1. Questions on §1.1.

Question 1.1. Show that in any category, every isomorphism is both a monomorphism and an epimorphism.

Question 1.2. Let $\mathcal{C}$ be a category and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category. Let $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, a morphism in $\mathcal{C}$, with corresponding morphism $f^{\mathsf{op}} \in \operatorname{Hom}_{\mathcal{C}^{\mathsf{op}}}(Y, X)$.

(1) Prove that f^{op} is a monomorphism in $\mathcal{C}^{\mathsf{op}}$ if and only if f is an epimorphism in $\mathcal{C}$.

Let $f = gh$ for $h \in \operatorname{Hom}_{\mathcal{C}}(X, Z)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(Z, Y)$.

(2) Prove that if f is an epimorphism (in $\mathcal{C}$) then g is an epimorphism. Using $\mathcal{C}^{\mathsf{op}}$ and what you have already shown, prove that if f is a monomorphism then h is a monomorphism.

Question 1.3. Let $\mathcal{C}$ be the category defined by a partially ordered set. Prove that every morphism is both a monomorphism and an epimorphism, and that the only isomorphisms are the identity morphisms.

Question 1.4. Let R be a unital ring. Let $\operatorname{ob}(\mathcal{C}) \coloneqq \{*\}$, a singleton $*$, and let $\operatorname{Hom}_{\mathcal{C}}(*, *) = R$.

(1) Using multiplication in R, make $\mathcal{C}$ into a category. Show that any left R-module defines a functor of the form $\mathcal{C} \to \mathbf{Ab}$. Show that any R-module homomorphism defines a natural transformation between functors of the form $\mathcal{C} \to \mathbf{Ab}$.

(2) Let $r \in R$. Show that r is a monomorphism (respectively, epimorphism) in $\mathcal{C}$, if and only if, r is not a left (respectively, right) zero-divisor. Give an example of $\mathcal{C}$ and $0 \neq r \in R$ which is not a monomorphism and not an epimorphism.

Question 1.5. Let $\mathbf{Grp}$ be the category of groups and consider the product $\mathbf{Grp} \times \mathbf{Grp}$ of $\mathbf{Grp}$ with itself.

(1) Show that there is a functor $P\colon \mathbf{Grp} \times \mathbf{Grp} \to \mathbf{Grp}$ sending any pair (G', G) of groups to their product $G' \times G$. Show that there is a functor $Q\colon \mathbf{Grp} \to \mathbf{Grp} \times \mathbf{Grp}$ sending a group H to (H, H).

(2) Show that *forgetting that a group is abelian* defines a full and faithful functor $U\colon \mathbf{Ab} \to \mathbf{Grp}$.

Recall that for any $G \in \mathbf{Grp}$ the *commutator* is the subgroup $[G, G]$ defined by finite products of the form $[x_1, y_1] \dots [x_n, y_n]$ where $n \geq 1$, $x_i, y_i \in G$ and $[x, y] \coloneqq x^{-1}y^{-1}xy$. You are given $[G, G]$ is normal in G.

(3) Show that sending $G \in \operatorname{ob}(\mathbf{Grp})$ to $G/[G, G] \in \operatorname{ob}(\mathbf{Ab})$ defines a dense functor $V\colon \mathbf{Grp} \to \mathbf{Ab}$.

Question 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors such that $FG = \operatorname{Id}_{\mathcal{D}}$ and $GF = \operatorname{Id}_{\mathcal{C}}$. Prove that F is an equivalence of categories.