5. Questions on §1.7 and §1.8.

Question 5.1. Let $\mathcal{C}$ be an abelian category, X be an object in $\mathcal{C}$ and consider the covariant functors

$$\mathrm{Hom}_{\mathcal{C}}(X,-)\colon \mathcal{C}\to \mathbf{Ab},\quad \mathrm{Hom}_{\mathcal{C}}(-,X)\colon \mathcal{C}^{\mathsf{op}}\to \mathbf{Ab}.$$

(1) Prove that $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is exact if and only if for any monomorphism $\theta\colon L\to M$ and any morphism $\sigma\colon L\to X$ there is a morphism $\phi\colon M\to X$ such that $\phi\theta=\sigma$. Describe when $\mathrm{Hom}_{\mathcal{C}}(X,-)$ is exact.

Let $n\in\mathbb{Z}$ with $n\neq -1,0,1$ and consider the abelian category $\mathcal{C}=\mathbf{Ab}$ of abelian groups.

(2) Recall the classification of finitely generated abelian groups. Explain why the full subcategory of $\mathbf{Ab}$ consisting of finite abelian groups does not have coproducts and does not have a generator.

(3) Prove that for any morphism $\varphi\colon \mathbb{Z}/n^2\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ we have $n\mathbb{Z}/n^2\mathbb{Z}\subseteq\ker(\varphi)$. Write down an isomorphism $n\mathbb{Z}/n^2\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ in $\mathbf{Ab}$ and hence explain why $\mathrm{Hom}_{\mathbf{Ab}}(-,\mathbb{Z}/n\mathbb{Z})$ is not exact.

(4) Explain why $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})=0$ and why $\mathbb{Z}/n\mathbb{Z}$ is finitely presented. Hence prove that the functor $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z},-)$ is not exact. Recall that, for a prime $p>0$, the *Prüfer group* $\mathbb{Z}_{p^\infty}$ $\mathbb{Z}_{p^\infty}$ is the directed colimit of the morphisms $\mathbb{Z}/p^i\mathbb{Z}\to\mathbb{Z}/p^{i+1}\mathbb{Z}$ given by $\ell+p^i\mathbb{Z}\mapsto p\ell+p^{i+1}\mathbb{Z}$. Prove

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}_{p^\infty})\cong \mathrm{colim}_{i>0}(\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/p^i\mathbb{Z})).$$

You may use results from the course.

Question 5.2. Using results in the course prove that $R-\mathsf{Mod}$ is Grothendieck where R is a ring.

Question 5.3. Define a category $\mathcal{I}$ by $\mathrm{ob}(\mathcal{I})=\{\ell,m,n\}$ with non-identity morphisms $\alpha\colon m\to\ell$ and $\beta\colon m\to n$. Let R be a ring, let $F\colon\mathcal{I}\to R-\mathsf{Mod}$ be a functor and use the following notation

$$L:=F(\ell),\quad M:=F(m),\quad N:=F(n),\quad a:=F(\alpha),\quad b:=F(\beta)$$

(1) Compute the colimit of F by considering the cokernel of the map $M\to L\oplus N$, $x\mapsto(a(x),-b(x))$.

(2) Let $\phi\colon F\to F'$ be a natural transformation between $\mathcal{I}$-diagrams in $R-\mathsf{Mod}$. Describe the morphism

$$\mathrm{colim}_{i\in\mathcal{I}}(\phi)\colon \mathrm{colim}_{i\in\mathcal{I}}(F)\to\mathrm{colim}_{i\in\mathcal{I}}(F')$$

(3) Considering $(L,M,N)=(R,0,R)$ and $(L',M',N')=(R,R,R)$, find a natural transformation ϕ of $\mathcal{I}$-diagrams in $R-\mathsf{Mod}$ such that ϕ_ℓ, ϕ_m and ϕ_n are injective but $\mathrm{colim}_{i\in\mathcal{I}}(\phi)$ is not.

(4) Using a counter-example, explain why colimits of short exact sequences are not always exact. Explain why your example is consistent with the fact that filtered colimits of exact sequences are exact.

Question 5.4. Find an example of an abelian category with a generator that is not a Grothendieck category.