2. Questions on §1.1 and §1.2.

Question 2.1. Let **Domain** be the category of integral domains (commutative rings for which 0 is the only zero-divisor) with injective ring homomorphisms. Let **Field** be the category of fields with field homomorphisms. Any integral domain R has a *field of fractions* $\mathrm{Frac}(R) = \{\frac{r}{s} \mid 0 \neq s \in R \ni r\}$.

1. Show that objects and morphisms in **Field** define objects and morphisms in **Domain**, and hence explain why there is a full and faithful functor $I\colon \mathbf{Field} \to \mathbf{Domain}$. Explain why I is not dense.
2. Prove that any morphism $R \to R'$ in **Domain** defines a morphism $\mathrm{Frac}(R) \to \mathrm{Frac}(R')$ in **Field**.
3. Show that there is a dense functor $\mathrm{Frac}\colon \mathbf{Domain} \to \mathbf{Field}$ taking any domain to its field of fractions.
4. Prove that Frac is left adjoint to I.

Question 2.2. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, let $E, F\colon \mathcal{C} \to \mathcal{D}$ and $G, H\colon \mathcal{D} \to \mathcal{E}$ be functors, and let $\alpha\colon E \to F$ and $\beta\colon G \to H$ be natural transformations.

1. Show that the morphisms $G(\alpha_X)$ with $X \in \mathrm{ob}(\mathcal{C})$ define a natural transformation $G(\alpha_-)\colon GE \to GF$.
2. Show that the morphisms $\beta_{F(X)}$ with $X \in \mathrm{ob}(\mathcal{C})$ define a natural transformation $\beta_{F(-)}\colon GF \to HF$.
3. Show that there are functors $G(?_-)\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$ and $?_{F(-)}\colon \mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$.

Question 2.3. Let $\mathcal{C}$ be a category with one object $*$ and assume that every morphism in $\mathcal{C}$ is an isomorphism.

1. Prove that the set $G \coloneqq \mathrm{Hom}_{\mathcal{C}}(*,*)$ is a group, and that any functor from $\mathcal{C}$ to the category **Set** of sets defines a G-set (meaning a set equipped with a G-action).
2. Describe the G-set corresponding to the functor $\mathrm{Hom}_{\mathcal{C}}(*,-)$. What does an equivariant function between G-sets correspond to? Translate Yoneda's lemma into the language of groups and actions.

Question 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. Consider the functors $\mathrm{Hom}_{\mathcal{D}}(F(-),?)$ and $\mathrm{Hom}_{\mathcal{C}}(-,G(?))$, both of the form $\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathbf{Set}$. Let $\alpha\colon \mathrm{Hom}_{\mathcal{D}}(F(-),?) \to \mathrm{Hom}_{\mathcal{C}}(-,G(?))$ be a natural transformations (and hence a morphism in $\mathrm{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{D}, \mathbf{Set})$).

1. Show that, by setting $\eta_X \coloneqq \alpha_{X,F(X)}(\mathrm{Id}_{F(X)})$ for any $X \in \mathrm{ob}(\mathcal{C})$, one defines a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \to GF$ of functors of the form $\mathcal{C} \to \mathcal{C}$.
2. Prove that $\alpha_{X,Y}(p) = G(p)\eta_X$ for all $X \in \mathrm{ob}(\mathcal{C})$, $Y \in \mathrm{ob}(\mathcal{D})$ and $p \in \mathrm{Hom}_{\mathcal{D}}(F(X),Y)$.
3. Let $\beta\colon \mathrm{Hom}_{\mathcal{C}}(-,G(?)) \to \mathrm{Hom}_{\mathcal{D}}(F(-),?)$ be a natural transformation. Define a natural transformation $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$ with $\beta_{X,Y}(q) = \varepsilon_Y F(q)$ for all $X \in \mathrm{ob}(\mathcal{C})$, $Y \in \mathrm{ob}(\mathcal{D})$ and $q \in \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$.

Now assume also that β is an inverse of α in the category $\mathrm{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{D}, \mathbf{Set})$, so F is left adjoint to G.

4. In terms of Question 2.2, prove that $(G\varepsilon_-) \circ \eta_{G(-)} = \mathrm{Id}_G$ and $\varepsilon_{F(-)} \circ (F\eta_-) = \mathrm{Id}_F$.
5. Prove that any functor that is right adjoint to F is naturally isomorphic to G. Similarly, prove that any functor that is left adjoint to G is naturally isomorphic to F.

Hint: begin by assuming that $G'\colon \mathcal{D} \to \mathcal{C}$ is another functor which is right adjoint to F. This means (F, G) and (F, G') are adjoint pairs. Hence $\alpha'\colon \mathrm{Hom}_{\mathcal{D}}(F(-),?) \to \mathrm{Hom}_{\mathcal{C}}(-,G'(?))$ and $\beta'\colon \mathrm{Hom}_{\mathcal{C}}(-,G'(?)) \to \mathrm{Hom}_{\mathcal{D}}(F(-),?)$ are natural isomorphisms such that $\alpha' = (\beta')^{-1}$ giving a diagram

$$\mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \underset{\alpha_{X,Y}}{\overset{\beta_{X,Y}}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \underset{\beta'_{X,Y}}{\overset{\alpha'_{X,Y}}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}}(X,G'(Y))$$

for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. It follows that, for any such Y, specifying $X = G(Y)$ and considering Id_X on the left-hand side produces a morphism $G(Y) \to G'(Y)$ on the right-hand side.