Practice questions

Question 14.4. Explain why the category of finite-dimensional vector spaces is finitely complete and finitely cocomplete, but not complete and not cocomplete.

Question 14.5. By means of a counterexample, show that not all colimits in module categories are left exact. Hint: your counterexample must involve an $\mathcal{I}$-diagram $\mathcal{I} \to R-\mathsf{Mod}$ where $\mathcal{I}$ is not filtered (why?).

Question 14.6. Let $\mathcal{I}$ and $\mathcal{C}$ be skeletally small categories and let $\mathcal{D}$ be a category.

(1) Show that any functor $F\colon \mathcal{I} \times \mathcal{C} \to \mathcal{D}$ defines a functor $\hat{F}\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ such that if $I \in \mathrm{ob}(\mathcal{I})$ then $\hat{F}(I)$ sends $C \in \mathrm{ob}(\mathcal{C})$ to $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $F(\mathrm{id}_I, g)$.

(2) Define, for any natural transformation between functors $F, G\colon \mathcal{I} \times \mathcal{C} \to \mathcal{D}$, a natural transformation between the functors $\hat{F}, \hat{G}\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ from (1).

Now let $S\colon \mathcal{I} \to [\mathcal{C}, \mathcal{D}]$ be a functor.

(3) Show that for any $i \in \mathcal{I}(I, J)$ and any $g \in \mathcal{C}(A, B)$ we have $S(J)(g) \circ S(i)_A = S(i)_B \circ S(I)(g)$.

(4) Define a functor $\mathcal{I} \times \mathcal{C} \to \mathcal{D}$ where $\mathrm{ob}(\mathcal{I} \times \mathcal{C}) \ni (I, A) \mapsto (S(I))(A)$.

(5) Finally, prove that the categories $\mathrm{Fun}(\mathcal{I} \times \mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ are isomorphic.

Question 14.7. Recall: the category **Grp** of groups and the category **Ab** of abelian groups.

(1) For the functors $P\colon \mathbf{Grp} \times \mathbf{Grp} \to \mathbf{Grp}$ and $Q\colon \mathbf{Grp} \to \mathbf{Grp} \times \mathbf{Grp}$ from Question 1.5 that satisfy $P(G', G) = G' \times G$ and $Q(H) = (H, H)$, prove that Q is right adjoint to P.

Recall from Question 1.5 the commutator $[G, G]$ of any group G

(2) Let $G \in \mathbf{Grp}$ and $A \in \mathbf{Ab}$. Prove any homomorphism $G \to A$ is the composition of the quotient $G \to G/[G, G]$ and a homomorphism $G/[G, G] \to A$.

(3) Show that, in the notation from Question 1.5, V is left adjoint to U.

Question 14.8. Let $\mathcal{I}$ be a small category, let $\mathcal{C}$ be a skeletally small category and let $\mathcal{D}$ be a category such that the colimit of any $\mathcal{I}$-diagram in $\mathcal{D}$ exists. Recall Question 14.6.

(1) Prove that there is a functor $\operatorname*{colim}_{\mathcal{I}\to\mathcal{D}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \to \mathcal{D}$ taking any $\mathcal{I}$-diagram to its colimit.

(2) Prove that colimits of $\mathcal{I}$-diagrams in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ exists, and describe $\operatorname*{colim}_{\mathcal{I}\to\mathrm{Fun}(\mathcal{C},\mathcal{D})}$ in terms of $\operatorname*{colim}_{\mathcal{I}\to\mathcal{D}}$.

Question 14.9. Let $\mathcal{C}$ be a (preadditive, namely a) $\mathbb{Z}$-linear category. Consider the collection $\mathrm{ob}(\mathcal{C}_{\boxplus})$ of symbols of the form 0 or of the form $X_1 \boxplus \cdots \boxplus X_n$ with $X_i \in \mathrm{ob}(\mathcal{C})$.

(1) Define a category $\mathcal{C}_{\boxplus}$ whose objects are $\mathrm{ob}(\mathcal{C}_{\boxplus})$ by defining morphisms, composition and identity morphisms. Do so in such a way that $\mathcal{C}_{\boxplus}$ is additive with zero object 0 and direct sums given by $\boxplus$.

(2) Construct a(n additive, namely a) $\mathbb{Z}$-linear functor $i_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}_{\boxplus}$ such that, for any additive category $\mathcal{D}$ and any additive functor $F\colon \mathcal{C} \to \mathcal{D}$ there is an additive functor $F_{\boxplus}\colon \mathcal{C}_{\boxplus} \to \mathcal{D}$ such that $F_{\boxplus} \circ i_{\mathcal{C}} = F$.

Now assume $\mathcal{C} = K\mathcal{P}$ where $\mathcal{P}$ is the category defined by a partially ordered set.

(3) Explain how a morphism between non-zero objects in $\mathcal{C}_{\boxplus}$ uniquely corresponds to a matrix with entries in K, in such a way that composition corresponds to matrix multiplication.

(4) Let $0 \neq X \in \mathrm{ob}(\mathcal{C}_{\boxplus})$ and $f \in \mathrm{End}_{\mathcal{C}_{\boxplus}}(X)$. By considering Smith-normal form, prove that if K is a field and if $f^2 = f$ then $f = gh$ where hg is an identity morphism.

Thus, this exercise shows that the category $K\mathcal{P}_{\boxplus}$ has *split idempotents* when K is a field and $\mathcal{P}$ is a poset.

Question 14.10. Let $\mathcal{C}$ be a category with all coequalisers and all coproducts. Prove that $\mathcal{C}$ is *cocomplete*.

Question 14.11. Let $R = \mathrm{Hom}_{\mathbf{Set}}(\mathbb{R}, K)$, the set of functions of the form $\mathbb{R} \to K$, where $\mathbb{R}$ denotes the (uncountable) set of real numbers, and K is some field. For any $f \in R$ let $\mathrm{Supp}(f) = \{x \in \mathbb{R} \colon f(x) \neq 0\}$. Let I be the subset of R consisting of functions $f \colon \mathbb{R} \to K$ with *countable support*, meaning that either $\mathrm{Supp}(f)$ is finite (and possibly empty) or that there is a sequence $(x_1, x_2, x_3, \dots) \in \mathbb{R}^{\mathbb{N}}$ where $\mathrm{Supp}(f) = \{x_i \colon 0 < i \in \mathbb{Z}\}$.

(1) Prove that R is a ring where addition, multiplication and the identities are defined point-wise.
(2) Prove that I is an ideal in the ring R that is not finitely generated as an R-module.

Let S be a set, M_s be an R-module for each $s \in S$, $M = \bigoplus_{s \in S} M_s$, and $\pi_s \colon M \to M_s$ be the projection for each $s \in S$. Let $\varphi \colon I \to M$ be an R-module homomorphism.

(3) Prove that if $S = \mathbb{N}$ and if $f_s \in I$ for each $s \in S$ then $\bigcup_{s \in S} \mathrm{Supp}(f_s)$ is countable and hence define an element $\delta \in I$ such that $f_s \delta = f_s$ for each $s \in S$.
(4) Prove, by contradiction, that there can only be finitely many $s \in S$ such that $\pi_s \varphi = 0$.
(5) Prove that $\mathrm{Hom}_R(I, M) \cong \bigoplus_{s \in S} \mathrm{Hom}_R(I, M_s)$.

Thus this question provides an example of a *compact* module that is not finitely generated.

Question 14.12. Prove the Snake Lemma for an abelian category (that is not a module category, so without diagram chasing). Hint: use Question 14.8(3) and its dual.

Question 14.13. Let $\mathcal{C}$ be an abelian category and let $\chi_i \colon X_i \hookrightarrow A$ $(i \in I)$ be a set of monomorphisms in $\mathcal{C}$. Assume the coproduct $\coprod_{i \in I} X_i$, equipped with morphisms $\iota_i \colon X_i \to \coprod_{i \in I} X_i$, exists (as a colimit) in $\mathcal{C}$.

(1) Prove that there is a morphism $\chi \colon \coprod_{i \in I} X_i \to A$ such that $\chi \iota_i$ for each i.

Denote the cokernel of χ by $\gamma \colon A \to \mathrm{cok}(\chi)$. Denote the *image* of χ, so the kernel of γ, by $\delta \colon \mathrm{im}(\chi) \hookrightarrow A$.

(2) Prove that there is a monomorphism $\kappa_i \colon X_i \hookrightarrow \mathrm{im}(\chi)$ such that $\chi_i = \delta \kappa_i$ for each i.
(3) Suppose there exist monomorphisms $\mu \colon K \to A$ and $\lambda_i \colon X_i \hookrightarrow K$ such that $\chi_i = \mu \lambda_i$ for each i.
(4) Prove that there is a monomorphism $\varepsilon \colon \mathrm{im}(\chi) \to K$ such that $\mu \varepsilon = \delta$.

Thus, this exercise shows how to define the sum $\mathrm{im}(\chi) \coloneqq \sum_i X_i$ of A of subobjects.

Question 14.14. Let R be a commutative ring and let A and B be R-algebras.

(1) Prove that the R-R-bimodule $A \otimes_R B$ has the structure of an R-algebra.
(2) Let 1_A and 1_B be the multiplicative identities of the rings A and B, respectively. Prove that if M is an A-B-bimodule such that $(r \cdot 1_A)m = m(r \cdot 1_B)$ then M has the structure of a left $A \otimes_R B^{\mathsf{op}}$-module.

Question 14.15. Let Q, R and S be rings, A an S-R-bimodule, B an R-Q-bimodule, and C an S-Q-bimodule. Prove that there is an S-S-bimodule isomorphism $\mathrm{Hom}_Q(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_Q(B, C))$.

Question 14.16. Let $\mathcal{A}$ be an abelian category. Let $L, M \in \mathrm{ob}(\mathcal{A})$. Let $i_L \colon L \to L \oplus M$ and $i_M \colon M \to L \oplus M$ be the canonical morphisms defining the coproduct. Assume that L and M are subobjects of an object N, meaning there are monomorphisms $j_L \colon L \to N$ and $j_M \colon M \to N$.

(1) Prove that there is a morphism $\varphi_{L,M} \colon L \oplus M \to N$ such that $\varphi_{L,M} i_L = j_L$ and $\varphi_{L,M} i_M = j_M$.
(2) Let $L \cap M = \ker(\varphi_{L,M})$. Prove that if L is a subobject of M then $L \cap M \cong L$.
(3) Defining $L + M$ as in Question 14.13, prove that if $L \cap M = 0$ then $L + M \cong L \oplus M$.
(4) Define the meaning of an *essential extension* in $\mathcal{A}$, so that it recovers the definition for modules when $\mathcal{A} = R-\mathsf{Mod}$ for a ring R.
(5) Let $X, Y, Z \in \mathrm{ob}(\mathcal{A})$. Suppose that Z is an essential extension of Y. Prove that if Y is an essential extension of X then Z is an essential extension of X. Prove that if $Z = X \oplus Y$ then $X = 0$.

Question 14.17. Let $\mathcal{A}$ be an abelian category (if you prefer, consider instead $\mathcal{A} = R-\mathsf{Mod}$ where R is a ring). Let $\sigma \colon P \to L$ be a projective cover and a let $\tau \colon Q \to L$ be an epimorphism.

(1) Prove that there exists a homomorphism $\rho\colon P \to Q$ with $\tau\rho = \sigma$ and $\ker(\rho)$ small in P.
(2) Prove that if $\ker(\tau)$ is small in Q then ρ is a projective cover.
(3) Prove that if τ is a projective cover then ρ is an isomorphism.

Question 14.18. Let $\mathcal{A}$ be an abelian category and let $\mathcal{C} = C(\mathcal{A})$. Let I be a set and let M_i be an object in $\mathcal{C}$ for each $i \in I$. Assume that for each $n \in \mathbb{Z}$ the coproduct $\bigoplus_{i\in I} M_i^n$ exists in $\mathcal{A}$.

(1) Prove that there is an object in $\mathcal{C}$ of the form $\cdots \to \bigoplus_{i\in I} M_i^n \to \bigoplus_{i\in I} M_i^{n+1} \to \cdots$ that defines the coproduct in $\mathcal{C}$ of the objects M_i. Conjecture how the product is defined.
(2) Prove that, for each $n \in \mathbb{Z}$, there is an isomorphism $H^n(\bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} H^n(M_i)$ in $\mathcal{A}$.

For part (3) below assume that $\mathcal{A} = R-\mathsf{Mod}$, the category of left modules over a ring R. Let $X \in \mathrm{ob}(\mathcal{A})$.

(3) For each $N \in \mathrm{ob}(\mathcal{C})$ define $N \otimes_R X$ is a similar way to the definition that was done for chain complexes. Thus prove that $H^n((\bigoplus_{i\in I} M_i) \otimes_R X) \cong \bigoplus_{i\in I} H^n(M_i \otimes_R X)$ in $\mathcal{A}$ for each $n \in \mathbb{Z}$.

Question 14.19. Continue the notation from Question 10.1.

(1) Prove that f is null-homotopic if and only if there exists $e \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{cone}(\mathrm{id}_X), Y)$ such that $e\iota_{\mathrm{id}_X} = f$.

Define and denote the *mapping cyclinder* of a morphism a by $\mathrm{cyl}(a) := \mathrm{cone}(\Sigma^{-1}\pi_a)$. Consider the morphisms $\Delta\colon X \to X \oplus X$ and $\rho\colon X \oplus X \to X$ defined by $\Delta := (\mathrm{id}_X, \mathrm{id}_X)^{\boldsymbol{t}}$ and $\rho := (\mathrm{id}_X, 0)$.

(2) Prove that there is a monomorphism $i\colon X \oplus X \to \mathrm{cyl}(\mathrm{id}_X)$ in $\mathcal{C}$ with cokernel $\mathrm{cok}(i) = \Sigma X$.
(3) Prove that there is a morphism $j\colon \mathrm{cyl}(\mathrm{id}_X) \to \mathrm{cone}(\mathrm{id}_X)$ such that $ji\Delta = \iota_{\mathrm{id}_X}$.
(4) Prove that f is homotopic to $f' \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ if and only if there is a morphism $k\colon \mathrm{cyl}(\mathrm{id}_X) \to Y$ such that $ki = (f, -f')$. Hint: find a morphism $k\colon \mathrm{cone}(\mathrm{id}_X) \to \mathrm{cyl}(\mathrm{id}_X)$ such that $jk = \mathrm{id}_{\mathrm{cone}(\mathrm{id}_X)}$.

Question 14.20. Let $f\colon A \to B$ be a homomorphism of rings. Recall the functor Res_A from Question 6.3(1). Prove that if P is a projective B-module then $\mathrm{proj.\,dim}_A(\mathrm{Res}_A(P)) \leq \mathrm{proj.\,dim}_A(\mathrm{Res}_A(B))$.

Question 14.21. Let $n \in \mathbb{Z}$ such that $n \neq -1, 0, 1$. Without calculating the image of $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ on a $\mathbb{Z}$-module, explain why there must exist some $\mathbb{Z}$-module X such that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, X) \neq 0$.

Question 14.22. Let R be a ring.

(1) Let $n > 0$ and X and Y be R-modules. Prove that if $f \in \mathrm{Hom}_R(R^n, X)$ and $g \in \mathrm{Hom}_R(Y, X)$ and $\mathrm{im}(f) \subseteq \mathrm{im}(g)$ then there exists $h \in \mathrm{Hom}(R^n, Y)$ such that $f = gh$.

From now on assume that $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules.

(2) Prove that if L and N are finitely generated then so is M.
(3) Prove that if N is finitely presented and if M is finitely generated then L is finitely generated. Hint: apply the snake lemma to the exact sequence and a given presentation of N. Then use (2).
(4) Prove that if M is finitely presented and if L is finitely generated then N is finitely presented.
(5) Prove that if L and N are finitely presented modules then so is M. Hint: use the snake lemma.

Question 14.23. Let R be a ring. Let M, P and Q be R-modules and let $f \in \mathrm{Hom}_R(P, M)$ and $g \in \mathrm{Hom}_R(Q, M)$ be surjective. Let $S = \{(p, q) \in P \oplus Q \mid f(p) = g(q)\}$.

(1) Prove that the maps $h \in \mathrm{Hom}_R(S, P)$ and $k \in \mathrm{Hom}_R(Q, M)$ defined by $h(p, q) = p$ and $k(p, q) = q$ are surjective, and prove that $\ker(h) \cong \ker(g)$ and $\ker(k) \cong \ker(f)$.
(2) Prove *Schanuel's lemma*: that if P and Q are projective then $P \oplus \ker(g) \cong Q \oplus \ker(f)$.

From now on assume I is an ideal in R.

(3) Use Schanuel's lemma to prove that if R/I is finitely presented then I is finitely generated.
(4) Construct an example of a finitely generated module that is not finitely presented.