11. Questions on §4.2, §4.3, §4.4

Question 11.1. Let R be a ring and fix the following short exact sequences in the category $R-\mathsf{Mod}$,

$$\xi\colon 0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0, \qquad \theta\colon 0 \longrightarrow A \xrightarrow{\ f'\ } B' \xrightarrow{\ g'\ } C \longrightarrow 0.$$

(1) Consider $(f,-f')^{\boldsymbol{t}}\colon A \to B \oplus B'$ and $(g,-g')\colon B \oplus B' \to C$. Let $L := \operatorname{im}((f,-f')^{\boldsymbol{t}})$ and $N = \ker((g,-g'))$. Prove that $L \subseteq N$. Define $M := N/L$, $f''(a) := (f(a),0)+L$ and $g''((b,b')+L) := g(b)$ for $a \in A$, $b \in B$ and $b' \in B$. Show that the sequence $\psi\colon 0 \to A \to M \to C \to 0$ is short exact.

(2) Prove that if $f' = -f$ and $g' = g$ then ψ is split. Prove that if ξ is split then $0 \to A \to B \to C \to 0$ is split then θ and ψ are equivalent.

Let P be a projective module and $z\colon P \to C$ be an epimorphism with kernel $i\colon K \to P$.

(3) Prove that there exist $x \in \operatorname{Hom}_R(P,B)$ and $x' \in \operatorname{Hom}_R(P,B')$ such that $gx = g'x' = z$. Explain why $\operatorname{im}((x,x')^{\boldsymbol{t}}) \subseteq N$. Prove that there exist $y, y' \in \operatorname{Hom}_R(K,A)$ such that $fy = xi$ and $f'y' = x'i$.

(4) Define $h\colon P \to M$ by $h(p) := (x(p),x'(p)) + L$ for each $p \in P$. Show $g''h = z$ and $f''(y+y') = hi$.

(5) Carefully following the proof of the Theorem on page 66, prove that $\hat{\xi} + \hat{\theta} = \hat{\psi}$ in $\operatorname{Ext}^1_R(C,A)$.

The operation discussed here is called the *Baer sum*. There is a version for $\operatorname{Ext}^n_R(C,A)$.

Question 11.2. Let $f\colon A \to B$ be a homomorphism of rings. Recall the functor $\operatorname{Res}_A\colon B-\mathsf{Mod} \to A-\mathsf{Mod}$ from Question 6.3(1) that reconsiders any B-module as an A-module using f.

(1) For any B-module M define a B-balanced map $\operatorname{Res}_A(B) \times M \to \operatorname{Res}_A(M)$ and thus prove that Res_A and $\operatorname{Res}_A(B) \otimes_B -$ are naturally isomorphic. Hence explain why there is an adjunction

$$\operatorname{Hom}_A(\operatorname{Res}_A(M),T) \cong \operatorname{Hom}_B(M,\operatorname{Hom}_A(\operatorname{Res}_A(B),T)), \quad (M \in \operatorname{ob}(B-\mathsf{Mod}),\ T \in \operatorname{ob}(A-\mathsf{Mod})).$$

(2) Using that B is finitely presented in $B-\mathsf{Mod}$, Question 6.3(4), and a result in the notes, explain why Res_A preserves colimits. Hence prove that Res_A is an exact functor.

From now on assume that $\operatorname{Res}_A(B)$ is a projective A-module.

(3) Prove that $\operatorname{Res}_A(P)$ is a projective A-module for any projective B-module P. Prove that if $\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$ is a projective resolution of a B-module M then applying Res_A gives a projective resolution $\cdots \to \operatorname{Res}_A(P^2) \to \operatorname{Res}_A(P^1) \to \operatorname{Res}_A(P^0) \to \operatorname{Res}_A(M) \to 0$ of $\operatorname{Res}_A(M)$.

Let S and T be A-modules, let M and N be B-modules, let $\theta\colon \operatorname{Res}_A(M) \to S$ be an A-module homomorphism, and let $\varphi\colon \operatorname{Hom}_A(\operatorname{Res}_A(B),T) \to N$ be a B-module homomorphism.

(4) Let $\cdots \to Q^2 \to Q^1 \to Q^0 \to S \to 0$ and $\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$ be projective resolutions (in different module categories). Construct a commutative diagram of abelian groups of the form

$$\begin{array}{ccccccc}
\operatorname{Hom}_A(S,T) & \longrightarrow & \operatorname{Hom}_A(Q^0,T) & \longrightarrow & \operatorname{Hom}_A(Q^1,T) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Hom}_A(\operatorname{Res}_A(M),T) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Res}_A(P^0),T) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Res}_A(P^1),T) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Hom}_B(M,N) & \longrightarrow & \operatorname{Hom}_B(P^0,N) & \longrightarrow & \operatorname{Hom}_B(P^1,N) & \longrightarrow & \cdots
\end{array}$$

(5) Construct, for each integer $n \geq 0$, a homomorphism $\operatorname{Ext}^n_{\mathrm{Res}}(\theta,\varphi)\colon \operatorname{Ext}^n_A(S,T) \to \operatorname{Ext}^n_B(M,N)$. Deduce the *Eckmann–Shapiro lemma* for rings, which states that

$$\operatorname{Ext}^n_A(\operatorname{Res}_A(L),R) \cong \operatorname{Ext}^n_B(L,\operatorname{Hom}_A(\operatorname{Res}_A(B),R)), \quad (L \in \operatorname{ob}(B-\mathsf{Mod}),\ R \in \operatorname{ob}(A-\mathsf{Mod})).$$