## 9. Questions on §3.1, §3.2 and §3.3

**Question 9.1.** Let $\mathcal{A}$ be an abelian category where every short exact sequence splits.

(1) Recall Question 4.3. Let $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(Y,Z)$ such that $gf = 0$. Write down the meaning of the symbol $\ker(g)/\mathrm{im}(f)$. Prove that $Y \cong \mathrm{im}(f) \oplus \mathrm{im}(g) \oplus (\ker(g)/\mathrm{im}(f))$ in $\mathcal{A}$.

(2) Let $\mathcal{C} = C(\mathcal{A})$, the category of cochain complexes in $\mathcal{A}$, and $L \in \mathrm{ob}(\mathcal{C})$. Construct $M \in \mathrm{ob}(\mathcal{C})$ and $a \in \mathrm{Hom}_{\mathcal{C}}(L,M)$ such that $M^n = H^n(L)$ and $H^n(a)$ is an isomorphism for each $n \in \mathbb{Z}$.

**Question 9.2.** Define an orientated simplicial complex $K$ with 5 vertices 1, 2, 3, 4 and 5, and setting

$$x_1 = [1], \quad x_2 = [2], \quad x_3 = [3], \quad x_4 = [4], \quad x_5 = [5],$$
$$y_1 = [1,2], \quad y_2 = [1,3], \quad y_3 = [1,4], \quad y_4 = [2,3], \quad y_5 = [2,4], \quad y_6 = [3,4], \quad y_7 = [3,5], \quad y_8 = [4,5],$$
$$z_1 = [1,2,3], \quad z_2 = [1,2,4], \quad z_3 = [1,3,4], \quad z_4 = [2,3,4].$$

Consider matrices $A$ and $B$ over $\mathbb{Q}$ with null spaces $K_A$ and $K_B$ and column spaces $I_A$ and $I_B$

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}^{\boldsymbol{t}}$$

$$K_A = \mathrm{span}_{\mathbb{Q}}\{(1,-1,0,1,0,0,0,0)^{\boldsymbol{t}}, (1,0,-1,0,1,0,0,0)^{\boldsymbol{t}}, (0,1,-1,0,0,1,0,0)^{\boldsymbol{t}}, (0,-1,1,0,0,0,-1,1)^{\boldsymbol{t}}\},$$
$$K_B = \mathrm{span}_{\mathbb{Q}}\{(-1,1,-1,1)^{\boldsymbol{t}}\},$$
$$I_A = \mathrm{span}_{\mathbb{Q}}\{(-1,1,0,0,0)^{\boldsymbol{t}}, (-1,0,1,0,0)^{\boldsymbol{t}}, (-1,0,0,1,0)^{\boldsymbol{t}}, (0,0,-1,0,1)^{\boldsymbol{t}}\},$$
$$I_B = \mathrm{span}_{\mathbb{Q}}\{1,-1,0,1,0,0,0,0)^{\boldsymbol{t}}, (1,0,-1,0,1,0,0,0)^{\boldsymbol{t}}, (0,1,-1,0,0,1,0,0)^{\boldsymbol{t}}\}.$$

(1) Draw a picture that describes $K$. Using your picture, write down the number of path connected components, the number of holes whose boundaries are edges (respectively, triangles).

(2) For the complex $S := C(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $C(\mathbb{Q}-\mathsf{Mod})$ compute the image of each $d_1^S(y_j \otimes 1)$ (respectively, $d_2^S(z_k \otimes 1)$) as a linear combination of $x_1 \otimes 1, \ldots, x_5 \otimes 1$ (respectively, $y_1 \otimes 1, \ldots, y_8 \otimes 1$).

(3) Using the values of $A$, $B$, $K_A$, $K_B$, $I_A$ and $I_B$, compute the homology of $C(K)$ with coefficients in $\mathbb{Q}$ in each degree, and find an element of $\ker(d_1^S) \setminus \mathrm{im}(d_2^S)$.

(4) Explain how your calculations in (3) relate to your sketch and observations in (1).

**Question 9.3.** Let $\mathcal{A}$ be an abelian category, $\mathcal{C} = C(\mathcal{A})$, and $\mathcal{B}$ be the category defined by $\{0,1\}$ with $0 < 1$.

(1) Prove $\mathrm{Fun}(\mathcal{B},\mathcal{A})$ is abelian by describing the zero object, direct sums, kernels and cokernels.

(2) Define a functor $D\colon \mathrm{Fun}(\mathcal{B},\mathcal{A}) \to \mathcal{C}$ such that, for all $F \in \mathrm{Fun}(\mathcal{B},\mathcal{A})$, we have that $H^0(D(F)) = \ker(F(0 \to 1))$, $H^1(D(F)) = \mathrm{cok}(F(0 \to 1))$ and $H^n(D(F)) = 0$ for all $n \in \mathbb{Z}$ with $n \neq 0, 1$.

(3) Define a functor $E\colon \mathcal{A} \to \mathrm{Fun}(\mathcal{B},\mathcal{A})$ such that $H^n(D(E(M))) = 0$ for all $M \in \mathrm{ob}(\mathcal{A})$.

(4) Consider the category $\mathrm{Fun}(\mathcal{B},\mathcal{C})$ and suppose we have a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

which has exact rows. Prove that there is a commutative diagram in $\mathcal{A}$ with exact rows of the form

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H^{n-1}(N) & \longrightarrow & H^n(L) & \longrightarrow & H^n(M) & \longrightarrow & H^n(N) & \longrightarrow & H^{n+1}(L) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^{n-1}(Z) & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Z) & \longrightarrow & H^{n+1}(X) & \longrightarrow & \cdots \end{array}$$