6. Questions on §2.1 and §2.2

Question 6.1. Consider the category of abelian groups, that is, the category of $\mathbb{Z}$-modules.

(1) Let $n \in \mathbb{Z}$ with $n \neq -1, 0, 1$. Without using the classification of finitely generated modules over a principal ideal domain, explain why $\mathbb{Z}/n\mathbb{Z}$ is not a projective $\mathbb{Z}$-module.

For parts (2) and (3) below we consider $\mathbb{Q}$ as a $\mathbb{Z}$-module.

(2) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ and hence explain why $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \bigoplus_{i \in I} \mathbb{Z}) = 0$ for any set I. Hence prove by contradiction that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.

(3) Consider the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \dots]$ in infinitely many variables, as a $\mathbb{Z}$-module. Show that there is a surjective $\mathbb{Z}$-module homomorphism $p\colon \mathbb{Z}[x_1, x_2, x_3, \dots] \to \mathbb{Q}$ that sends x_n to $\frac{1}{n}$. Prove that there does not exist a $\mathbb{Z}$-module homomorphism $i\colon \mathbb{Q} \to \mathbb{Z}[x_1, x_2, x_3, \dots]$ such that $pi = \mathrm{id}_{\mathbb{Q}}$.

Question 6.2. Let Q, R, S be rings, A an S-R-bimodule, B an R-Q-bimodule and C an S-Q-bimodule.

(1) Prove that the abelian group $A \otimes_R B$ has the structure of an S-Q-bimodule. Prove that the abelian group $\mathrm{Hom}_S(A, C)$ has the structure of an R-Q-bimodule.

(2) Prove that there is an Q-Q-bimodule isomorphism $\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$.

(3) Assume that B is projective as a left R-module, and that A is projective as a left S-module. Prove that $A \otimes_R B$ is projective as a left S-module.

Question 6.3. Let $f\colon A \to B$ be a homomorphism of rings.

(1) Prove that any left B-module N has the structure of an A-module, denoted $\mathrm{Res}_A(N)$, and defined by $a \cdot n := f(a)n$. Explain why this defines a functor $\mathrm{Res}_A\colon B-\mathsf{Mod} \to A-\mathsf{Mod}$.

(2) Explain how B can be considered as a B-A-bimodule. Prove that for any $M \in \mathrm{ob}(A-\mathsf{Mod})$ and $\theta \in \mathrm{Hom}_B(B \otimes_A M, N)$ there exists $\varphi \in \mathrm{Hom}_A(M, \mathrm{Res}_A(N))$ given by $\varphi(m) = \theta(1_B \otimes m)$.

(3) Prove that for any $N \in \mathrm{ob}(B-\mathsf{Mod})$ and $\varphi \in \mathrm{Hom}_A(M, \mathrm{Res}_A(N))$ there exists an A-balanced map $B \times M \to N$ given by $(b, m) \mapsto b\varphi(m)$. Show that this defines a morphism $B \otimes_A M \to N$ in $B-\mathsf{Mod}$.

(4) Prove that Res_A and $\mathrm{Hom}_B(B, -)$ are naturally isomorphic as functors $B-\mathsf{Mod} \to A-\mathsf{Mod}$.