3. Questions on §1.3 and §1.4.

Question 3.1. Consider the set $\mathbb{N}_{>0}$ as a partially ordered set, and consider the category $\mathbb{N}_{>0}^{\mathsf{op}}$.

(1) To each relation $n \leq n+1$ assign the function $\pi_n\colon \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ given by $\pi_n(a + p^{n+1}\mathbb{Z}) = a + p^n\mathbb{Z}$. Explain why this defines an $\mathbb{N}_{>0}^{\mathsf{op}}$-diagram ρ in the category **Ring** of rings.

Let $\widehat{\mathbb{Z}}_p \coloneqq \{(a_n + p^n\mathbb{Z}) \in \prod_{n>0} \mathbb{Z}/p^n\mathbb{Z} \mid a_n - a_{n+1} \in p^n\mathbb{Z} \text{ for all } n > 0\}$.

(2) Explain why $\widehat{\mathbb{Z}}_p$ is a subring of $\prod_{n>0} \mathbb{Z}/p^n\mathbb{Z}$ and prove that there exist ring homomorphisms $\tau_n\colon \widehat{\mathbb{Z}}_p \to \mathbb{Z}/p^n\mathbb{Z}$ such that $\pi_n\tau_{n+1} = \tau_n$ for each $n > 0$.

(3) Prove that $\widehat{\mathbb{Z}}_p$ equipped with $(\tau_n\colon n > 0)$ defines a limit of ρ.

Thus this exercise shows that the *p-adic integers* $\widehat{\mathbb{Z}}_p$ are *the inverse limit* of $\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \cdots$

Question 3.2. Let $\mathcal{C}$ be a preadditive category.

(1) Prove that for any $X \in \mathrm{ob}(\mathcal{C})$ the set $\mathrm{End}_{\mathcal{C}}(X) \coloneqq \mathrm{Hom}_{\mathcal{C}}(X, X)$ of *endomorphisms* of X is a ring.

(2) Let $X, Y \in \mathrm{ob}(\mathcal{C})$. Show that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an $\mathrm{End}_{\mathcal{C}}(Y)$-$\mathrm{End}_{\mathcal{C}}(X)$-bimodule.

(3) Let $X, Y \in \mathrm{ob}(\mathcal{C})$ such that their *direct sum* $X \oplus Y$ exists in $\mathcal{C}$. Using that the direct sum is a product and a coproduct, describe (in as much detail as you wish) the ring $\mathrm{End}_{\mathcal{C}}(X \oplus Y)$.

(4) Let $\mathcal{C}$ be a K-*category*. So, by definition, for each $X, Y, Z \in \mathrm{ob}(\mathcal{C})$ the following statements hold.
   (a) The set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of a K-module.
   (b) For each $\theta \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ given by $\varphi \mapsto \varphi\theta$ is K-linear.
   (c) For each $\varphi \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ given by $\theta \mapsto \varphi\theta$ is K-linear.

   Prove that for each $X \in \mathrm{ob}(\mathcal{C})$ there is a ring homomorphism $K \to \mathrm{End}_{\mathcal{C}}(X)$.

Question 3.3. Let K be a commutative ring and recall the functor $\mathrm{Free}_K\colon \mathbf{Set} \to K-\mathsf{Mod}$.

(1) Find a K-module isomorphism $\mathrm{Free}_K(H) \times \mathrm{Free}_K(H') \cong \mathrm{Free}_K(H \sqcup H')$ for any $H, H' \in \mathrm{ob}(\mathbf{Set})$.

Now also let $\mathcal{C}$ be a category.

(2) By setting $\mathrm{ob}(K\mathcal{C}) \coloneqq \mathrm{ob}(\mathcal{C})$ and $\mathrm{Hom}_{K\mathcal{C}}(X, Y) \coloneqq \mathrm{Free}_K(\mathrm{Hom}_{\mathcal{C}}(X, Y))$ for each $X, Y \in \mathrm{ob}(\mathcal{C})$, define a category $K\mathcal{C}$. Prove that $K\mathcal{D}$ is a K-algebra when $\mathcal{D}$ is a category with only one object.

(3) Prove that $K\mathcal{C}$ is a K-category.

(4) Show that every functor $\mathcal{C} \to K-\mathsf{Mod}$ defines a of K-linear functor of the form $K\mathcal{C} \to K-\mathsf{Mod}$. Similarly, show that natural transformations between functors of the form $\mathcal{C} \to K-\mathsf{Mod}$ give rise to natural transformations between K-linear functors of the form $K\mathcal{C} \to K-\mathsf{Mod}$.

(5) Now assume $\mathcal{C}$ is the category defined by a partially ordered set. Explain how morphisms in $K\mathcal{C}$ correspond to elements of K. Show that, here, the ring map from Question 3.2(4) is an isomorphism.