7. Questions on §2.3 and §2.4

Question 7.1. Let R be a ring and consider the category $R-\mathsf{Mod}$ of left R-modules.

(1) Let $F\colon R-\mathsf{Mod} \to R-\mathsf{Mod}$ be a functor. Define the binary operations and identity elements that make the set $\mathrm{End}(F)$ of natural transformations of the form $F \to F$ into a ring.

(2) By considering multiplication by ring elements as homomorphisms, prove that there is a ring isomorphism between $\mathrm{End}(\mathrm{Id}_{R-\mathsf{Mod}})$ and the *centre* $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$ of R.

(3) If $X, Y \in R-\mathsf{Mod}$ then prove that the following set has the structure of a ring

$$\begin{pmatrix} \mathrm{End}_R(X) & \mathrm{Hom}_R(Y,X) \\ \mathrm{Hom}_R(X,Y) & \mathrm{End}_R(Y) \end{pmatrix}.$$

Prove this ring is isomorphic to $\mathrm{End}_R(X \oplus Y)$. Compute $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z})$ for a prime $p > 0$.

For (4) and (5) below let S be another ring and recall the theorem of Morita equivalence.

(4) Prove that if $R-\mathsf{Mod}$ and $S-\mathsf{Mod}$ are equivalent categories then $Z(R) \cong Z(S)$ as rings.

(5) Prove that if $R-\mathsf{Mod}$ and $S-\mathsf{Mod}$ are equivalent categories then there exists $n \in \mathbb{N}_{>0}$ and $e \in M_n(R)$ such that $e^2 = e$, $M_n(R)eM_n(R) = M_n(R)$ and $S \cong eM_n(R)e$ as rings.

(6) Prove that if there exists $n \in \mathbb{N}_{>0}$ and $e \in M_n(R)$ such that $e^2 = e$, $M_n(R)eM_n(R) = M_n(R)$ and $S \cong eM_n(R)e$ as rings, then $R-\mathsf{Mod}$ and $S-\mathsf{Mod}$ are equivalent categories.

Question 7.2. Let R be a ring and X, Y and Z be left R-modules.

(1) Suppose that Z is an essential extension of Y. Prove that if Y is an essential extension of X then Z is an essential extension of X. Prove that if $Z = X \oplus Y$ then $X = 0$.

(2) Assume that every short exact sequence of the form $0 \to X \to Y \to Z \to 0$ splits, and let $\alpha\colon M \to W$ and $\beta\colon M \to X$ be R-module homomorphisms where α is a monomorphism. Without using the Proposition/Definition at the start of §2.4, prove that there exists an R-module homomorphism $\gamma\colon W \to X$ such that $\gamma\alpha = \beta$. Hence prove that $\mathrm{Hom}_R(-, X)$ is exact.

(3) Let $R = \mathbb{Z}$ and $X = \mathbb{Z}/n\mathbb{Z}$ for an integer $n \neq -1, 0, 1$. Explain why X is not injective. Find an exact sequence of abelian groups $0 \to X \to Y \to Z \to 0$ that does not split.

Question 7.3. Let $\mathcal{C}$ be an abelian category, let I be a set, let $M_i \in \mathrm{ob}(\mathcal{C})$ for each $i \in I$, and assume the product $M = \prod_{i \in I} M_i$ exists in $\mathcal{C}$. Using the Proposition/Definition at the start of §2.4, prove that M is injective if and only if M_i is injective for each $i \in I$.