8. Questions on §2.5 and §2.6

Question 8.1. Let $p > 0$ be prime. In Question 3.1 we considered a ring called the *p-adic integers*, denoted $\widehat{\mathbb{Z}}_p$, which arose as the limit of an $\mathbb{N}^{\mathsf{op}}_{>0}$-diagram $\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \cdots$ in the category of rings.

(1) Prove that $\widehat{\mathbb{Z}}_p$ is an integral domain, and hence why $\widehat{\mathbb{Z}}_p \setminus \{0\}$ is a multiplicative subset.

If $n > 0$ then recall that any ideal in $\mathbb{Z}/p^n\mathbb{Z}$ is generated by a coset of the form $p^d + p^n\mathbb{Z}$ with $0 \leq d \leq n$.

(2) Using the ring projections $\widehat{\mathbb{Z}}_p \to \mathbb{Z}/p^n\mathbb{Z}$ show that $\widehat{\mathbb{Z}}_p$ is a noetherian principal ideal domain.

Consider the *p-adic numbers*, defined (for simplicity) by the field of fractions $\widehat{\mathbb{Q}}_p = \mathrm{Frac}(\widehat{\mathbb{Z}}_p)$. Let I be a set.

(3) Recalling the inclusion $\bigoplus_{i\in I} \widehat{\mathbb{Z}}_p \subseteq \prod_{i\in I} \widehat{\mathbb{Z}}_p$, prove that $\mathrm{Hom}_{\widehat{\mathbb{Z}}_p-\mathsf{Mod}}(\widehat{\mathbb{Q}}_p, \bigoplus_{i\in I} \widehat{\mathbb{Z}}_p) = 0$.
(4) Consider $\widehat{\mathbb{Q}}_p$ as a $\widehat{\mathbb{Z}}_p$-module. Explain why $\widehat{\mathbb{Q}}_p$ is a filtered colimit of finitely generated free $\widehat{\mathbb{Z}}_p$-modules. Explain why $\widehat{\mathbb{Q}}_p$ is cannot be projective, nor finitely generated, as a $\widehat{\mathbb{Z}}_p$-module.
(5) Show that $\prod_{i\in I} \mathrm{Hom}_{\widehat{\mathbb{Z}}_p-\mathsf{Mod}}(\widehat{\mathbb{Q}}_p, \widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p)$ and $\bigoplus_{i\in I} \mathrm{Hom}_{\widehat{\mathbb{Z}}_p-\mathsf{Mod}}(\widehat{\mathbb{Q}}_p, \widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p)$ are injective $\widehat{\mathbb{Z}}_p$-modules.
(6) Explain why there cannot exist a field K such that $\widehat{\mathbb{Z}}_p$ is a finite-dimensional K-algebra.

Thus this exercise shows that the ring $\widehat{\mathbb{Z}}_p$ is not *perfect*. However, it is *semiperfect*.

Question 8.2. Let R be a ring and M be a left R-module. A submodule X of M is said to be *small* in M if for any submodule Y of M with $Y \neq M$ we have $X + Y \neq M$.

(1) Let X be small in M. Prove that any submodule of X is small in M. Prove that if M is a submodule of a module N then X is small in N. Prove that if W is small in M then $W + X$ is small in M.
(2) Let L and N be R-modules, $\alpha \in \mathrm{Hom}_{R-\mathsf{Mod}}(N, M)$ and $\beta \in \mathrm{Hom}_{R-\mathsf{Mod}}(M, L)$. Prove that if β and $\beta\alpha$ are both surjective, and if $\ker(\beta)$ is small in M, then α is surjective. Prove that if $\ker(\beta\alpha)$ is small in N then $\ker(\alpha)$ is small in N.

From now on let P be an object in the category $\mathcal{C}$ of projective left R-modules and let $\pi \in \mathrm{Hom}_{R-\mathsf{Mod}}(P, M)$.

(3) Recall the generator in Question 5.2. Prove that π is surjective if and only if π is a $\mathcal{C}$-precover.
(4) Prove that if $\ker(\pi)$ is small in P then π is right minimal.

Be warned: one *cannot* dualise the first theorem from §2.6, since it there are rings over which not *every* module has a projective cover. In other words, not every ring is perfect.