4. Questions on §1.5 and §1.6.

Question 4.1. Let R be a ring, $\mathcal{C}$ be the category of left R-modules and $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

(1) Show that the inclusion of the R-submodule $\{x \in X \mid f(x) = 0\}$ into X defines the kernel of f. Write down an R-module C and an R-module homomorphism $Y \to C$ giving the cokernel of f in $\mathcal{C}$.

(2) Prove that f is a monomorphism (respectively, epimorphism) if and only if f is injective (respectively, surjective). Prove that $\mathcal{C}$ an abelian category.

A left R-module M is *noetherian* provided each submodule of M is finitely generated, and that if R is *left-noetherian* (so noetherian as a left module over itself) then every finitely generated module is noetherian.

(3) Let R be left-noetherian and let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ consisting of finitely generated left R-modules. Prove that $\mathcal{A}$ is an abelian category.

(4) Find an example of a short exact sequence in an abelian category which is not a split exact sequence.

Question 4.2. Let $\mathcal{C}$ be an additive category. You are given that $\mathcal{C}^{\mathsf{op}}$ is additive. Prove that the kernel of any morphism in $\mathcal{C}$ is a monomorphism, and that the cokernel of any morphism in $\mathcal{C}$ is a epimorphism.

Question 4.3. Let $\mathcal{C}$ be an additive category such that every morphism has a kernel and cokernel in $\mathcal{C}$.

(1) Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ with kernel $h\colon K \to X$ and cokernel $q\colon Y \to C$. Let $p\colon X \to D$ be the *coimage* of f (cokernel of h). Let $i\colon J \to Y$ be the *image* of f (kernel of q). Prove that there is a morphism $f'\colon D \to J$ such that $f = if'p$. Prove that if $\mathcal{C}$ is abelian then f' is an isomorphism. Hint: begin by using the argument from the proof of the Lemma at the bottom of page 14 in the notes. This argument should tell you that if' is a monomorphism.

(2) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathcal{C}$. Prove that if $\mathcal{C}$ is abelian and $gf = 0$ then there is a monomorphism from the image of f to the kernel of g. When is $X \to Y \to Z$ exact at Y?

Question 4.4. Let $\mathcal{C}$ be an abelian category and suppose we have a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccccc}
 & & A & \xrightarrow{f} & M & \xrightarrow{g} & Y & \longrightarrow & 0 \\
 & & {\scriptstyle c}\,\downarrow & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle x} & & \\
0 & \longrightarrow & B & \xrightarrow[p]{} & N & \xrightarrow[q]{} & Z & &
\end{array}$$

where you only know the solid arrows exist. Assume the rows are exact at M, Y, B and N.

(1) Prove that if $q\ell f = 0$ then c and x exist such that the diagram commutes (so $\ell f = pc$ and $xg = q\ell$).

Assume from now on that the dashed arrows exist such that the diagram commutes.

(2) Prove that if ℓ and q are epimorphisms then x is a epimorphism. Prove that if ℓ and f are monomorphisms then c is a monomorphism.

(3) Prove that if ℓ is an isomorphism, if f is a monomorphism and if q is an epimorphism then the kernel of x is isomorphic to the cokernel of c.