12. Questions on §4.5, §4.6, §4.7

Question 12.1. Let K be a field and let $R = K[x]/\langle x^n \rangle$ for some integer $n > 1$. Consider the R-module $M = R/I$ where I is the ideal generated by $\overline{x}^d = x^d + \langle x^n \rangle$ where $1 \leq d < n$. Hence $\dim_K(M) = d$. By computing a projective resolution of the R-module M, calculate the global dimension $\mathrm{gl.dim}(R)$ of R.

Question 12.2. Let $m, n \in \mathbb{Z}$ with $m, n \neq -1, 0, 1$. Prove that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Question 12.3. Let R be a ring and let $M \in \mathsf{Mod}-R$. Recall that $\mathrm{Tor}_n^R(M, -) = 0$ if M is flat.

(1) Prove that $\mathrm{Tor}_n^R(M, -) \colon R-\mathsf{Mod} \to \mathbf{Ab}$ is an additive functor. Using that $\mathsf{Mod}-R = R^{\mathsf{op}}-\mathsf{Mod}$ prove that if N is a left R-module then $\mathrm{Tor}_n^R(-, N) \colon \mathsf{Mod}-R \to \mathbf{Ab}$ is an additive functor.

(2) Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a flat resolution, $\Omega_0 M = M$ and $\Omega_n M = \mathrm{im}(P_n \to P_{n-1})$ for $n \geq 1$. Prove that $\mathrm{Tor}_i^R(\Omega_{n+1}M, X) \cong \mathrm{Tor}_{i+1}^R(\Omega_n M, X)$ for all $n \geq 0$, $i \geq 1$ and $X \in R-\mathsf{Mod}$.

(3) Prove that the following statements are equivalent.

   (a) There is a flat resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$.

   (b) $\mathrm{Tor}_m^R(M, X) = 0$ for any $m > n$ and any $X \in R-\mathsf{Mod}$.

   (c) $\mathrm{Tor}_{n+1}^R(M, X) = 0$ for any $X \in R-\mathsf{Mod}$.

Question 12.4. Let $f \colon A \to B$ be a homomorphism of rings. Recall the functor $\mathrm{Res}_A \colon B-\mathsf{Mod} \to A-\mathsf{Mod}$ from Question 6.3(1) that reconsiders any B-module as an A-module using f.

(1) Using Question 11.2 prove that if $\mathrm{Res}_A(B)$ is projective and if R is an A-module then

$$\mathrm{inj.dim}_A(R) \geq \mathrm{inj.dim}_B(\mathrm{Hom}_A(\mathrm{Res}_A(B), R))$$

(2) Prove that if $\mathrm{Res}_A(B)$ is flat then $\mathrm{inj.dim}_B(L) \geq \mathrm{inj.dim}_A(\mathrm{Res}_A(L))$ for any B-module L.

Question 12.5. Let $C(K)$ denote the chain complex associated to a simplicial complex K.

(1) Prove that if $n \in \mathbb{Z}$ then $H_n(C(K)) \cong \mathbb{Z}^{f_n} \oplus \bigoplus_{i=1}^{d(n)} \mathbb{Z}/t_{ni}\mathbb{Z}$ for some $f_n \geq 0$ and $t_{n1}, \ldots t_{nd(n)} > 0$.

(2) Prove that if M is a $\mathbb{Z}$-module then there is an isomorphism of $\mathbb{Z}$-modules

$$H^n(C(K); M) \cong M^{f_n} \oplus \left(\bigoplus_{i=1}^{d(n-1)} M/t_{(n-1)i}M\right) \oplus \left(\bigoplus_{i=1}^{d(n)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/t_{ni}\mathbb{Z}, M)\right)$$

(3) Calculate $H^n(C(K); \mathbb{Z}/p\mathbb{Z})$ and $H_n(C(K); \mathbb{Z}/p\mathbb{Z})$ where $p > 0$ is prime.