N}_0$ put $w_\Lambda = \exp(-e_\Lambda)$, so $w = \{w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is an $\mathbb{F}$-multiplicative family. By Theorem 4.1 there is then the $\mathbb{F}$-specification $\mathcal{U}_w = \{\lambda^w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$, where

$$\lambda^w_\Lambda(f) = (\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $\Lambda \in \mathcal{N}_0$. Let us assume that λ is H-invariant and that the family e is H-invariant, i.e., that $e_\Lambda(x) = e_{\psi(\Lambda)}(\psi(x))$ for all $x \in X$, $\Lambda \in \mathcal{N}_0$, $\psi \in H$. Then w is H-invariant and so by Propositions 7.4 and 7.5 the specification $\mathcal{U}_w$ is H-invariant.

As in Chapter 5 a mapping $u \in \mathrm{M}^\Diamond(\mathcal{F})$ is called quasi-local if for each $\varepsilon > 0$ there exists $\Lambda \in \mathcal{N}$ and a bounded mapping $v \in \mathrm{M}^\Diamond(\mathcal{F}^\Lambda)$ such that $|u - v| \leq \varepsilon$ (and so in particular u is also bounded).

Proposition 7.7 *Suppose the mapping e_Λ is quasi-local for each $\Lambda \in \mathcal{N}_0$. Then the set of H-invariant Gibbs states $\mathcal{G}_H(\mathcal{U}_w)$ is non-empty. Moreover, for each sequence $\{\mu_n\}_{n\geq 1}$ from $\mathcal{G}_H(\mathcal{U}_w)$ there exists a subsequence $\{n_j\}_{j\geq 1}$ and a Gibbs state $\mu \in \mathcal{G}_H(\mathcal{U}_w)$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges locally to μ.*

Proof The proof of Proposition 5.1 shows that $\mathcal{U}_w$ is a quasi-local specification and that $\mathcal{G}(\mathcal{V})$ is a non-empty locally equicontinuous subset of $\mathrm{P}(\mathcal{F})$. The result thus follows from Theorem 7.2. □

Now fix a non-empty subset $\mathcal{C}$ of $\mathcal{A}_\delta$ for which the inclusion order is directed and countably generated. We say that a subset Q of $\mathrm{P}(\mathcal{F})$ is *closed under $\mathcal{C}$-local convergence* if whenever $\{\mu_n\}_{n\geq 1}$ is a sequence from Q which converges $\mathcal{C}$-locally to $\mu \in \mathrm{P}(\mathcal{F})$ then $\mu \in Q$.

Proposition 7.8 *Suppose that Q is a non-empty convex locally equicontinuous $\mathcal{C}$-tight subset of $\mathrm{P}(\mathcal{F})$ which is closed under $\mathcal{C}$-local convergence and H-invariant. Then the set*

$$\mathrm{Fix}_H(Q) = \{\mu \in Q : \psi_*\mu = \mu \text{ for all } \psi \in H\}$$

is non-empty. Moreover, if $\{\mu_n\}_{n\geq 1}$ is any sequence of elements from $\mathrm{Fix}_H(Q)$ then there exists a subsequence $\{n_j\}_{j\geq 1}$ and $\mu \in \mathrm{Fix}_H(Q)$ such that $\{\mu_{n_j}\}_{j\geq 1}$ converges $\mathcal{C}$-locally to μ.

Proof If $\{\mu_n\}_{n\geq 1}$ is any sequence from Q which converges locally to $\mu \in \mathrm{P}(\mathcal{F})$ then by Proposition 5.5 $\{\mu_n\}_{n\geq 1}$ also converges $\mathcal{C}$-locally to μ, since Q is $\mathcal{C}$-tight. The result thus follows from Proposition 7.6. □

Proposition 7.8 can be applied to show that $\mathcal{G}_H(\mathcal{V})$ is non-empty for reasonable particle models (although the choice of Q depends very much on the particulars of the model). Note, however, that it can only be applied with $Q = \mathcal{G}_H(\mathcal{V})$ in rather trivial cases.

Finally, we end the chapter by looking at the relationship between H-invariance and the representation theory considered in Chapter 6. Let $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be an H-invariant $\mathbb{F}$-specification for which $\mathcal{G}(\mathcal{V})$ is non-empty and assume that $(X, \mathcal{F})$ is substandard Borel (although we no longer need Σ to have the Kolmogorov extension property). We do not assume that H is abelian, nor that H separates $\mathcal{N}$, so the extreme points of $\mathcal{G}_H(\mathcal{V})$ need not be H-ergodic.

Recall that if π is a quasi-probability kernel then $\mathcal{I}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu = \mu\pi\}$; moreover, if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ and π is $\mathcal{E}$-measurable then

$$\mathcal{J}(\pi) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu(gf) = \mu(g\pi(f)) \text{ for all } g \in \mathrm{M}(\mathcal{E}),\ f \in \mathrm{M}(\mathcal{F})\}\ .$$

Thus $\mathcal{J}(\pi) \subset \mathcal{I}(\pi)$, with equality when π is a strict $\mathcal{E}$-measurable kernel. Again put $P_\pi = \{x \in X : \pi(x, X) = 1\}$, thus $I_{P_\pi} = \pi(1)$ and, as noted twice before, $\mu(P_\pi) = 1$ for each $\mu \in \mathcal{I}(\pi)$.

Let $\pi \in \mathrm{K}(\mathcal{F})$ be the kernel given in Theorem 6.2, thus π is a $\mathcal{F}_\infty$-measurable quasi-probability kernel satisfying:

(1) $\mathcal{G}(\mathcal{V}) = \mathcal{J}(\pi) = \mathcal{I}(\pi)$.

(2) $\pi\pi = \pi$, and hence the mapping $\mu \mapsto \mu\pi$ maps $\{\mu \in \mathrm{P}(\mathcal{F}_\infty) : \mu(P_\pi) = 1\}$ onto $\mathcal{G}(\mathcal{V})$.

(3) $\pi(x, \cdot) \in \operatorname{ext} \mathcal{G}(\mathcal{V})$ for each $x \in P_\pi$.

(4) If $\mu \in \operatorname{ext} \mathcal{G}(\mathcal{V})$ then $\mu(\{x \in X : \pi(x, \cdot) = \mu\}) = 1$, and so in particular there exists $x \in P_\pi$ such that $\mu = \pi(x, \cdot)$.

Theorem 7.3 *Put*

$$\hat{X} = \{x \in X : \pi(f)(x) = (\psi^\diamond \pi)(f)(x) \text{ for all } f \in \mathrm{M}(\mathcal{F}),\ \psi \in H\}\ .$$

Then

(1) $\hat{X} \in \mathcal{F}_\infty \cap \mathcal{I}_H$, *and* $\mu(\hat{X}) = 1$ *for each* $\mu \in \mathcal{G}(\mathcal{V})$.

(2) $\mathcal{G}_H(\mathcal{V})$ *is empty if and only if* $\nu(\hat{X}) = 0$ *for all* $\nu \in \mathrm{P}_H(\mathcal{F})$.

(3) If $\nu \in \mathrm{P}_H(\mathcal{F})$ *with* $\nu(\hat{X}) = 1$ *and* $\mu = \nu\pi$ *then* $\mu \in \mathcal{G}_H(\mathcal{V})$*; if* ν *is also* H*-ergodic then* $\mu \in \operatorname{ext} \mathcal{G}_H(\mathcal{V})$.

(Note that if ν *is any* H*-ergodic element of* $\mathcal{G}_H(\mathcal{V})$ *then (1) implies that* $\nu(\hat{X})$ *must be either* 0 *or* 1.*)*

Proof This is given below. □

Remarks: Parts (1) and (2) of Theorem 7.3 show that either $\mathcal{G}_H(\mathcal{V})$ is non-empty or $\mathcal{G}(\mathcal{V})$ and $\mathrm{P}_H(\mathcal{F})$ are uniformly mutually singular in that $\mu(\hat{X}) = 1$ for all $\mu \in \mathcal{G}(\mathcal{V})$ and $\mu(\hat{X}) = 0$ for all $\mu \in \mathrm{P}_H(\mathcal{F})$.

(2) Theorem 7.3 (3) shows that if both $\mathcal{G}_H(\mathcal{V})$ and $\operatorname{ext} \mathrm{P}_H(\mathcal{F})$ are non-empty then $\operatorname{ext} \mathcal{G}_H(\mathcal{V})$ is also non-empty.

In order to prove Theorem 7.3 some preparatory lemmas are needed.

Lemma 7.13 *For each* $\psi \in H$ *let*

$$\hat{X}_\psi = \{x \in X : \pi(f)(x) = (\psi^\diamond \pi)(f)(x) \text{ for all } f \in \mathrm{M}(\mathcal{F})\}\ .$$

Then $\hat{X}_\psi \in \mathcal{F}_\infty$ *and* $\mu(\hat{X}_\psi) = 1$ *for each* $\mu \in \mathcal{G}(\mathcal{V})$.

Proof Let $\mathcal{A}$ be a countable algebra with $\mathcal{F} = \sigma(\mathcal{A})$; then $\mathcal{F}$ can be replaced by $\mathcal{A}$ in the definition of $\hat{X}_\psi$, and thus it is only necessary to show for each $F \in \mathcal{F}$ that $\{x \in P_\pi : \pi(x, F) = \psi^\diamond \pi(x, F)\} \in \mathcal{F}_\infty$ and that $\pi(\cdot, F) = \psi^\diamond \pi(\cdot, F)$ μ-a.e. for each $\mu \in \mathcal{G}(\mathcal{V})$. The first of these two statements is clearly true since both $\pi(\cdot, F)$ and $\psi^\diamond \pi(\cdot, F)$ are in $\mathrm{M}(\mathcal{F}_\infty)$ (the latter because H is $\mathbb{F}$-adapted). To show the second holds consider $\mu \in \mathcal{G}(\mathcal{V})$, $f \in \mathrm{M}(\mathcal{F})$ and $g \in \mathrm{M}(\mathcal{F}_\infty)$; then using (1) and the fact that $\psi_*^{-1}\mu \in \mathcal{G}(\mathcal{V})$ and $\psi^* g \in \mathcal{F}_\infty$ it follows that

$$\begin{aligned}
\mu(g(\psi^\diamond\pi)(f)) &= \mu(g(\psi^*)^{-1}(\pi(\psi^* f))) = \mu((\psi^{-1})^*((\psi^* g)\pi(\psi^* f))) \\
&= (\psi_*^{-1}\mu)((\psi^* g)\pi(\psi^* f)) = (\psi_*^{-1}\mu)((\psi^* g)(\psi^* f)) \\
&= (\psi_*^{-1}\mu)(\psi^*(gf)) = \mu(gf) = \mu(g\pi(f))\ ,
\end{aligned}$$

and hence $(\psi^\diamond\pi)(f) = \pi(f)$ μ-a.e. for each $f \in \mathrm{M}(\mathcal{F})$. In particular this means that $\psi^\diamond\pi(\cdot, F) = \pi(\cdot, F)$ μ-a.e. for each $F \in \mathcal{F}$. □

Lemma 7.14 $\hat{X} \in \mathcal{I}_H$.

Proof Let $x \in \hat{X}$, $\psi \in H$; then for all $\varphi \in H$, $f \in \mathrm{M}(\mathcal{F})$

$$\begin{aligned}
(\varphi^\diamond\pi)(f)(\psi(x)) &= \pi(\varphi^* f)(\varphi^{-1}(\psi(x))) \\
&= \pi(\varphi^* f)((\psi^{-1}\circ\varphi)^{-1}(x)) \\
&= \pi((\psi^{-1}\circ\varphi)^*(\psi^* f))((\psi^{-1}\circ\varphi)^{-1}(x)) \\
&= \pi((\psi^{-1}\circ\varphi)^*(\psi^* f))((\psi^{-1}\circ\varphi)^{-1}(x)) \\
&= ((\psi^{-1}\circ\varphi)^\diamond\pi)(\psi^* f)(x) = \pi(\psi^* f)(x) \\
&= ((\psi^{-1})^\diamond\pi)(\psi^*(f))(x) = \pi((\psi^{-1})^*(\psi^* f))(\psi(x)) \\
&= \pi(f)(\psi(x))
\end{aligned}$$

and thus $\psi(x) \in \hat{X}$. Hence by Lemma 7.1 $\hat{X} \in \mathcal{I}_\psi$ for all $\psi \in H$, i.e., $\hat{X} \in \mathcal{I}_H$. □

Lemma 7.15 *Let $\mu \in \mathrm{P}_H(\mathcal{F})$ with $\mu(\hat{X} \cap P_\pi) = 1$. Then $\mu\pi \in \mathcal{G}_H(\mathcal{V})$.*

Proof Since $\mu(P_\pi) = 1$ it follows that $(\mu\pi)(X) = \mu(P_\pi) = 1$, i.e., $\mu\pi \in \mathrm{P}(\mathcal{F})$ and thus by (1) and (2) $\mu\pi \in \mathcal{G}(\mathcal{V})$, since $(\mu\pi)\pi = \mu(\pi\pi) = \mu\pi$. It is thus enough to show that $\mu\pi \in \mathrm{P}_H(\mathcal{F})$. But if $\psi \in H$ and $f \in \mathrm{M}(\mathcal{F})$ then

$$\begin{aligned}
(\psi_*(\mu\pi))(f) &= (\mu\pi)(\psi^* f) = \mu(\pi(\psi^* f)) = \mu(I_{\hat{X}}\pi(\psi^* f)) \\
&= \mu(I_{\hat{X}}((\psi^{-1})^\diamond\pi)(\psi^* f)) = \mu(((\psi^{-1})^\diamond\pi)(\psi^* f)) \\
&= \mu(\psi^*(\pi(f))) = (\psi_*\mu)(\pi(f)) = \mu(\pi(f)) = (\mu\pi)(f)
\end{aligned}$$

and thus $\mu\pi \in \mathrm{P}_H(\mathcal{F})$. □

Lemma 7.16 *Let $\mu \in \mathrm{P}_H(\mathcal{F})$ be H-ergodic with $\mu(\hat{X} \cap P_\pi) = 1$. Then $\mu\pi$ is an extreme point of $\mathcal{G}_H(\mathcal{V})$.*

Proof Let $F \in \mathcal{F}_\infty \cap \mathcal{I}_H$; then for any $\psi \in H$ and $x \in \hat{X}$

$$\pi(I_F)(\psi(x)) = \pi((\psi^{-1})^* I_F)(\psi(x)) = ((\psi^{-1})^\Diamond \pi)(I_F)(x) = \pi(I_F)(x) \; .$$

Thus $\pi(I_F)$ is μ-almost ψ-invariant for each $\psi \in H$, since $\mu(\hat{X}) = 1$, and hence by Lemma 7.4 $\pi(I_F)$ is μ-a.e. equal to a constant (because μ is H-ergodic). But $\pi(x, \cdot) \in \operatorname{ext} \mathcal{G}(\mathcal{V})$ for each $x \in P_\pi$ and $F \in \mathcal{F}_\infty$, and therefore by Theorem 6.1 $\pi(I_F)(x) = \pi(x, F)$ is either 0 or 1. Hence either $\pi(\cdot, F) = 1$ μ-a.e. or $\pi(\cdot, F) = 0$ μ-a.e., which means $(\mu\pi)(F)$ is either 0 or 1; i.e., $\mu\pi$ is trivial on $\mathcal{F}_\infty \cap \mathcal{I}$, and therefore by Proposition 7.3 $\mu\pi$ is an extreme point of $\mathcal{G}_H(\mathcal{V})$. □

Proof of Theorem 7.3: (1) This follows directly from Lemmas 7.13 and 7.14.

(3) This follows directly from Lemmas 7.15 and 7.16.

(2) This also follows from Lemma 7.15: If there exists a measure $\nu \in \mathrm{P}_H(X, \mathcal{F})$ with $\nu(\hat{X}) > 0$ then $\breve{\nu}\pi \in \mathcal{G}_H(\mathcal{V})$, where $\breve{\nu} = \nu{\cdot}g$ with $g = (\nu(\hat{X}))^{-1} I_{\hat{X}}$, and thus $\mathcal{G}_H(\mathcal{V}) \neq \varnothing$. Conversely, if $\mathcal{G}_H(\mathcal{V})$ is non-empty then $\mu \in \mathrm{P}_H(X, \mathcal{F})$ and also $\mu(\hat{X}) = 1$ for any $\mu \in \mathcal{G}_H(\mathcal{V})$. □

8 One-point Gibbs states

In this chapter we consider a spatial system $\Sigma = (S, \mathcal{N}, X, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}})$ in which S is a countably infinite set and $\mathcal{N}$ is the ring of finite subsets of S (and so we have a situation typically associated with a lattice model). As usual let $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be the associated family of σ-algebras.

Let $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be an $\mathbb{F}$-specification. We are interested here in the following question: Given a *one-point Gibbs state*, i.e., a measure $\mu \in \mathrm{P}(\mathcal{F})$ satisfying $\mu\pi_{\{s\}} = \mu$ for all $s \in S$, when can we conclude that μ is actually a Gibbs state, i.e., that $\mu\pi_\Lambda = \mu$ for all $\Lambda \in \mathcal{N}_0$?

This question arises, for instance, when employing a technique of Dobrushin [5] for establishing the existence of Gibbs states. Dobrushin's method essentially only constructs one-point Gibbs states and so there remains the problem of showing that these measures are actually 'real' Gibbs states.

In explicit models the question has an affirmative answer if all the conditional probabilities are strictly positive. A precise formulation of this statement is given in Theorem 1.33 of Georgii [11], which is a direct extension of a remark of Dobrushin [5].

Without the assumption of strict positivity things are more complicated, and in this chapter we rework Georgii's proof to obtain Theorem 8.1 below, in which the assumption of strict positivity can be relaxed somewhat.

We restrict our attention to specifications of the form $\mathcal{U}_w = \{\lambda^w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ defined in terms of an independent measure λ and a family of mappings $w = \{w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$. (Recall that a measure $\lambda \in \mathrm{P}(\mathcal{F})$ is independent if whenever $\Lambda,\ \Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$ then $\lambda(E \cap F) = \lambda(E)\lambda(F)$ for all $E \in \mathcal{F}^\Lambda$, $F \in \mathcal{F}^\Delta$.) Thus in what follows let $\lambda \in \mathrm{P}(\mathcal{F})$ be an independent measure for which the associated family of kernels $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ exists. This means that λ_Λ is the unique kernel with $\lambda_\Lambda(gf) = g\lambda(f)$ for all $g \in \mathrm{M}(\mathcal{F}_\Lambda)$, $f \in \mathrm{M}(\mathcal{F}^\Lambda)$, and by Proposition 4.1 $\mathcal{U}$ is a simple $\mathbb{F}$-specification. (In any reasonable example this specification will exist.) Also let $w = \{w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ be a family from $\mathrm{M}(\mathcal{F})$ such that $\mathcal{U}_w = \{\lambda^w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ is an $\mathbb{F}$-specification, where $\lambda^w_\Lambda : \mathrm{M}(\mathcal{F}) \to \mathrm{M}(\mathcal{F})$ is the strict $\mathcal{F}_\Lambda$-measurable quasi-probability kernel defined by

$$\lambda^w_\Lambda(f) = (\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda f)$$

for each $f \in \mathrm{M}(\mathcal{F})$. The family w must thus satisfy the conditions given in Proposition 4.3, which will usually be achieved by w being $\mathbb{F}$-multiplicative.

Let us note that the only special properties we require $\mathcal{U}$ to have (other than it is a specification) are that it is simple (i.e., each λ_Λ is a probability kernel and not just a quasi-probability kernel) and that if $\Lambda,\ \Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$ then

$\lambda_{\Lambda\cup\Delta} = \lambda_\Delta\lambda_\Lambda$. This later property is crucial, and Lemma 4.3 shows it holds for the specification $\mathcal{U}$.

To increase the legibility we will just write π_Λ instead of λ^w_Λ, and so we will be working with an $\mathbb{F}$-specification $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ with

$$\pi_\Lambda(f) = (\lambda_\Lambda(w_\Lambda))^{-1}\lambda_\Lambda(w_\Lambda f)$$

for each $f \in \mathrm{M}(\mathcal{F})$. For each $\Lambda \in \mathcal{N}_0$ put

$$G_\Lambda = \{x \in X : 0 < w_\Lambda(x) < \infty \text{ and } 0 < \lambda_\Lambda(w_\Lambda)(x) < \infty\} ,$$

let $\gamma_\Lambda = \lambda_\Lambda(I_{G_\Lambda})$ (so $\gamma_\Lambda \in \mathrm{M}(\mathcal{F}_\Lambda)$ with $\gamma_\Lambda \leq 1$) and let

$$Q = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu(\gamma_\Lambda) = \mu(\pi_\Lambda(\gamma_\Delta)) = 1 \text{ for all } \Lambda,\, \Delta \in \mathcal{N}_0 \text{ with } \Delta \subset \Lambda\} .$$

The simplest possible case here is with $G_\Lambda = X$ for each $\Lambda \in \mathcal{N}_0$, and then $Q = \mathrm{P}(\mathcal{F})$. Note that

$$Q \cap \mathcal{G}(\mathcal{V}) = \{\mu \in \mathcal{G}(\mathcal{V}) : \mu(\gamma_\Lambda) = 1 \text{ for all } \Lambda \in \mathcal{N}_0\} .$$

Lemma 8.1 below implies that if $\mu \in \mathcal{G}(\mathcal{V})$ then $\mu(G_\Lambda) = 1$ for each $\Lambda \in \mathcal{N}_0$. The condition $\mu(\gamma_\Lambda) = 1$, however, says that $(\mu\lambda_\Lambda)(G_\Lambda) = 1$.

Theorem 8.1 $\{\mu \in Q : \mu\pi_{\{s\}} = \mu$ *for all* $s \in S\} \subset \mathcal{G}(\mathcal{V})$.

Proof It is easier not to use the family $\{w_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ directly but to employ instead a family $\{v_\Lambda\}_{\Lambda\in\mathcal{N}_0}$ already occurring implicitly in Proposition 3.1. For each $\Lambda \in \mathcal{N}_0$ let $v_\Lambda = \alpha_\Lambda j_\Lambda w_\Lambda$, where $\alpha_\Lambda \in \mathrm{M}(\mathcal{F}_\Lambda)$ and $j_\Lambda \in \mathrm{M}(\mathcal{F})$ are given by

$$\alpha_\Lambda(x) = \begin{cases} (\lambda_\Lambda(w_\Lambda)(x))^{-1} & \text{if } 0 < \lambda_\Lambda(w_\Lambda)(x) < \infty , \\ 0 & \text{otherwise} , \end{cases}$$

$$j_\Lambda(x) = \begin{cases} 1 & \text{if } 0 < w_\Lambda(x) < \infty , \\ 0 & \text{otherwise} . \end{cases}$$

Then it was shown in Proposition 3.1 that

$$\pi_\Lambda(f) = \lambda_\Lambda(v_\Lambda f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. Note that by definition $v_\Lambda(x) < \infty$ for all $x \in X$ and that

$$G_\Lambda = \{x \in X : v_\Lambda(x) > 0\}$$

for each $\Lambda \in \mathcal{N}_0$. Note also that $\lambda_\Lambda(v_\Lambda) = \pi_\Lambda(1)$ and so the mapping $\lambda_\Lambda(v_\Lambda)$ can only take on the values 0 and 1. For each $\Lambda \in \mathcal{N}_0$ define $v^\diamond_\Lambda : X \to \mathbb{R}^+_\infty$ by

$$v^\diamond_\Lambda(x) = \begin{cases} (v_\Lambda(x))^{-1} & \text{if } v_\Lambda(x) > 0 , \\ 0 & \text{otherwise} ; \end{cases}$$

thus $v^\diamond_\Lambda \in \mathrm{M}(\mathcal{F}_\Lambda)$ and $v_\Lambda v^\diamond_\Lambda = I_{G_\Lambda}$, since $G_\Lambda = \{x \in X : v_\Lambda(x) > 0\}$ (and since $v_\Lambda(x) < \infty$ for all $x \in X$).

Lemma 8.1 *If $\mu \in \mathrm{P}(\mathcal{F})$ and $\Lambda \in \mathcal{N}_0$ with $\mu\pi_\Lambda = \mu$ then $\mu(G_\Lambda) = 1$.*

Proof Since $G_\Lambda = \{x \in X : v_\Lambda(x) > 0\}$ it follows that

$$\begin{aligned}\mu(G_\Lambda) = (\mu\pi_\Lambda)(G_\Lambda) &= \mu(\pi_\Lambda(I_{G_\Lambda})) = \mu(\lambda_\Lambda(v_\Lambda I_{G_\Lambda}))\\ &= \mu(\lambda_\Lambda(v_\Lambda)) = \mu(\pi_\Lambda(1)) = (\mu\pi_\Lambda)(1) = \mu(X) = 1\ . \quad \square\end{aligned}$$

For each $\Lambda \in \mathcal{N}_0$ let $\mathcal{G}_\Lambda = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu\pi_\Lambda = \mu\}$. In Lemmas 8.2 and 8.3 let $\Lambda,\ \Delta \in \mathcal{N}_0$ with $\Lambda \cap \Delta = \varnothing$ and put $Q_{\Lambda,\Delta} = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu(\gamma_\Lambda) = \mu(\gamma_\Delta) = 1\}$.

Lemma 8.2 *There exists $u_{\Lambda,\Delta} \in \mathrm{M}(\mathcal{F})$ such that*

$$\mu(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta} f)) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$ whenever $\mu \in Q_{\Lambda,\Delta} \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$.

Proof Let $h = v^\diamond_\Lambda \lambda_\Lambda(v_\Lambda v^\diamond_\Delta)$, and define $u_{\Lambda,\Delta} \in \mathrm{M}(\mathcal{F})$ by

$$u_{\Lambda,\Delta}(x) = \begin{cases} (h(x))^{-1} & \text{if } x \in D = \{x \in X : 0 < h(x) < \infty\}\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

(and so $hu_{\Lambda,\Delta} = I_D$). Let $\mu \in Q_{\Lambda,\Delta} \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$; then, since $\mu\pi_\Lambda = \mu = \mu\pi_\Delta$,

$$\begin{aligned}(\mu\lambda_\Lambda)(v_\Lambda f) &= \mu(\lambda_\Lambda(v_\Lambda f)) = \mu(\pi_\Lambda(f)) = (\mu\pi_\Lambda)(f)\\ &= \mu(f) = (\mu\pi_\Delta)(f) = \mu(\pi_\Delta(f)) = \mu(\lambda_\Delta(v_\Delta f))\\ &= (\mu\lambda_\Delta)(v_\Delta f)\end{aligned}$$

for all $f \in \mathrm{M}(\mathcal{F})$. Moreover, since $\mu \in Q_{\Lambda,\Delta}$,

$$(\mu\lambda_\Lambda)(G_\Lambda) = \mu(\gamma_\Lambda) = 1 = \mu(\gamma_\Delta) = (\mu\lambda_\Delta)(G_\Delta)\ .$$

Thus by Lemmas 3.3 (3) and 4.2 and Proposition M.10.6 (3)

$$\begin{aligned}\mu(\lambda_{\Lambda\cup\Delta}(f)) &= \mu(\lambda_\Delta(\lambda_\Lambda(f))) = (\mu\lambda_\Delta)(\lambda_\Lambda(f)) = (\mu\lambda_\Delta)(I_{G_\Delta}\lambda_\Lambda(f))\\ &= (\mu\lambda_\Delta)(v_\Delta v^\diamond_\Delta \lambda_\Lambda(f)) = \mu(v^\diamond_\Delta\lambda_\Lambda(f)) = (\mu\lambda_\Lambda)(v_\Lambda v^\diamond_\Delta\lambda_\Lambda(f))\\ &= \mu(\lambda_\Lambda(v_\Lambda v^\diamond_\Delta \lambda_\Lambda(f))) = \mu(\lambda_\Lambda(f\lambda_\Lambda(v_\Lambda v^\diamond_\Delta)))\\ &= (\mu\lambda_\Lambda)(f\lambda_\Lambda(v_\Lambda v^\diamond_\Delta)) = (\mu\lambda_\Lambda)(I_{G_\Lambda} f\lambda_\Lambda(v_\Lambda v^\diamond_\Delta))\\ &= (\mu\lambda_\Lambda)(v_\Lambda v^\diamond_\Lambda f\lambda_\Lambda(v_\Lambda v^\diamond_\Delta)) = \mu(v^\diamond_\Lambda f\lambda_\Lambda(v_\Lambda v^\diamond_\Delta))\\ &= \mu(v^\diamond_\Lambda\lambda_\Lambda(v_\Lambda v^\diamond_\Delta)f) = \mu(hf)\end{aligned}$$

for all $f \in \mathrm{M}(\mathcal{F})$. In particular, applying Proposition M.10.6 (4) with $f = 1$ shows that $\mu(\{x \in X : h(x) = \infty\}) = 0$. Let $B = \{x \in X : \lambda_\Lambda(v_\Lambda v^\diamond_\Delta)(x) = 0\}$; then $B \in \mathcal{F}_\Lambda$ and (since $\mu\pi_\Lambda = \mu$)

$$\begin{aligned}0 = \mu(I_B\lambda_\Lambda(v_\Lambda v^\diamond_\Delta)) &= \mu(\lambda_\Lambda(v_\Lambda I_B v^\diamond_\Delta))\\ &= \mu(\pi_\Lambda(I_B v^\diamond_\Delta)) = (\mu\pi_\Lambda)(I_B v^\diamond_\Delta) = \mu(I_B v^\diamond_\Delta)\ .\end{aligned}$$

But $\{x \in X : v^\diamond_\Delta(x) > 0\} = \{x \in X : v_\Delta(x) > 0\} = G_\Delta$ and by Lemma 8.1 $\mu(G_\Delta) = 1$ (since $\mu\pi_\Delta = \mu$), and hence by Proposition M.10.6 (1) $\mu(B) = 0$. In the same way $\mu(\{x \in X : v^\diamond_\Lambda(x) > 0\} = \mu(G_\Lambda) = 1$ (since $\mu\pi_\Lambda = \mu$). Therefore $\mu(\{x \in X : h(x) = 0\}) = 0$ and thus $\mu(\{x \in X : 0 < h(x) < \infty\}) = 1$, i.e., $\mu(D) = 1$. Now by Proposition M.10.6 (3) (since $hu_{\Lambda,\Delta} = I_D$ and $\mu(D) = 1$)

$$\mu(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)) = \mu(hu_{\Lambda,\Delta}f) = \mu(I_D f) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. □

Lemma 8.3 $Q \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta \subset \mathcal{G}_{\Lambda\cup\Delta}$.

Proof By Lemma M.14.2 $\pi_{\Lambda\cup\Delta}(x,\cdot)\pi_\Lambda = (\pi_{\Lambda\cup\Delta}\pi_\Lambda)(x,\cdot) = \pi_{\Lambda\cup\Delta}(x,\cdot)$ and in the same way $\pi_{\Lambda\cup\Delta}(x,\cdot)\pi_\Delta = \pi_{\Lambda\cup\Delta}(x,\cdot)$ for all $x \in X$. Hence if $\pi_{\Lambda\cup\Delta}(x,\cdot) \in \mathrm{P}(\mathcal{F})$ then $\pi_{\Lambda\cup\Delta}(x,\cdot) \in \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$. Put $B = \{x \in X : \pi_{\Lambda\cup\Delta}(\gamma_\Lambda)(x) = \pi_{\Lambda\cup\Delta}(\gamma_\Lambda)(x) = 1\}$; if $x \in B$ then by Proposition M.14.2 $\pi_{\Lambda\cup\Delta}(x,\cdot)(\gamma_\Lambda) = \pi_{\Lambda\cup\Delta}(x,\cdot)(\gamma_\Delta) = 1$, which means that $\pi_{\Lambda\cup\Delta}(x,\cdot) \in Q_{\Lambda,\Delta}$, i.e., $\pi_{\Lambda\cup\Delta}(x,\cdot) \in Q_{\Lambda,\Delta} \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$ for all $x \in B$.

Let $u_{\Lambda,\Delta} \in \mathrm{M}(\mathcal{F})$ be the mapping given in Lemma 8.2. Therefore if $x \in B$ then $\pi_{\Lambda\cup\Delta}(x,\cdot)(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)) = \pi_{\Lambda\cup\Delta}(x,\cdot)(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, and so it follows from Proposition M.14.2 and Lemma 3.3 (2) that

$$\begin{aligned}\pi_{\Lambda\cup\Delta}(f)(x) &= \pi_{\Lambda\cup\Delta}(x,\cdot)(f) = \pi_{\Lambda\cup\Delta}(x,\cdot)(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)) \\ &= \pi_{\Lambda\cup\Delta}(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f))(x) = (\pi_{\Lambda\cup\Delta}\lambda_{\Lambda\cup\Delta})(u_{\Lambda,\Delta}f)(x) \\ &= \pi_{\Lambda\cup\Delta}(1)(x)\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)(x) = \lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)(x)\end{aligned}$$

for all $x \in B$ and all $f \in \mathrm{M}(\mathcal{F})$. Next let $\mu \in Q$; then $\mu(\pi_{\Lambda\cup\Delta}(\gamma_\Lambda)) = 1$ and $\pi_{\Lambda\cup\Delta}(\gamma_\Lambda) \le 1$ and thus $\mu(\{x \in X : \pi_{\Lambda\cup\Delta}(\gamma_\Lambda)(x) = 1\}) = 1$. The same is also true when γ_Λ is replaced by γ_Δ and this shows that $\mu(B) = 1$.

Now let $\mu \in Q \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$; then $\mu \in Q_{\Lambda,\Delta} \cap \mathcal{G}_\Lambda \cap \mathcal{G}_\Delta$ and $\mu(B) = 1$. Therefore by Lemma 8.2 and Proposition M.10.8 (3)

$$\begin{aligned}(\mu\pi_{\Lambda\cup\Delta})(f) = \mu(\pi_{\Lambda\cup\Delta}(f)) &= \mu(I_B\pi_{\Lambda\cup\Delta}(f)) \\ &= \mu(I_B\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)) = \mu(\lambda_{\Lambda\cup\Delta}(u_{\Lambda,\Delta}f)) = \mu(f)\end{aligned}$$

for all $f \in \mathrm{M}(\mathcal{F})$, i.e., $\mu\pi_{\Lambda\cup\Delta} = \mu$, and so $\mu \in \mathcal{G}_{\Lambda\cup\Delta}$. □

Theorem 8.1 now follows immediately from Lemma 8.3. □

9 More on the construction of specifications

We continue the analysis started at the end of Chapter 4. If $\mathcal{V} = \{\pi_A\}_{A\in J}$ is an $\mathbb{F}$-specification and $w = \{w_A\}_{A\in J}$ is a family from $\mathrm{M}(\mathcal{F})$ then as in Chapter 4 we define a family of kernels $\{\pi_A^w\}_{A\in J}$ by letting $\pi_A^w(f) = (\pi_A(w_A))^{-1}\pi_A(w_A f)$ for each $f \in \mathrm{M}(\mathcal{F})$. Thus π_A^w is a strict $\mathcal{F}_A$-measurable quasi-probability kernel for each $A \in J$ and Proposition 4.3 gives necessary and sufficient conditions for $\mathcal{V}_w = \{\pi_A^w\}_{A\in J}$ to be an $\mathbb{F}$-specification. In particular, this will be the case if the family w is $\mathbb{F}$-multiplicative. In this chapter we present a uniform (if somewhat abstract) general method of constructing families $w = \{w_A\}_{A\in J}$ so that $\mathcal{V}_w = \{\pi_A^w\}_{A\in J}$ is an $\mathbb{F}$-specification.

Let $(X, \mathcal{F})$ be a measurable space. The constructions in this chapter will involve elements of $\mathrm{M}(\mathcal{F} \times \mathcal{F})$, i.e., mappings $f : X \times X \to \mathbb{R}^+_\infty$ with $f^{-1}(\mathcal{B}^+_\infty) \subset \mathcal{F} \times \mathcal{F}$ and so we need to start with a technical fact about the relationship between mappings having two arguments and kernels.

Recall that if $f \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$ then by Proposition M.2.7 the sections $y \mapsto f(x, y)$ and $y \mapsto f(y, x)$ define elements of $\mathrm{M}(\mathcal{F})$ for each $x \in X$, and we denote these sections by $f(x, \diamond)$ and $f(\diamond, x)$ respectively. For each kernel $\pi \in \mathrm{K}(\mathcal{F})$ we can thus define a mapping $\Psi_\pi : \mathrm{M}(\mathcal{F} \times \mathcal{F}) \to \mathrm{M}(X \times X)$ by

$$\Psi_\pi(f)(x, y) = \pi(f(y, \diamond))(x)$$

for all $f \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$, $(x, y) \in X \times X$. Then Ψ_π is clearly linear and continuous and hence it is a pre-kernel.

Lemma 9.1 *If the kernel π is finite then Ψ_π is a finite kernel.*

Proof Since $\Psi_\pi(1)(x, y) = \pi(1)(x) < \infty$ for all $(x, y) \in X \times X$, the pre-kernel Ψ_π is finite. Let F_1, $F_2 \in \mathcal{F}$; then

$$\Psi_\pi(I_{F_1\times F_2})(x, y) = \pi(I_{F_1}(y)I_{F_2})(x) = I_{F_1}(y)\pi(I_{F_2})(x)$$

and so $\Psi_\pi(I_{F_1\times F_2}) \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$. Hence by Proposition M.14.5 Ψ_π is a kernel. □

Let $\delta : X \to X \times X$ be the mapping given by $\delta(x) = (x, x)$; then $\delta^{-1}(\mathcal{F}\times\mathcal{F}) = \mathcal{F}$. In fact, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are sub-σ-algebras of $\mathcal{F}$ then $\delta^{-1}(\mathcal{E}_1 \times \mathcal{E}_2) = \mathcal{E}_1 \vee \mathcal{E}_2$, since $\delta^{-1}(E_1 \times E_2) = E_1 \cap E_2$ for all $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$.

Now for each kernel $\pi \in \mathrm{K}(\mathcal{F})$ define a mapping $\psi_\pi : \mathrm{M}(\mathcal{F} \times \mathcal{F}) \to \mathrm{M}(X)$ by

$$\psi_\pi(f)(x) = \pi(f(x, \diamond))(x)$$

for all $f \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$, $x \in X$, thus $\psi_\pi(f) = \Psi_\pi(f) \circ \delta$ for all $f \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$.

Lemma 9.2 *If the kernel π is finite then ψ_π is a finite kernel.*

Proof This follows immediately from Lemma 9.1 together with the fact that $\psi_\pi(f) = \Psi_\pi(f) \circ \delta$ for all $f \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$. $\square$

Now let J be a non-empty set equipped with a directed and countably generated partial order $\preceq$, let $(X, \mathcal{F})$ be an arbitrary measurable space and $\mathbb{F} = \{\mathcal{F}_A\}_{A \in J}$ be a decreasing family of sub-σ-algebras of $\mathcal{F}$. In what follows let us fix an $\mathbb{F}$-specification $\mathcal{V} = \{\pi_A\}_{A \in J}$. If $w = \{w_A\}_{A \in J}$ is a family from $\mathrm{M}(\mathcal{F})$ then the family of kernels $\{\pi^w_A\}_{A \in J}$ is defined by letting $\pi^w_A(f) = (\pi_A(w_A))^{-1}\pi_A(w_A f)$ for each $f \in \mathrm{M}(\mathcal{F})$, $A \in J$. By Proposition 3.1 π^w_A is a strict $\mathcal{F}_A$-measurable quasi-probability kernel for each $A \in J$.

If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then $\mathrm{M}_{\mathsf{F}}(\mathcal{E})$ will denote the mappings in $\mathrm{M}(\mathcal{E})$ which omit the value ∞. A family $v = \{v_A\}_{A \in J}$ is said to be *normed* if $v_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F})$ for all $A \in J$ with $\pi_{\mathrm{A}}(v_A)(x)$ either 0 or 1 for each $x \in X$. (Of course, this depends on $\mathcal{V}$, but we are considering $\mathcal{V}$ to be fixed.) If $v = \{v_A\}_{A \in J}$ is normed then

$$\pi^v_A(f) = \pi_A(v_A f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$, since $\pi_A(v_A f)(x) = 0$ whenever $\pi_A(v_A)(x) = 0$.

If $w = \{w_A\}_{A \in J}$ is any family from $\mathrm{M}(\mathcal{F})$ then Proposition 3.1 shows that the family $v = \{v_A\}_{A \in J}$ defined by $v_A = \alpha^w_A j^w_A w_A$ is normed and $\pi^v_A = \pi^w_A$ for each $A \in J$, where as in Chapters 3 and 4

$$\alpha^w_A(x) = \begin{cases} (\pi_A(w_A)(x))^{-1} & \text{if } 0 < \pi_A(w_A)(x) < \infty \, , \\ 0 & \text{otherwise} \, , \end{cases}$$

$$j^w_A(x) = \begin{cases} 1 & \text{if } 0 < w_A(x) < \infty \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

If $v = \{v_A\}_{A \in J}$ is a normed family then $\{\pi^v_A\}_{A \in J}$ being an $\mathbb{F}$-specification means

$$\pi_B(v_B f) = \pi_B(v_B \pi_A(v_A f))$$

must hold for all $f \in \mathrm{M}(\mathcal{F})$ and all A, $B \in J$ with $A \preceq B$.

Now let us call a family $w = \{w_A\}_{A \in J}$ from $\mathrm{M}(\mathcal{F})$ an $\mathbb{F}$-*specification family* if $\{\pi^w_A\}_{A \in J}$ is an $\mathbb{F}$-specification. This is the case if and only if the conditions in Proposition 4.3 are satisfied for all A, $B \in J$ with $A \preceq B$. In this chapter we present a method of constructing *normed* $\mathbb{F}$-*specification families*, i.e., normed families $v = \{v_A\}_{A \in J}$ which are also $\mathbb{F}$-specification families.

Let us start with a simple example which perhaps gives a hint of what lies behind the construction. In order to eliminate any technical problems assume for this example that the specification $\mathcal{V}$ is simple (i.e., $\pi_A(1) = 1$ for all $A \in J$.) Consider an $\mathbb{F}$-multiplicative family $w = \{w_A\}_{A \in J}$ with $0 < w_A(x) < \infty$ for all $x \in X$, $A \in J$. (In particular, the assumptions made here imply that $\pi_A(w_A)(x) > 0$ for all $x \in X$, $A \in J$.) For each $A \in J$ define a mapping $G_A : X \times X \to \mathbb{R}^+$ by

$$G_A(x, y) = w_A^{-1}(x) w_A(y) ;$$

thus $G_A \in \mathrm{M}(\mathcal{F} \times \mathcal{F})$ and

(1) $G_A(x, z) = G_A(x, y) G_A(y, z)$ for all x, y, $z \in X$.

Lemma 9.3 *Let* A, $B \in J$ *with* $A \preceq B$ *and define* $\sigma_{A,B} : X \to \mathbb{R}^+_\infty$ *by*

$$\sigma_{A,B}(x) = \pi_A(G_B(x, \diamond))(x)$$

for each $x \in X$. *Then* $\pi^w_A(f) = \pi_A(\sigma^{-1}_{A,B} f)$ *for all* $f \in \mathrm{M}(\mathcal{F})$.

Proof There exists a mapping $w_{B,A} \in \mathrm{M}(\mathcal{F}_A)$ with $w_B = w_{B,A} w_A$, and then also $0 < w_{B,A}(x) < \infty$ for all $x \in X$. Now

$$\sigma_{A,B}(x) = \pi_A(G_B(x, \diamond))(x) = \pi_A(w_B^{-1}(x) w_B)(x) = w_B^{-1}(x) \pi_A(w_B)(x)$$

for each $x \in X$, i.e., $\sigma_{A,B} = w_B^{-1} \pi_A(w_B)$, thus

$$\sigma_{A,B} = w_B^{-1} \pi_A(w_B) = w_B^{-1} \pi_A(w_{B,A} w_A) = w_B^{-1} w_{B,A} \pi_A(w_A) = w_A^{-1} \pi_A(w_A)$$

and therefore $\sigma^{-1}_{A,B} = (\pi_A(w_A))^{-1} w_A$. Hence for all $f \in \mathrm{M}(\mathcal{F})$

$$\pi^w_A(f) = (\pi_A(w_A))^{-1} \pi_A(w_A f) = \pi_A((\pi_A(w_A))^{-1} w_A f) = \pi_A(\sigma^{-1}_{A,B} f) . \quad \square$$

Looking at the expression for $\pi^w_A(f)$ in Lemma 9.3 we might hope to replace the mapping G_B by a single mapping G corresponding to taking B to be 'infinitely large'. In fact, this can be done: Choose an increasing order-generating sequence $\{A_n\}_{n \geq 1}$ from J, define $G \in \mathrm{M}_{\mathsf{F}}(\mathcal{F} \times \mathcal{F})$ by

$$G(x, y) = \begin{cases} \lim_{n \to \infty} G_{A_n}(x, y) & \text{if the limit exists and is finite ,} \\ 0 & \text{otherwise ,} \end{cases}$$

and for each $A \in J$ let $\sigma_A : X \to \mathbb{R}^+_\infty$ be given by

$$\sigma_A(x) = \pi_A(G(x, \diamond))(x)$$

for each $x \in X$. Then it will follow from the results below that $\pi^w_A(f) = \pi_A(\sigma_A^{-1} f)$ for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$.

Note that the mapping G defined above will in general no longer satisfy (1). However, it does satisfy the following two conditions:

(2) $G(x,y) > 0$ whenever $G(y,x) > 0$,

(3) $G(x,z) = G(x,y)G(y,z)$ whenever $G(x,y)G(y,z) > 0$.

The construction to be given below involves mappings satisfying (2) and (3). Thus now let $G \in \mathrm{M}_{\mathsf{F}}(\mathcal{F} \times \mathcal{F})$ be a mapping satisfying these two conditions. Put

$$D = \{x \in X \times X : G(x,y) > 0\} \quad \text{and} \quad D_* = \{x \in X : G(x,x) > 0\} ;$$

by Lemma M.9.6 (and the fact that $\delta^{-1}(\mathcal{F} \times \mathcal{F}) = \mathcal{F}$) it follows that $D \in \mathcal{F} \times \mathcal{F}$ and $D_* \in \mathcal{F}$ and, of course, if $x \in D_*$ then $G(x,x) = 1$. For each $x \in X$ let

$$D_x = \{y \in X : G(x,y) > 0\} ;$$

thus by Proposition M.2.7 $D_x \in \mathcal{F}$. By (2) and (3) the sets D_x, $x \in X$, have the following properties:

(4) $D_* \subset D \times D$ and $D_x \subset D_*$ for all $x \in X$.

(5) $D_x = \varnothing$ for all $x \in X \setminus D_*$.

(6) $x \in D_x$ for all $x \in D_*$, and if x, $y \in D_*$ then either $D_x \cap D_y = \varnothing$ or $D_x = D_y$.

For each $A \in J$ let

$$\Delta_A = \{x \in X : \pi_A(x, D_*) > 0 \text{ and } \pi_A(x, D_* \setminus D_x) = 0\}$$

and so by Lemmas 9.2 and M.9.6 $\Delta_A \in \mathcal{F}$. Moreover, (5) implies that $\Delta_A \subset D_*$. We also need the mapping $\sigma_A : X \to \mathbb{R}^+_\infty$ defined by letting

$$\sigma_A(x) = \pi_A(G(x,\diamond))(x)$$

for each $x \in X$, and so by Lemma 9.2 $\sigma_A \in \mathrm{M}(\mathcal{F})$. Note that if $x \in \Delta_A$ then $\sigma_A(x) > 0$, since $\sigma_A(x) = \pi_A(I_{D_x} G(x,\diamond))(x)$, $\pi_A(x, D_x) = \pi_A(x, D_*) > 0$ and $G(x,y) > 0$ for all $y \in D_x$. There is thus a mapping $v_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F})$ defined by

$$v_A = I_{\Delta_A} \sigma_A^{-1} .$$

Theorem 9.1 *The family $v = \{v_A\}_{A \in J}$ is a normed $\mathbb{F}$-specification family.*

Proof Later. □

Theorem 9.1 gives the basic construction. If $v = \{v_A\}_{A \in J}$ is the family occurring above then we write π_A^G instead of π_A^v; thus (since $v = \{v_A\}_{A \in J}$ is normed)

$$\pi_A^G f = \pi_A(I_{\Delta_A} \sigma_A^{-1} f)$$

for all $f \in \mathrm{M}(\mathcal{F})$ and by Theorem 9.1 $\mathcal{V}_G = \{\pi_A^G\}_{A \in J}$ is an $\mathbb{F}$-specification.

In order to apply the above construction mappings are needed which satisfy conditions (2) and (3), and the natural way to obtain them is as in the example considered at the beginning of the chapter. We will restrict our attention here to mappings in $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$, since the value ∞ introduces additional complications which we want to avoid. Thus let $w = \{w_A\}_{A\in J}$ be a family from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ and for each $A \in J$ define $G_A : X \times X \to \mathbb{R}^+$ by

$$G_A(x,y) = \begin{cases} w_A^{-1}(x)w_A(y) & \text{if } w_A(x) > 0 \;, \\ 0 & \text{otherwise} \;, \end{cases}$$

and so $G_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F} \times \mathcal{F})$. Now choose an increasing order-generating sequence $\{A_n\}_{n\geq 1}$ from J and define a mapping $G : X \times X \to \mathbb{R}^+$ by

$$G(x,y) = \begin{cases} \lim\limits_{n\to\infty} G_{A_n}(x,y) & \text{if the limit exists and is finite} \;, \\ 0 & \text{otherwise} \;; \end{cases}$$

thus by Lemmas M.9.4 and M.9.7 $G \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}\times\mathcal{F})$ and G clearly satisfies (2) and (3). For $A \in J$ put $C_A^w = \{x \in X : w_A(x) > 0\}$; then we here have

$$D_* = \{x \in X : G(x,x) > 0\} = \bigcup_{m\geq 1}\bigcap_{n\geq m} C_{A_n}^w \;.$$

If G is obtained in this way from w then G is said to be *derived* from w. (The definition of G also depends on the choice of the sequence $\{A_n\}_{n\geq 1}$, but this sequence is considered to be fixed throughout the chapter.)

This raises the question: If $w = \{w_A\}_{A\in J}$ was already an $\mathbb{F}$-specification family and G is derived from w then what is the relationship between $\mathcal{V}_w$ and $\mathcal{V}_G$? If w is $\mathbb{F}$-multiplicative then the following result shows that $\mathcal{V}_w$ and $\mathcal{V}_G$ are essentially the same specification. Note that if w is $\mathbb{F}$-multiplicative then $C_B^w \subset C_A^w$ whenever $A \preceq B$ and thus $D_* = \bigcap_{A\in J} C_A^w$.

Theorem 9.2 *Let $w = \{w_A\}_{A\in J}$ be an $\mathbb{F}$-multiplicative family from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ and let G be derived from w. Then*

$$\pi_A^G(f) = \pi_A(I_{D_*}\sigma_A^{-1}f) = \pi_A^w(I_{D_*}f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$. Moreover, $\mu(D_*) = 1$ for each $\mu \in \mathcal{G}(\mathcal{V}_w)$ and so in particular $\mathcal{G}(\mathcal{V}_G) = \mathcal{G}(\mathcal{V}_w)$.

Proof Later. □

For a general $\mathbb{F}$-specification family $w = \{w_A\}_{A\in J}$ there seems no way to obtain an explicit relationship between $\mathcal{V}_w$ and $\mathcal{V}_G$. However, there is an explicit relationship between their Gibbs states:

Theorem 9.3 *Let* $w = \{w_A\}_{A\in J}$ *be an* $\mathbb{F}$*-specification family from* $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ *and let* G *be derived from* w*. For each* $A \in J$ *let* $R^w_A = \{x \in X : \pi_A(w_A)(x) < \infty\}$ *and put* $R^w = \bigcup_{m\ge 1}\bigcap_{n\ge m} R^w_{A_n}$*. Then*

$$\mathcal{G}(\mathcal{V}_w) = \{\mu \in \mathcal{G}(\mathcal{V}_G) : \mu(R^w) = 1\} .$$

In particular, $\mathcal{G}(\mathcal{V}_w) = \mathcal{G}(\mathcal{V}_G)$ *if* $R^w = X$*, which will be the case if the family* w *is normed.*

Proof Later □

If $w = \{w_A\}_{A\in J}$ is any $\mathbb{F}$-specification family from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ then we can apply Theorem 9.3 to the normed family $u = \{u_A\}_{A\in J}$, where $u_A = \alpha^w_A w_A$ with α^w_A as at the beginning of the chapter. (Note that we don't need the factor j^w_A here, because we are assuming that $w_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F})$.) Then $\pi^u_A = \pi^w_A$ and u is a normed $\mathbb{F}$-specification family. Thus by Theorem 9.3

$$\mathcal{G}(\mathcal{V}_w) = \mathcal{G}(\mathcal{V}_u) = \mathcal{G}(\mathcal{V}_{G'}) ,$$

where G' is derived from u, and clearly $G' = I_{T\times T}G$, where

$$T = \bigcup_{m\ge 1}\bigcap_{n\ge m} \{x \in X : 0 < \pi_{A_n}(w_{A_n})(x) < \infty\} .$$

We now give the first application of Theorem 9.3. This will answer the question: Given $\mu \in \mathrm{P}(\mathcal{F})$, when is $\mu \in \mathcal{G}(\mathcal{V}_w)$ for some $\mathbb{F}$-specification $\mathcal{V}_w = \{\pi^w_A\}_{A\in J}$? There is a fairly obvious necessary condition which such Gibbs states must satisfy. For if $w = \{w_A\}_{A\in J}$ is any $\mathbb{F}$-specification family and $u_A = \alpha^w_A w_A$ then

$$(\mu\pi^w_A)(f) = (\mu\pi^u_A)(f) = \mu(\pi^u_A(f)) = \mu(\pi_A(u_A f)) = (\mu\pi_A)(u_A f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $\mu \in \mathrm{P}(\mathcal{F})$, and thus $\mu = \mu\pi^w_A$ if and only if $\mu \ll \mu\pi_A$ and $\mu = (\mu\pi_A)\cdot u_A$. In particular, this implies $\mathcal{G}(\mathcal{V}_w) \subset \mathrm{P}_{\mathcal{V}}(\mathcal{F})$, where

$$\mathrm{P}_{\mathcal{V}}(\mathcal{F}) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu \ll \mu\pi_A \text{ for all } A \in J\} .$$

Let $W(\mathcal{V})$ denote the set of all $\mathbb{F}$-specification families from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$; then the above implies that $\mathcal{G}(\mathcal{V}_w) \subset P_{\mathcal{V}}(\mathcal{F})$ for each $w \in W(\mathcal{V})$. Perhaps somewhat surprisingly, it turns out that

$$\bigcup_{w\in W(\mathcal{V})} \mathcal{G}(\mathcal{V}_w) = \mathrm{P}_{\mathcal{V}}(\mathcal{F}) ;$$

i.e., every element of $\mathrm{P}(\mathcal{F})$ which could be a Gibbs state corresponding to some specification of the type being considered here, is one. This follows from:

Theorem 9.4 *Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$ and for each $A \in J$ let $h_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F})$ be such that $\mu = (\mu\pi_A){\cdot}h_A$. Then $\mu \in \mathcal{G}(\mathcal{V}_G)$, where G is derived from the family $\{h_A\}_{A\in J}$.*

Proof Later. □

Finally, we look at a further way of obtaining $\mathbb{F}$-specification families. This construction may seem somewhat artificial, but we give an important example of it at the end of the chapter. If $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$ then let

$$\mathrm{M}^+_{\mathsf{F}}(\mathcal{E}) = \{f \in \mathrm{M}(\mathcal{E}) : 0 < f(x) < \infty \text{ for all } x \in X\} .$$

Theorem 9.5 *Let $u = \{u_A\}_{A\in J}$ be a family from $\mathrm{M}^+_{\mathsf{F}}(\mathcal{F})$ and suppose that for each A, $B \in J$ with $A \preceq B$ there exist $u_{B,A} \in \mathrm{M}^+_{\mathsf{F}}(\mathcal{F})$ and $u'_{B,A} \in \mathrm{M}^+_{\mathsf{F}}(\mathcal{F}_A)$ with $u_B = u'_{B,A}u_{B,A}$ such that the limit $w_A = \lim_n u_{A_n,A}$ exists and is in $\mathrm{M}^+_{\mathsf{F}}(\mathcal{F})$ for each A. Then $w = \{w_A\}_{A\in J}$ is an $\mathbb{F}$-specification family and $\mathcal{V}_w = \mathcal{V}_G$, where G is derived from u. Moreover, $\pi^G_A(f) = \pi_A(\sigma_A^{-1}f)$ for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$.*

Proof Later. □

With a certain amount of effort Theorem 9.5 can be generalised to apply to certain families from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ (i.e., allowing the value 0).

We now start to prove the above results.

Proof of Theorem 9.1: We need the following extension of Lemma 9.2:

Lemma 9.4 *Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$ and π be a strict $\mathcal{E}$-measurable finite kernel. Then $\Psi_\pi(g) \in \mathrm{M}(\mathcal{E}\times\mathcal{F})$ for all $f \in \mathrm{M}(\mathcal{F}\times\mathcal{F})$. Moreover, if $f \in \mathrm{M}(\mathcal{E}\times\mathcal{F})$ then $\pi((f\circ\delta)g)(x) = \pi(f(x,\diamond)g)(x)$ for all $x \in X$ and all $g \in \mathrm{M}(\mathcal{F})$.*

Proof If F_1, $F_2 \in \mathcal{F}$; then $\Psi_\pi(I_{F_1\times F_2})(x,y) = \pi(I_{F_1}(y)I_{F_2})(x) = I_{F_1}(y)\pi(I_{F_2})(x)$ and so $\Psi_\pi(I_{F_1\times F_2}) \in \mathrm{M}(\mathcal{E}\times\mathcal{F})$. Hence by Proposition M.14.5 $\Psi_\pi(g) \in \mathrm{M}(\mathcal{E}\times\mathcal{F})$ for all $f \in \mathrm{M}(\mathcal{F}\times\mathcal{F})$. Moreover, if $E \in \mathcal{E}$, $F \in \mathcal{F}$ and $f = I_{E\times F}$ then

$$\begin{aligned}\pi((f\circ\delta)g)(x) &= \pi((I_{E\times F}\circ\delta)g)(x) = \pi(I_{E\cap F}g)(x) = \pi(I_E I_F g)(x)\\ &= I_E(x)\pi(I_F g)(x) = \pi(I_E(x)I_F g)(x) = \pi(f(x,\diamond)g)(x)\end{aligned}$$

and hence by Proposition M.14.4 $\pi((f\circ\delta)g)(x) = \pi(f(x,\diamond)g)(x)$ for all $x \in X$, $g \in \mathrm{M}(\mathcal{F})$ for each $f \in \mathrm{M}(\mathcal{E}\times\mathcal{F})$. □

For each $A \in J$ let

$$\underline{\Delta}_A = \{(x,y) \in X\times X : \pi_A(x,D_*) > 0 \text{ and } \pi_A(x,D_*\setminus D_y) = 0\} ;$$

thus by Lemmas 9.4 and M.9.6 $\underline{\Delta}_A \in \mathcal{F}_A \times \mathcal{F}$, and by (5) $\underline{\Delta}_A \subset X \times D_*$. Define a mapping $\underline{\sigma}_A : X \times X \to \mathbb{R}^+_\infty$ by

$$\underline{\sigma}_A(x, y) = \pi_A(G(y, \diamond))(x)$$

for all $(x, y) \in X \times X$, and so by Lemma 9.4 $\underline{\sigma}_A \in \mathrm{M}(\mathcal{F}_A \times \mathcal{F})$. Exactly as above it follows that $\underline{\sigma}_A(x, y) > 0$ if $(x, y) \in \underline{\Delta}_A$, since $\underline{\sigma}_A(x, y) = \pi_A(I_{D_y} G(y, \diamond))(x)$, $\pi_A(x, D_y) = \pi_A(x, D_*) > 0$ and $G(y, z) > 0$ for all $z \in D_y$, and therefore there is a mapping $\underline{v}_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A \times \mathcal{F})$ defined by

$$\underline{v}_A = I_{\underline{\Delta}_A} \underline{\sigma}_A^{-1} \ .$$

Of course, $v_A = \underline{v}_A \circ \delta$.

Lemma 9.5 *Let $A \in J$ and $x \in X$.*

(1) If $(x, y) \in \underline{\Delta}_A$ for some $y \in D_*$ then $I_{\underline{\Delta}_A}(x, \diamond) = I_{D_y}$.

(2) If y, $z \in X$ with $G(y, z) > 0$ then $I_{\underline{\Delta}_A}(x, y) = I_{\underline{\Delta}_A}(x, z)$.

Proof (1) Let $(x, y) \in \underline{\Delta}_A$. If $z \in D_y$ then by (6) $D_z = D_y$ and thus $(x, z) \in \underline{\Delta}_A$. Conversely, if $(x, z) \in \underline{\Delta}_A$ then $\pi_A(x, D_* \setminus D_z) = \pi_A(x, D_* \setminus D_y) = 0$, thus

$$\pi_A(x, D_z \cap D_y) \geq \pi_A(x, D_*) - \pi_A(x, D_* \setminus D_z) - \pi_A(x, D_* \setminus D_y) = \pi_A(x, D_*) > 0 \ ,$$

therefore $D_z \cap D_y \neq \varnothing$ and hence by (6) $z \in D_z = D_y$. From this it follows that $\{z \in D_* : (x, z) \in \underline{\Delta}_A\} = D_y$ and thus that $I_{\underline{\Delta}_A}(x, \diamond) = I_{G_y}$.

(2) Let y, $z \in X$ with $G(y, z) > 0$; if one of (x, y) and (x, z) is in $\underline{\Delta}_A$ then by (1) so is the other (since y, $z \in D_y = D_z$) and thus $I_{\underline{\Delta}_A}(x, y) = 1 = I_{\underline{\Delta}_A}(x, z)$. But if neither of (x, y) and (x, z) is in $\underline{\Delta}_A$ then trivially $I_{\underline{\Delta}_A}(x, y) = 0 = I_{\underline{\Delta}_A}(x, z)$. □

Lemma 9.6 *If y, $z \in X$ with $G(y, z) > 0$ (and so in fact y, $z \in D_*$) then*

$$\underline{\sigma}_A^{-1}(x, z) = G(y, z) \underline{\sigma}_A^{-1}(x, y)$$

for all $A \in J$, $x \in X$.

Proof By (2) and (3) $G(y, w) = G(y, z)G(z, w)$ for all $w \in X$, thus

$$\underline{\sigma}_A(x, y) = \pi_A(G(y, \diamond))(x) = G(y, z)\pi_A(G(z, \diamond))(x) = G(y, z)\underline{\sigma}_A(x, z)$$

and hence $\underline{\sigma}_A^{-1}(x, z) = G(y, z)\underline{\sigma}_A^{-1}(x, y)$ for all $A \in J$, $x \in X$. □

Lemma 9.7 *Let $A \in J$, $x \in X$ and let $y \in D_*$ with $0 < \underline{\sigma}_A(x, y) < \infty$. Then $0 < \underline{\sigma}_A(x, z) < \infty$ for all $z \in D_y$ and $\pi_A(I_{D_y} \underline{\sigma}_A^{-1}(x, \diamond))(x) = 1$.*

Proof Lemma 9.6 implies that $0 < \underline{\sigma}_A(x,z) < \infty$ for all $z \in D_y$ and

$$\begin{aligned}\pi_A(I_{D_y}\underline{\sigma}_A^{-1}(x,\diamond))(x) &= \pi_A(I_{D_y}G(y,\diamond)\underline{\sigma}_A^{-1}(x,y))(x)\\ &= \underline{\sigma}_A^{-1}(x,y)\pi_A(G(y,\diamond))(x) = 1\ . \ \square\end{aligned}$$

Lemma 9.8 *Let $A \in J$ and $x \in X$. Then either $\underline{v}_A(x,y) = 0$ for all $y \in X$, in which case $\pi_A(v_A)(x) = 0$, or $\pi_A(v_A)(x) = 1$, and so in particular the family $\{v_A\}_{A\in J}$ is normed. Moreover, $\pi_A(v_A I_{D_*} f) = \pi_A(v_A f)$ for all $f \in \mathrm{M}(\mathcal{F})$.*

Proof If $\underline{v}_A(x,y) = 0$ for all $y \in X$ then by Lemma 9.4

$$\pi_A(v_A)(x) = \pi_A(\underline{v}_A \circ \delta)(x) = \pi_A(\underline{v}_A(x,\diamond))(x) = 0\ .$$

Thus suppose there exists $y \in X$ with $\underline{v}_A(x,y) > 0$, so $0 < \underline{\sigma}_A(x,y) < \infty$ and $(x,y) \in \underline{\Delta}_A$. Then by Lemmas 9.4, 9.5 (1) and 9.7

$$\begin{aligned}\pi_A(v_A)(x) &= \pi_A(\underline{v}_A \circ \delta)(x) = \pi_A(\underline{v}_A(x,\diamond))(x)\\ &= \pi_A(I_{\underline{\Delta}_A}(x,\diamond)\underline{\sigma}_A^{-1}(x,\diamond))(x) = \pi_A(I_{D_y}\underline{\sigma}_A^{-1}(x,\diamond))(x) = 1\ .\end{aligned}$$

Moreover, in the same way

$$\begin{aligned}\pi_A(v_A I_{X\setminus D_*})(x) &= \pi_A((\underline{v}_A \circ \delta)I_{X\setminus D_*})(x)\\ &= \pi_A(\underline{v}_A(x,\diamond)I_{X\setminus D_*})(x) = \pi_A(I_{X\setminus D_*}I_{\underline{\Delta}_A}(x,\diamond)\underline{\sigma}_A^{-1}(x,\diamond))(x)\\ &= \pi_A(I_{X\setminus D_*}I_{D_y}\underline{\sigma}_A^{-1}(x,\diamond))(x) = 0\ ,\end{aligned}$$

since by (4) $D_y \subset D_*$, and thus $\pi_A(v_A I_{X\setminus D_*}) = 0$, since $\pi_A(v_A I_{X\setminus D_*})(x) = 0$ holds trivially if $\pi_A(v_A)(x) = 0$. Therefore by Proposition M.10.6 (3)

$$\pi_A(v_A f) = \pi_A(v_A I_{D_*} f) + \pi_A(v_A I_{X\setminus D_*} f) = \pi_A(v_A I_{D_*} f) + 0 = \pi_A(v_A I_{D_*} f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. $\square$

Lemma 9.9 *If y, $z \in X$ with $G(y,z) > 0$ then for all $A \in J$, $x \in X$*

$$\underline{v}_A(x,z) = G(y,z)\underline{v}_A(x,y)\ .$$

Proof By Lemma 9.5 (2) and Lemma 9.6

$$\underline{v}_A(x,z) = I_{\underline{\Delta}_A}(x,z)\underline{\sigma}_A^{-1}(x,z) = G(y,z)I_{\underline{\Delta}_A}(x,y)\underline{\sigma}_A^{-1}(x,y) = G(y,z)\underline{v}_A(x,y)\ . \ \square$$

For each $A \in J$ put $E_A = \{x \in X : \pi_A(x, D_*) > 0\}$; thus $\pi_A(v_A f) = I_{E_A}\pi_A(v_A f)$ for all $f \in \mathrm{M}(\mathcal{F})$, since by Lemma 9.8 $\pi_A(v_A f) = \pi_A(v_A I_{D_*} f)$.

For the rest of the proof of Theorem 9.1 let A, $B \in J$ with $A \preceq B$.

Lemma 9.10 *If* $(x,y) \in \underline{\Delta}_B$ *then* $(z,y) \in \underline{\Delta}_A$ *for* $\pi_B(x,\cdot)$*-a.e.* $z \in E_A$*.*

Proof If $(x,y) \in \underline{\Delta}_B$ then $\pi_B(x, D_* \setminus D_y) = 0$ and therefore

$$0 = (\pi_B \pi_A)(I_{D_*} - I_{D_y})(x) = \pi_B(\pi_A(I_{D_*} - I_{D_y}))(x) \ .$$

Hence $\pi_A(z, D_* \setminus D_y) = 0$ for $\pi_B(x,\cdot)$-a.e. $z \in X$, and from this it follows that $(z,y) \in \underline{\Delta}_A$ for $\pi_B(x,\cdot)$-a.e. $z \in E_A$. □

Lemma 9.11 *If* $\underline{\sigma}_B(x,y) < \infty$ *then* $\underline{\sigma}_A(\cdot,y) < \infty$ $\pi_B(x,\cdot)$*-a.e.*

Proof This follows from the fact that

$$\begin{aligned}\underline{\sigma}_B(x,y) = \pi_B(G(y,\diamond))(x) &= (\pi_B\pi_A)(G(y,\diamond))(x) \\ &= \pi_B(\pi_A(G(y,\diamond)))(x) = \pi_B(\underline{\sigma}_A(\diamond,y))(x) \ . \ \square\end{aligned}$$

Lemma 9.12 $\pi_B(v_B f) = \pi_B(\pi_A(v_A)\pi_A(v_B f))$ *for all* $f \in \mathrm{M}(\mathcal{F})$*.*

Proof Let $x \in X$; if $\pi_B(v_B)(x) = 0$ then $\pi_B(v_B f)(x) = 0$ and, since $\pi_A(v_A) \le 1$, it follows that $\pi_B(\pi_A(v_A) v_A(v_B f))(x) \le \pi_B(\pi_A(v_B f))(x) = \pi_B(v_B f)(x) = 0$, and so in this case $\pi_B(v_B f)(x) = 0 = \pi_B(\pi_A(v_A)\pi_A(v_B f))(x)$. It can thus be assumed that $\pi_B(\underline{v}_B(x,\diamond))(x) = \pi_B(v_B)(x) = 1$. There then exists $y \in X$ with $\underline{v}_B(x,y) > 0$, and therefore $0 < \underline{\sigma}_B(x,y) < \infty$ and $(x,y) \in \underline{\Delta}_B$. By Lemmas 9.10 and 9.11 this implies that $\underline{v}_A(\cdot,y) > 0$ $\pi_B(x,\cdot)$-a.e. on E_A, and hence from Lemma 9.8 $\pi_A(v_A) = 1$ $\pi_B(x,\cdot)$-a.e. on E_A. Again using Lemma 9.8 it now follows that

$$\begin{aligned}\pi_B(v_B f)(x) &= \pi_B(\pi_A(v_B f))(x) \ge \pi_B(\pi_A(v_A)\pi_A(v_B f))(x) \\ &\ge \pi_B(I_{E_A}\pi_A(v_B f))(x) \ge \pi_B(I_{E_A}\pi_A(v_B I_{D_*} f))(x) \\ &= \pi_B(\pi_A(v_B I_{D_*} f))(x) = \pi_B(v_B I_{D_*} f)(x) = \pi_B(v_B f)(x)\end{aligned}$$

and hence $\pi_B(v_B f)(x) = \pi_B(\pi_A(v_A)\pi_A(v_B f))(x)$. □

Lemma 9.13 *Let* $z \in X$ *and for* $C = A,\ B$ *put* $S^z_C = \{y \in X : (y,z) \in \underline{\Delta}_C\}$*. Then for each* $y \in S^z_A \cap S^z_B$ *and for all* $z_1,\ z_2 \in X$

$$\underline{v}_A(y,z_1)\underline{v}_B(y,z_2) = \underline{v}_B(y,z_1)\underline{v}_A(y,z_2) \ .$$

Proof Let z_1, $z_2 \in X$ and suppose first that $G(z_1, z_2) > 0$. Then by Lemma 9.9 $\underline{v}_A(y, z_1) = G(z_2, z_1)\underline{v}_A(y, z_2)$ and $\underline{v}_B(y, z_1) = G(z_2, z_1)\underline{v}_B(y, z_2)$ and thus in this case $\underline{v}_A(y, z_1)\underline{v}_B(y, z_2) = \underline{v}_B(y, z_1)\underline{v}_A(y, z_2)$ holds for all $y \in X$.

Now let $y \in S^z_A \cap S^z_B$ and thus by Lemma 9.5 (1) $I_{\underline{\Delta}_A}(y, \diamond) = I_{\underline{\Delta}_B}(y, \diamond) = I_{D_z}$. Hence if $G(z_1, z_2) = 0$ then

$$\begin{aligned}\underline{v}_A(y, z_1)\underline{v}_B(y, z_2) &= I_{\underline{\Delta}_A}(y, z_1)\underline{\sigma}_A^{-1}(y, z_1)I_{\underline{\Delta}_B}(y, z_2)\underline{\sigma}_B^{-1}(y, z_2)\\ &= I_{D_z}(z_1)I_{D_z}(z_2)\underline{\sigma}_A^{-1}(y, z_1)\underline{\sigma}_B^{-1}(y, z_2) = 0\end{aligned}$$

and in the same way $\underline{v}_B(y, z_1)\underline{v}_A(y, z_2) = 0$. In particular, if $y \in S^z_A \cap S^z_B$ and $G(z_1, z_2) = 0$ then $\underline{v}_A(y, z_1)\underline{v}_B(y, z_2) = \underline{v}_B(y, z_1)\underline{v}_A(y, z_2)$. □

Lemma 9.14 *$\pi_B(v_B f) = \pi_B(v_B \pi_A(v_A f))$ for all $f \in \mathrm{M}(\mathcal{F})$.*

Proof Let $f \in \mathrm{M}(\mathcal{F})$ and $x \in X$; if $\underline{v}_B(x, y) = 0$ for all $y \in X$ then by Lemma 9.4

$$\begin{aligned}\pi_B(v_B f)(x) &= \pi_B((\underline{v}_B \circ \delta)f)(x) = \pi_B(\underline{v}_B(x, \diamond)f)(x)\\ &= 0 = \pi_B(\underline{v}_B(x, \diamond)\pi_A(v_A f))(x)\\ &= \pi_B((\underline{v}_B \circ \delta)\pi_A(v_A f))(x) = \pi_B(v_B \pi_A(v_A f))(x)\end{aligned}$$

and so we can assume that $\underline{v}_B(x, y) > 0$ for some $y \in X$. Thus by Lemma 9.5 (1) there exists $z \in X$ such that $I_{\underline{\Delta}}(x, \diamond) = I_{D_z}$. Let $y \in S^z_A \cap S^z_B$; then by Lemma 9.13 $\underline{v}_A(y, z_1)\underline{v}_B(y, z_2)f(z_2) = \underline{v}_B(y, z_1)\underline{v}_A(y, z_2)f(z_2)$ for all z_1, $z_2 \in X$. Hence, again making use of Lemma 9.4,

$$\begin{aligned}\underline{v}_A(y, z_1)\pi_A(v_B f)(y) &= \underline{v}_A(y, z_1)\pi_A((\underline{v}_B \circ)f)(y) = \underline{v}_A(y, z_1)\pi_A(\underline{v}_B(y, \diamond)f)(y)\\ &= \underline{v}_B(y, z_1)\pi_A(\underline{v}_A(y, \diamond)f)(y) = \underline{v}_B(y, z_1)\pi_A((\underline{v}_A \circ \delta)f)(y)\\ &= \underline{v}_B(y, z_1)\pi_A(v_A f)(y)\end{aligned}$$

for all $z_1 \in X$ and therefore by Lemma 9.4

$$\begin{aligned}(\pi_A(v_A)\pi_A(v_B f))(y) &= \pi_A(v_A)(y)\pi_A(v_B f)(y)\\ &= \pi_A(\underline{v}_A(y, \diamond))(y)\pi_A(v_B f)(y) = \pi_A(\underline{v}_B(y, \diamond))(y)\pi_A(v_A f)(y)\\ &= \pi_A(v_B)(y)\pi_A(v_A f)(y) = (\pi_A(v_B)\pi_A(v_A f))(y)\end{aligned}$$

for all $y \in S^z_A \cap S^z_B$. Now by Lemma 9.10 $\pi_B(x, E_A \setminus S^z_A) = 0$ and thus, since $I_{E_A}\pi_A(v_A) = \pi_A(v_A)$, $I_{E_A}\pi_A(v_A f) = \pi_A(v_A f)$ and $I_{\underline{\Delta}_B}(x, z) = 1$, it follows that

$$\begin{aligned}\pi_B(\pi_A(v_A)\pi_A(v_B f))(x) &= I_{\underline{\Delta}_B}(x, z)\pi_B(\pi_A(v_A)\pi_A(v_B f))(x)\\ &= \pi_B(I_{\underline{\Delta}_B}(\diamond, z)\pi_A(v_A)\pi_A(v_B f))(x) = \pi_B(I_{S^z_{\underline{\Delta}}}\pi_A(v_A)\pi_A(v_B f))(x)\\ &= \pi_B(I_{S^z_{\underline{\Delta}}} I_{E_A}\pi_A(v_A)\pi_A(v_B f))(x) = \pi_B(I_{S^z_{\underline{\Delta}}} I_{E_A}\pi_A(v_B)\pi_A(v_A f))(x)\\ &= \pi_B(I_{S^z_{\underline{\Delta}}}\pi_A(v_B)\pi_A(v_A f))(x) = \pi_B(I_{\underline{\Delta}_B}(\diamond, z)\pi_A(v_B)\pi_A(v_A f))(x)\\ &= I_{\underline{\Delta}_B}(x, z)\pi_B(\pi_A(v_B)\pi_A(v_A f))(x) = \pi_B(\pi_A(v_B)\pi_A(v_A f))(x)\ .\end{aligned}$$

Together with Lemma 9.12 this implies $\pi_B(v_B f)(x) = \pi_B(v_B \pi_A(v_A f))(x)$. $\square$

Theorem 9.1 now follows from Lemma 9.14. $\square$

We next prepare for the proof of Theorem 9.2. Let us say that $G \in \mathrm{M}_{\mathsf{F}}(\mathcal{F} \times \mathcal{F})$ satisfying (2) and (3) is *regular* if these exists $\hat{D} \in \bigcap_{A \in J}(\mathcal{F}_A \times \mathcal{F}_A)$ such that $D = (D_* \times D_*) \cap \hat{D}$.

Lemma 9.15 *If G is regular then $\pi_A(x, D_* \setminus D_x) = 0$ for all $A \in J$, $x \in D_*$. In particular, this implies that if G is regular then $\Delta_A = E_A \cap D_*$ for each $A \in J$, where again $E_A = \{x \in X : \pi_A(x, D_*) > 0\}$.*

Proof Let $A \in J$, $x \in D_*$; then $D_x = \{z \in X : (z, x) \in \hat{D}\} \cap D_*$ and thus, using the fact that $\hat{D}_x = \{z \in X : (z, x) \in \hat{D}\} \in \mathcal{F}_A$, it follows that

$$\pi_A(x, D_* \setminus D_x) = \pi_A(x, D_* \setminus \hat{D}_x) = I_{X \setminus \hat{D}_x}(x)\pi_A(x, D_*) = 0 ,$$

because $x \in \hat{D}_x$ for each $x \in D_*$. $\square$

Proposition 9.1 *If G is regular then*

$$\pi_A^G(f) = \pi_A(I_{D_*}\sigma_A^{-1} f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$.

Proof Using Lemma 9.15 and the fact that $E_A \in \mathcal{F}_A$

$$\pi_A(x, \Delta_A) = \pi_A(x, E_A \cap D_*) = I_{E_A}(x)\pi_A(x, D_*) = \pi_A(x, D_*)$$

and therefore $\pi_A(x, D_* \setminus \Delta_A) = 0$ for each $x \in X$. $\square$

Proposition 9.2 *Let $w = \{w_A\}_{A \in J}$ be an $\mathbb{F}$-multiplicative family from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ and let G be derived from w. Then G is regular.*

Proof For each A, $B \in J$ with $A \preceq B$ there exists a mapping $w_{B,A} \in \mathrm{M}(\mathcal{F}_A)$ with $w_B = w_{B,A} w_A$, and we can assume $w_{B,A}$ is in $\mathrm{M}_{\mathsf{F}}(\mathcal{F}_A)$, since if $w'_{B,A} = I_{K_{B,A}} w_{B,A}$ with $K_{B,A} = \{x \in X : w_{B,A}(x) < \infty\}$ then $w'_{B,A} \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A)$ and $w_B = w'_{B,A} w_A$. For A, $B \in J$ with $A \preceq B$ define a mapping $G_B^A : X \times X \to \mathbb{R}^+$ by

$$G_B^A(x, y) = \begin{cases} w_{B,A}^{-1}(x) w_{B,A}(y) & \text{if } w_{B,A}(x) > 0 , \\ 0 & \text{otherwise} , \end{cases}$$

so $G_B^A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A \times \mathcal{F}_A)$, and each $n \geq 1$ let

$$\hat{D}_n = \{(x,y) \in X \times X : \lim_{m\to\infty} G_{A_m}^{A_n}(x,y) \text{ exists and is positive and finite}\} .$$

Put $\hat{D} = \bigcup_{m\geq 1} \bigcap_{n\geq m} \hat{D}_n$. Clearly $\hat{D}_n \in \mathcal{F}_{A_n} \times \mathcal{F}_{A_n}$, and so $\hat{D} \in \bigcap_{A\in J}\big(\mathcal{F}_A \times \mathcal{F}_A\big)$. Now if $n \leq m$ and $x \in D_* = \bigcap_{A\in J} C_A^w$ then

$$w_{A_m}^{-1}(x) w_{A_m}(y) = G_{A_m}^{A_n}(x,y) w_{A_n}^{-1}(x) w_{A_n}(y) ;$$

thus $D = \{(x,y) \in X \times X : G(x,y) > 0\} = (D_* \times D_*) \cap \hat{D}_n$ for each $n \geq 1$. Hence this also holds with $\hat{D}_n$ replaced by $\hat{D}$, i.e., G is regular. □

Proof of Theorem 9.2: We have an $\mathbb{F}$-multiplicative family $w = \{w_A\}_{A\in J}$ from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ and G is derived from w. If A, $B \in J$ with $A \preceq B$ there, as above, there exists $w_{B,A} \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A)$ with $w_B = w_{B,A} w_A$. Also let $G_B^A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A \times \mathcal{F}_A)$ be as in the proof of Proposition 9.2 and for each $A \in J$ define $G^A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A \times \mathcal{F}_A)$ by

$$G^A(x,y) = \begin{cases} \lim_{n\to\infty} G_{A_n}^A(x,y) & \text{if the limit exists and is finite ,} \\ 0 & \text{otherwise .} \end{cases}$$

If $y \in D_*$ then $w_A(y) > 0$, $w_B(y) > 0$ and $w_{B,A}(y) > 0$, and thus

$$w_A(y) G_B(y,z) = w_A(y) w_B^{-1}(y) w_B(y) = w_A(z) G_B^A(y,z)$$

for all $z \in X$, and from this it follows that

$$w_A(y) G(y,z) = w_A(z) G^A(y,z)$$

for all $y \in D_*$, $z \in X$. Therefore, since by Proposition M.2.7 $G^A(y,\diamond) \in \mathrm{M}(\mathcal{F}_A)$,

$$w_A(y) \pi_A(G(y,\diamond))(x) = \pi_A(w_A G^A(y,\diamond))(x) = G^A(y,x) \pi_A(w_A)(x)$$

for all $x \in X$, $y \in D_*$. But $w_{B,A}(y) > 0$ and hence $G^A(y,y) = 1$ for all $y \in D_*$, and this implies $w_A(y)\sigma_A(y) = \pi_A(w_A)(y)$ for all $y \in D_*$, i.e., $I_{D_*}\sigma_A^{-1} = I_{D_*}\alpha_A^w w_A$. It thus follows from Propositions 9.1 and 9.2 that

$$\pi_A^G(f) = \pi_A(I_{D_*}\sigma_A^{-1} f) = \pi_A(I_{D_*}\alpha_A^w w_A f) = \pi_A^w(I_{D_*} f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. Now let $\mu \in \mathcal{G}(\mathcal{V}_w)$; if $A \in J$ then

$$\begin{aligned} \mu(C_A^w) &= (\mu\pi_A^w)(C_A^w) \\ &= \mu(\pi_A(\alpha_A^w w_A I_{C_A^w})) = \mu(\pi_A(\alpha_A^w w_A)) = (\mu\pi_A^w)(X) = \mu(X) = 1 \end{aligned}$$

and hence $\mu(D_*) = 1$, since $D_* = \bigcap_{n\geq 1} C_{A_n}^w$.

It remains to show that $\mathcal{G}(\mathcal{V}_G) = \mathcal{G}(\mathcal{V}_w)$. Let $\mu \in \mathcal{G}(\mathcal{V}_w)$; then

$$(\mu\pi_A^G)(f) = \mu(\pi_A^G(f)) = \mu(\pi_A^w(I_{D_*}f)) = (\mu\pi_A^w)(I_{D_*}f) = \mu(I_{D_*}f) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$ (since $\mu(D_*) = 1$) and this implies that $\mu \in \mathcal{G}(\mathcal{V}_G)$. Conversely, let $\mu \in \mathcal{G}(\mathcal{V}_G)$; if $A \in J$ then

$$(\mu\pi_A^w)(I_{D_*}f) = \mu(\pi_A^w(I_{D_*}f)) = \mu(\pi_A^G(f)) = (\mu\pi_A^G)(f) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, and in particular (with $f = 1$) $(\mu\pi_A^w)(I_{D_*}) = 1$. Therefore $(\mu\pi_A^w)(f) = (\mu\pi_A^w)(I_{D_*}f) = \mu(f)$, which shows that $\mu \in \mathcal{G}(\mathcal{V}_w)$.

This completes the proof of Theorem 9.2. □

Theorem 9.3 will be a corollary of Proposition 9.3 below. In what follows let $w = \{w_A\}_{A\in J}$ be an $\mathbb{F}$-specification family from $\mathrm{M}_{\mathsf{F}}(\mathcal{F})$ and let G be derived from w. Again let $\mathrm{P}_{\mathcal{V}}(\mathcal{F}) = \{\mu \in \mathrm{P}(\mathcal{F}) : \mu \ll \mu\pi_A \text{ for all } A \in J\}$. Recall that $R^w = \bigcup_{m\geq 1}\bigcap_{n\geq m} R^w_{A_n}$, where $R^w_A = \{x \in X : \pi_A(w_A)(x) < \infty\}$.

Proposition 9.3 *Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$; then*

$$(\mu\pi_A^G)(I_{R^w}f) = (\mu\pi_A^w)(I_{R^w}I_{D_*}f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$.

Proof As in Chapter 4 define $i_A^w \in \mathrm{M}(\mathcal{F}_A)$ by

$$i_A^w(x) = \begin{cases} 1 & \text{if } 0 < \pi_A(w_A)(x) < \infty \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

Lemma 9.16 *Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$ and $A \in J$; then $(1 - i_A^w)I_{D_*}I_{R^w} = 0$ $\mu\pi_A$-a.e.*

Proof Let A, $B \in J$ with $A \preceq B$. By Proposition 4.3 $i_A^w\pi_B(w_B) = i_A^w\pi_B(i_A^w w_B)$ and hence $i_B^w\pi_B((1 - i_A^w)w_B) = 0$. But if $i_B^w(x) = 0$ (i.e., $\pi_B(w_B)(x) = 0$) then also $\pi_B((1-i_A^w)w_B)(x) = 0$, and this implies that $I_{R_B^w}\pi_B((1-i_A^w)w_B) = 0$, where $R_B^w = \{x \in X : \pi_B(w_B)(x) < \infty\}$. Thus, since $R_B^w \in \mathcal{F}_B$,

$$0 = I_{R_B^w}\pi_B((1 - i_A^w)w_B) = \pi_B((1 - i_A^w)w_B I_{R_B^w}) = (\pi_B\pi_A)((1 - i_A^w)w_B I_{R_B^w}) \ .$$

and therefore $(\mu\pi_B)(\pi_A((1-i_A^w)w_B I_{R_B^w})) = \mu((\pi_B\pi_A)((1-i_A^w)w_B I_{R_B^w})) = 0$ for all $\mu \in \mathrm{P}(\mathcal{F})$. If $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$ then $\mu \ll \mu\pi_B$ and in this case

$$(\mu\pi_A)((1 - i_A^w)w_B I_{R_B^w}) = \mu(\pi_A((1 - i_A^w)w_B I_{R_B^w})) = 0 \ ,$$

i.e., $(1 - i_A^w)w_B I_{R_B^w} = 0$ $\mu\pi_A$-a.e. In particular, this holds with $B = A_n$ for all large enough n, which implies that $(1 - i_A^w)I_{D_*}I_{R^w} = 0$ $\mu\pi_A$-a.e. □

For $A \in J$, $y \in X$ let $\pi_A^2(y,\cdot) = \pi_A(y,\cdot) \times \pi_A(y,\cdot)$ be the product measure on $\mathcal{F} \times \mathcal{F}$. Then $\pi_A^2 : X \times (\mathcal{F} \times \mathcal{F}) \to \mathbb{R}_\infty^+$ is a finite pre-kernel and therefore by Proposition M.14.5 π_A^2 is a kernel, since $\pi_A^2(\cdot, F_1 \times F_2) = \pi_A(\cdot, F_1)\pi_A(\cdot, F_2)$ is an element of $\mathrm{M}(\mathcal{F} \times \mathcal{F})$ for all F_1, $F_2 \in \mathcal{F}$. We thus consider π_A^2 as a mapping from $\mathrm{M}(\mathcal{F} \times \mathcal{F})$ to $\mathrm{M}(\mathcal{F})$. For A, $B \in J$ with $A \preceq B$ put

$$U_B^A = \{(z_1, z_2) \in X \times X : w_A(z_1)w_B(z_2) = w_B(z_1)w_A(z_2)\} .$$

Lemma 9.17 *Let A, $B \in J$ with $A \preceq B$. Then $(\mu\pi_A^2)((X \times X) \setminus U_B^A) = 0$ for all $\mu \in \mathrm{P}_\mathcal{V}(\mathcal{F})$.*

Proof By Proposition 4.3

$$\begin{aligned}\pi_B(\pi_A^2(w_A(\diamond_1)f(\diamond_1)w_B(\diamond_2)g(\diamond_2))) &= \pi_B(\pi_A(w_Af)\pi_A(w_Bg))\\ &= \pi_B(\pi_A(w_Bf)\pi_A(w_Ag)) = \pi_B(\pi_A^2(w_B(\diamond_1)f(\diamond_1)w_A(\diamond_2)g(\diamond_2)))\end{aligned}$$

for all f, $g \in \mathrm{M}(\mathcal{F})$, where $\diamond_1$ and $\diamond_2$ indicate which arguments are involved. In particular, for all F_1, $F_2 \in \mathcal{F}$

$$\pi_B(\pi_A^2(w_A(\diamond_1)w_B(\diamond_2)I_{F_1}(\diamond_1)I_{F_2}(\diamond_2))) = \pi_B(\pi_A^2(w_B(\diamond_1)w_A(\diamond_2)I_{F_1}(\diamond_1)I_{F_2}(\diamond_2)))$$

and therefore by Proposition M.14.4

$$\pi_B(\pi_A^2(w_A(\diamond_1)w_B(\diamond_2)I_C(\diamond_1, \diamond_2))) = \pi_B(\pi_A^2(w_B(\diamond_1)w_A(\diamond_2)I_C(\diamond_1, \diamond_2)))$$

for all $C \in \mathcal{F} \times \mathcal{F}$. This in turn implies that if $\mu \in \mathrm{P}(\mathcal{F})$ then

$$(\mu\pi_B\pi_A^2)(w_A(\diamond_1)w_B(\diamond_2)I_C(\diamond_1, \diamond_2)) = (\mu\pi_B\pi_A^2)(w_B(\diamond_1)w_A(\diamond_2)I_C(\diamond_1, \diamond_2))$$

for all $C \in \mathcal{F} \times \mathcal{F}$ and so by Lemma M.10.4

$$(\mu\pi_B)(\pi_A^2((X \times X) \setminus U_B^A)) = (\mu\pi_B\pi_A^2)((X \times X) \setminus U_B^A) = 0 .$$

Therefore if $\mu \in \mathrm{P}_\mathcal{V}(\mathcal{F})$ (and so $\mu \ll \mu\pi_B$) then also

$$(\mu\pi_A^2)((X \times X) \setminus U_B^A) = \mu(\pi_A^2((X \times X) \setminus U_B^A)) = 0 . \quad \square$$

Lemma 9.18 *Let $A \in J$; then $I_{D_*}\sigma_A w_A = I_{D_*}\pi_A(w_A)$ and $I_{\Delta_A} w_A = I_{D_*} w_A$ hold $\mu\pi_A$-a.e. for all $\mu \in \mathrm{P}_\mathcal{V}(\mathcal{F})$.*

Proof Fix $A \in J$ and define $\underline{\tau}_A$, $\underline{\tau}_A^* : X \times X \to \mathbb{R}^+$ by $\underline{\tau}_A(x, y) = \pi_A(x, D_y)$ and $\underline{\tau}_A^*(x, y) = I_{D_*}(y)\pi_A(x, D_*)$. Thus

$$\underline{\Delta}_A(x, y) = \{(x, y) \in X \times X : \underline{\tau}_A^*(x, y) > 0 \text{ and } \underline{\tau}_A(x, y) = \underline{\tau}_A^*(x, y)\} .$$

Note that $\underline{\tau}_A(x,y) = \pi_A(I_D(y,\diamond))(x)$ and $\underline{\tau}^*_A(x,y) = \pi_A(I_{D_*\times D_*}(y,\diamond))(x)$. Also let τ_A, $\tau^*_A : X \to \mathbb{R}^+$ be given by $\tau_A = \underline{\tau}_A \circ \delta$ and $\tau^*_A = \underline{\tau}^*_A \circ \delta$, and therefore

$$\Delta_A = E_A \cap D_* \cap \{x \in X : \tau_A(x) = \tau^*_A(x)\} ,$$

where $E_A = \{x \in X : \pi_A(x, D_*) > 0\}$. Let $B \in J$ with $A \preceq B$; if $(z_1, z_2) \in U^A_B$ and $w_B(z_2) > 0$ then $w_A(z_1) = G_B(z_2, z_1)w_A(z_2)$. Choose $p \geq 1$ with $A \preceq A_p$. If $(z_1, z_2) \in U = \bigcap_{n\geq p} U^A_{A_n}$ and $z_2 \in D_*$ then $w_{A_n}(z_2) > 0$ for all $n \geq q$ for some $q \geq p$ and hence $w_A(z_1) = G_{A_n}(z_2, z_1)w_A(z_2)$ for all $n \geq q$. This implies that

$$I_{D_*}(z_2)w_A(z_1) = I_{D_*}(z_2)G(z_2, z_1)w_A(z_2)$$

for all $(z_1, z_2) \in U$. Moreover, it then follows that

$$I_D(z_1, z_2)w_A(z_2) = I_{D_*\times D_*}(z_1, z_2)w_A(z_2)$$

for all $(z_1, z_2) \in U$, since $(z_2, z_1) \in U$, $D \subset D_* \times D_*$ and $I_D(z_1, z_2) = I_D(z_2, z_1)$.

Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$; then by Lemma 9.17 $(\mu\pi^2_A)((X \times X) \setminus U^A_{A_n}) = 0$ for all $n \geq p$ and therefore $(\mu\pi^2_A)((X \times X) \setminus U) = 0$.

Let $f \in \mathrm{M}(\mathcal{F})$ and $y \in X$; then by Lemma 9.4 and Theorem M.15.1

$$\begin{aligned}
&\pi^2_A(I_{D_*}(\diamond_2)G(\diamond_2, \diamond_1)w_A(\diamond_2)f(\diamond_2))(y) \\
&\qquad = \pi_A(I_{D_*}(\diamond_2)\pi_A(G(\diamond_2, \diamond_1))(y)w_A(\diamond_2)f(\diamond_2))(y) \\
&\qquad = \pi_A(I_{D_*}\underline{\sigma}_A(y, \diamond)w_A f)(y) = \pi_A(I_{D_*}\sigma_A w_A f)(y) ,
\end{aligned}$$

where again $\diamond_1$ and $\diamond_2$ indicate which arguments are involved, and

$$\pi^2_A(I_{D_*}(\diamond_2)w_A(\diamond_1)f(\diamond_2))(y) = \pi_A(I_{D_*}f)(y)\pi_A(w_A)(y) = \pi_A(I_{D_*}\pi_A(w_A)f)(y) .$$

Now since $I_{D_*}(z_2)w_A(z_1) = I_{D_*}(z_2)G(z_2, z_1)w_A(z_2)$ for all $(z_1, z_2) \in U$ it follows from Proposition M.10.6 (3) that for all $f \in \mathrm{M}(\mathcal{F})$

$$\begin{aligned}
(\mu\pi_A)(I_{D_*}\sigma_A w_A f) &= \mu(\pi_A(I_{D_*}\sigma_A w_A f)) \\
&= \mu(\pi^2_A(I_{D_*}(\diamond_2)G(\diamond_2, \diamond_1)w_A(\diamond_2)f(\diamond_2))) \\
&= (\mu\pi^2_A)(I_{D_*}(\diamond_2)G(\diamond_2, \diamond_1)w_A(\diamond_2)f(\diamond_2)) \\
&= (\mu\pi^2_A)(I_U(\diamond_1, \diamond_2)I_{D_*}(\diamond_2)G(\diamond_2, \diamond_1)w_A(\diamond_2)f(\diamond_2)) \\
&= (\mu\pi^2_A)(I_U(\diamond_1, \diamond_2)I_{D_*}(\diamond_2)w_A(\diamond_1)f(\diamond_2)) \\
&= (\mu\pi^2_A)(I_{D_*}(\diamond_2)w_A(\diamond_1)f(\diamond_2)) = \mu(\pi^2_A(I_{D_*}(\diamond_2)w_A(\diamond_1)f(\diamond_2))) \\
&= \mu(\pi_A(I_{D_*}\pi_A(w_A)f)) = (\mu\pi_A)(I_{D_*}\pi_A(w_A)f) .
\end{aligned}$$

Therefore by Lemma M.10.4 (2) $I_{D_*}\sigma_A w_A = I_{D_*}\pi_A(w_A)$ $\mu\pi_A$-a.e.

Again let $f \in \mathrm{M}(\mathcal{F})$ and $y \in X$; then by Lemma 9.4 and Theorem M.15.1

$$\begin{aligned}\pi_A^2(I_D(\diamond_2,\diamond_1)w_A(\diamond_2)f(\diamond_2))(y)&\\ &= \pi_A(\pi_A(I_D(\diamond_2,\diamond_1))(y)w_A(\diamond_2)f(\diamond_2))(y)\\ &= \pi_A(\underline{\tau}_A(y,\diamond)w_Af)(y) = \pi_A(\tau_A w_A f)(y)\ ,\end{aligned}$$

and in the same way $\pi_A^2(I_{D_*\times D_*}(\diamond_2,\diamond_1)w_A(\diamond_2)f(\diamond_2))(y) = \pi_A(\tau_A^* w_A f)(y)$. Then, since $I_D(z_1,z_2)w_A(z_2) = I_{D_*\times D_*}(z_1,z_2)w_A(z_2)$ for all $(z_1,z_2) \in U$ it follows that

$$(\mu\pi_A)(\tau_A w_A f) = (\mu\pi_A)(\tau_A^* w_A f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. Thus if $V_A = \{x \in X : \tau_A(x) = \tau_A^*(x)\}$ then by Lemma M.10.4 and Proposition M.10.6 (3) $(\mu\pi_A)(I_{X\setminus V_A} w_A) = 0$. But $\Delta_A = D_* \cap E_A \cap V_A$ and therefore by Proposition M.10.6 (3)

$$\begin{aligned}(\mu\pi_A)(I_{\Delta_A} w_A f) &= (\mu\pi_A)(I_{D_*}I_{E_A}I_{V_A}w_Af) = (\mu\pi_A)(I_{D_*}I_{E_A}w_Af)\\ &= \mu(\pi_A(I_{D_*}I_{E_A}w_Af)) = \mu(I_{E_A}\pi_A(I_{D_*}w_Af)) = \mu(\pi_A(I_{D_*}w_Af))\\ &= (\mu\pi_A)(I_{D_*}w_Af)\end{aligned}$$

for all $f \in \mathrm{M}(\mathcal{F})$. Thus by Lemma M.10.4 (2) $I_{\Delta_A} w_A = I_{D_*} w_A$ $\mu\pi_A$-a.e. □

Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$ and $A \in J$; then by Lemmas 9.16 and 9.18

$$I_{R^w} I_{\Delta_A} \sigma_A^{-1} = I_{R^w} I_{D_*} \alpha_A^w w_A \quad \mu\pi_A\text{-a.e.}$$

and therefore by Lemma M.10.4 (2)

$$\begin{aligned}(\mu\pi_A^G)(I_{R^w} f) &= \mu(\pi_A^G(I_{R^w} f))\\ &= \mu(\pi_A(I_{R^w} I_{\Delta_A}\sigma_A^{-1} f)) = \mu(\pi_A(I_{R^w} I_{D_*}\alpha_A^w w_A f))\\ &= \mu(\pi_A^w(I_{R^w} I_{D_*} f)) = (\mu\pi_A^w)(I_{R^w} I_{D_*} f)\end{aligned}$$

for all $f \in \mathrm{M}(\mathcal{F})$. This completes the proof of Proposition 9.3. □

Proof of Theorem 9.3: Let $\mu \in \mathcal{G}(\mathcal{V}_w)$; then for each $A \in J$

$$\begin{aligned}\mu(C_A^w \cap R_A^w) &= (\mu\pi_A^w)(C_A^w \cap R_A^w)\\ &= \mu(\pi_A(\alpha_A^w I_{R_A^w} w_A I_{C_A^w})) = \mu(\pi_A(\alpha_A^w w_A)) = (\mu\pi_A^w)(X) = \mu(X) = 1\end{aligned}$$

and hence $\mu(R^w \cap D_*) = 1$, since $R^w = \bigcap_{n\ge1} R_{A_n}^w$ and $D_* = \bigcap_{n\ge1} C_{A_n}^w$. This shows in particular that $\mu(R^w) = 1$ for all $\mu \in \mathcal{G}(\mathcal{V}_w)$. Let $A \in J$; then, since $\mathcal{G}(\mathcal{V}_w) \subset \mathrm{P}_{\mathcal{V}}(\mathcal{F})$, it follows from Proposition 9.3 that

$$(\mu\pi_A^G)(I_{R^w} f) = (\mu\pi_A^w)(I_{R^w} I_{D_*} f) = \mu(I_{R^w} I_{D_*} f) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, and in particular (with $f = 1$) $(\mu\pi_A^G)(I_{R^w}) = 1$. Therefore $(\mu\pi_A^G)(f) = (\mu\pi_A^G)(I_{R^w} f) = \mu(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$, which shows that $\mu \in \mathcal{G}(\mathcal{V}_G)$. Hence $\mathcal{G}(\mathcal{V}_w) \subset \{\mu \in \mathcal{G}(\mathcal{V}_G) : \mu(R^w) = 1\}$.

Conversely, let $\mu \in \mathcal{G}(\mathcal{V}_G)$ with $\mu(R^w) = 1$. Then, since $\mathcal{G}(\mathcal{V}_G) \subset \mathrm{P}_{\mathcal{V}}(\mathcal{F})$, it follows from Proposition 9.3 that

$$(\mu\pi_A^w)(I_{R^w} I_{D_*} f) = (\mu\pi_A^G)(I_{R^w} f) = \mu(I_{R^w} f) = \mu(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$, and in particular (with $f = 1$) $(\mu\pi_A^w)(I_{R^w} I_{D_*}) = 1$. Therefore $(\mu\pi_A^w)(f) = (\mu\pi_A^w)(I_{R^w} I_{D_*} f) = \mu(f)$ for all $f \in \mathrm{M}(\mathcal{F})$, $A \in J$, which shows that $\mu \in \mathcal{G}(\mathcal{V}_w)$. Hence $\{\mu \in \mathcal{G}(\mathcal{V}_G) : \mu(R^w) = 1\} \subset \mathcal{G}(\mathcal{V}_w)$.

This completes the proof of Theorem 9.3. □

Proof of Theorem 9.4: Let $\mu \in \mathrm{P}_{\mathcal{V}}(\mathcal{F})$ and for each $A \in J$ let $h_A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F})$ be such that $\mu = (\mu\pi_A){\cdot}h_A$. Moreover, let G be derived from the family $\{h_A\}_{A\in J}$.

For each $A \in J$ let $\pi_A^2 : \mathrm{M}(\mathcal{F} \times \mathcal{F}) \to \mathrm{M}(\mathcal{F})$ be the kernel defined in the proof of Proposition 9.3 with $\pi_A^2(y, \cdot) = \pi_A(y, \cdot) \times \pi_A(y, \cdot)$ for each $y \in X$. Moreover, for A, $B \in J$ with $A \preceq B$ put

$$U_B^A = \{(z_1, z_2) \in X \times X : h_A(z_1)h_B(z_2) = h_B(z_1)h_A(z_2)\} \ .$$

Lemma 9.19 *Let A, $B \in J$ with $A \preceq B$. Then $(\mu\pi_A^2)((X \times X) \setminus U_B^A) = 0$.*

Proof For each $f \in \mathrm{M}(\mathcal{F})$

$$\begin{aligned}\mu(\pi_A(h_A f)) = (\mu\pi_A)(h_A f) &= \mu(f) = (\mu\pi_B)(h_B f)\\ &= (\mu(\pi_B\pi_A))(h_B f) = (\mu\pi_B)(\pi_A(h_B f)) \ .\end{aligned}$$

Therefore if f, $g \in \mathrm{M}(\mathcal{F})$ then

$$\begin{aligned}\mu(\pi_A(h_A f)\pi_A(h_A g)) &= \mu(\pi_A(h_A \pi_A(h_A f) g))\\ &= (\mu\pi_B)(\pi_A(h_B \pi_A(h_A f) g)) = (\mu\pi_B)(\pi_A(h_A f)\pi_A(h_B g)) \ .\end{aligned}$$

Thus, since $\mu(\pi_A(h_A f)\pi_A(h_A g))$ is symmetric in f and g, it follows that

$$\begin{aligned}(\mu\pi_B)(\pi_A^2(h_A(\diamond_1)h_B(\diamond_2)f(\diamond_1)g(\diamond_2))) &= (\mu\pi_B)(\pi_A(h_A f)\pi_A(h_B g))\\ = (\mu\pi_B)(\pi_A(h_A g)\pi_A(h_B f)) &= (\mu\pi_B)(\pi_A^2(h_B(\diamond_1)h_A(\diamond_2)f(\diamond_1)g(\diamond_2)))\end{aligned}$$

for all f, $g \in \mathrm{M}(\mathcal{F})$ and exactly as in the proof of Lemma 9.17 this implies that $(\mu\pi_B)(\pi_A^2((X \times X) \setminus U_B^A)) = 0$. But $\mu \ll \mu\pi_B$ and therefore

$$(\mu\pi_A^2)((X \times X) \setminus U_B^A) = \mu(\pi_A^2((X \times X) \setminus U_B^A)) = 0 \ . \ \square$$

Let $A \in J$; from Lemma 9.19 it follows exactly as in the proof of Lemma 9.18 that $I_{D_*}\sigma_A h_A = I_{D_*}\pi_A(h_A)$ and $I_{\Delta_A} h_A = I_{D_*} h_A$ hold $\mu\pi_A$-a.e. Moreover, by Lemma 3.3 (3)

$$\begin{aligned}(\mu\pi_A)(g\pi_A(h_A)) = \mu(\pi_A(g\pi_A(h_A))) &= \mu(\pi_A(h_A\pi_A(g))) \\ &= (\mu\pi_A)(h_A\pi_A(g)) = \mu(\pi_A(g)) = (\mu\pi_A)(g)\end{aligned}$$

for all $g \in \mathrm{M}(\mathcal{F})$, and hence $\pi_A(h_A) = 1$ $\mu\pi_A$-a.e. Thus $v_A = I_{\Delta_A}\sigma_A^{-1} = I_{D_*}h_A$ $\mu\pi_A$-a.e., which implies that

$$(\mu\pi_A^G)(f) = \mu(\pi_A^G(f)) = \mu(\pi_A(v_A f)) = (\mu\pi_A)(v_A f) = (\mu\pi_A)(h_A I_{D_*} f) = \mu(I_{D_*} f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. But $\mu(I_{E_B} f) = (\mu\pi_B)(h_B I_{E_B} f) = (\mu\pi_B)(h_B f) = \mu(f)$ for all $B \in J$ and therefore $\mu(I_{D_*} f) = \mu(f)$. This shows that $\mu\pi_A^G = \mu$ for each $A \in J$, i.e., $\mu \in \mathcal{G}(\mathcal{V}_G)$, which completes the proof of Theorem 9.4. □

Proof of Theorem 9.5: We have a family $u = \{u_A\}_{A\in J}$ from $\mathrm{M}_{\mathsf{F}}^+(\mathcal{F})$ and for each $A,\, B \in J$ with $A \preceq B$ there exist $u_{B,A} \in \mathrm{M}_{\mathsf{F}}^+(\mathcal{F})$ and $u'_{B,A} \in \mathrm{M}_{\mathsf{F}}^+(\mathcal{F}_A)$ with $u_B = u'_{B,A}u_{B,A}$ such that the limit $w_A = \lim_n u_{A_n,A}$ exists and is in $\mathrm{M}_{\mathsf{F}}^+(\mathcal{F})$ for each A. For each $A \in J$ define $G_A \in \mathrm{M}_{\mathsf{F}}^+(\mathcal{F}\times\mathcal{F})$ by $G_A(x,y) = u_A(y)/u_A(x)$ and if $A \preceq B$ then let $G_B^A \in \mathrm{M}_{\mathsf{F}}^+(\mathcal{F}_A \times \mathcal{F}_A)$ be given by $G_B^A(x,y) = u'_{B,A}(y)/u'_{B,A}(x)$ for all $x,\, y \in X$. Thus by the definition of G

$$G(x,y) = \begin{cases} \lim\limits_{n\to\infty} G_{A_n}(x,y) & \text{if the limit exists and is finite ,} \\ \quad 0 & \text{otherwise ,} \end{cases}$$

and in particular $D_* = X$. Define $G^A \in \mathrm{M}_{\mathsf{F}}(\mathcal{F}_A \times \mathcal{F}_A)$ by

$$G^A(x,y) = \begin{cases} \lim\limits_{n\to\infty} G_{A_n}^A(x,y) & \text{if the limit exists and is finite ,} \\ \quad 0 & \text{otherwise .} \end{cases}$$

If $A \preceq B$ then $u_{B,A}(y)G_B(y,z) = u_{B,A}(z)G_B^A(y,z)$ and hence for each $A \in J$

$$w_A(y)G(y,z) = w_A(z)G^A(y,z)$$

for all $y,\, z \in X$. Thus, since by Proposition M.2.7 $G^A(y,\diamond) \in \mathrm{M}(\mathcal{F}_A)$,

$$w_A(y)\pi_A(G(y,\diamond))(x) = \pi_A(w_A G^A(y,\diamond))(x) = G^A(y,x)\pi_A(w_A)(x)$$

for all $x,\, y \in X$. But $G^A(y,y) = 1$ for all $y \in X$, and thus $w_A\sigma_A = \pi_A(w_A)$. Now the proof of Proposition 9.2 shows that G is regular and so by Proposition 9.1 (and the fact that $D_* = X$)

$$\pi_A^G(f) = \pi_A(\sigma_A^{-1} f) = \pi_A(\alpha_A^w w_A f) = \pi_A^w(f)$$

for all $f \in \mathrm{M}(\mathcal{F})$. Therefore $\pi^w_A = \pi^G_A$ for all $A \in J$, and in particular w is an $\mathbb{F}$-specification family. □

To end the chapter we give an application of Theorem 9.5 and this involves the situation considered the end of Chapter 1 (A lattice model).

Let $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ be a potential. Then for each $\Lambda \in \mathcal{N}_0$ the corresponding conditional energy $E^\Phi_\Lambda : X \to \mathbb{R}^\diamond$ was defined by

$$E^\Phi_\Lambda = \sideset{}{^*}\sum_{\Delta \cap \Lambda \neq \varnothing} \Phi_\Delta \circ \mathrm{p}_\Delta \ .$$

By Proposition 1.7 the family $\{E^\Phi_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is $\mathbb{F}$-additive and therefore the family $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ with $e_\Lambda = \exp(-E^\Phi_\Lambda)$ is $\mathbb{F}$-multiplicative.

Now for each $\Lambda \in \mathcal{N}_0$ define the *total energy* $U^\Phi_\Lambda : X \to \mathbb{R}^\diamond$ by

$$U^\Phi_\Lambda = \sum_{\Delta \subset \Lambda} \Phi_\Delta \circ \mathrm{p}_\Delta \ .$$

Also if $\Lambda \subset \Delta$ then define $U^\Phi_{\Delta,\Lambda}$, $\hat{U}^\Phi_{\Delta,\Lambda} : X \to \mathbb{R}^\diamond$ by

$$U^\Phi_{\Delta,\Lambda} = \sum_{\Gamma \subset \Delta,\, \Gamma \cap \Lambda \neq \varnothing} \Phi_\Gamma \circ \mathrm{p}_\Delta \ , \quad \hat{U}^\Phi_{\Delta,\Lambda} = \sum_{\Gamma \subset \Delta,\, \Gamma \cap \Lambda = \varnothing} \Phi_\Gamma \circ \mathrm{p}_\Delta \ ,$$

and note that each of these sums is over only finitely many terms. Then clearly U^Φ_Λ, $U^\Phi_{\Delta,\Lambda} \in \mathrm{M}^\diamond(\mathcal{F})$, $\hat{U}^\Phi_{\Delta,\Lambda} \in \mathrm{M}^\diamond(\mathcal{F}_\Lambda)$ and $U^\Phi_\Delta = \hat{U}^\Phi_{\Delta,\Lambda} + U^\Phi_{\Delta,\Lambda}$.

Put $u_\Lambda = \exp(-U^\Phi_\Lambda)$, $u_{\Delta,\Lambda} = \exp(-U^\Phi_{\Delta,\Lambda})$ and $u'_{\Delta,\Lambda} = \exp(-\hat{U}^\Phi_{\Delta,\Lambda})$; then u_Λ and $u_{\Delta,\Lambda}$ are both in $\mathrm{M}(\mathcal{F})$, $u'_{\Delta,\Lambda} \in \mathrm{M}(\mathcal{F}_\Lambda)$ and $u_\Delta = u'_{\Delta,\Lambda} u_{\Delta,\Lambda}$. If the mappings Φ_A, $\Lambda \in \mathcal{N}_0$, only take values in $\mathbb{R}$ (i.e., they omit the values $-\infty$ and ∞) then the same is true of the mappings U^Φ_Λ, $U^\Phi_{\Delta,\Lambda}$ and $\hat{U}^\Phi_{\Delta,\Lambda}$ (since they are defined by finite sums) and in this case we have u_Λ, $u_{\Delta,\Lambda} \in \mathrm{M}^+_{\mathsf{F}}(\mathcal{F})$ and $u'_{\Delta,\Lambda} \in \mathrm{M}^+_{\mathsf{F}}(\mathcal{F}_\Lambda)$.

Let $\{\Lambda_n\}_{n \ge 1}$ be an increasing sequence from $\mathcal{N}_0$ with $\bigcup_{n \ge 1} \Lambda_n = S$. If Φ satisfies the condition in Proposition 1.8 then it is easy to see that

$$E^\Phi_\Lambda = \sum_{\Delta \cap \Lambda \neq \varnothing} \Phi_\Delta \circ \mathrm{p}_\Delta = \lim_{n \to \infty} \sum_{\Delta \subset \Lambda_n, \Delta \cap \Lambda \neq \varnothing} \Phi_\Delta \circ \mathrm{p}_\Delta = \lim_{n \to \infty} \hat{U}^\Phi_{\Lambda_n,\Lambda}$$

and hence $e_\Lambda = \lim_n u'_{\Lambda_n,\Lambda}$; moreover, $e_\Lambda \in \mathrm{M}^+_{\mathsf{F}}(\mathcal{F})$. Thus the requirements in Theorem 9.5 are met. Hence $e = \{e_\Lambda\}_{\Lambda \in \mathcal{N}_0}$ is an $\mathbb{F}$-specification family (which we knew already since it is $\mathbb{F}$-multiplicative) and $\mathcal{V}_e = \mathcal{V}_G$, where G is derived from the family $\{u_\Lambda\}_{\Lambda \in \mathcal{N}_0}$. This means that the specification defined in terms of the potential Φ can be defined not only in terms of the conditional energy but also directly in terms of the total energy.

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Index