13. Questions on §5.1, §5.2, §5.3

Question 13.1. Let K be a field, $n > 0$ be an integer and $R = K[\![x_1, \ldots, x_n]\!]$, the power series.

(1) Prove that, for each $i = 1, \ldots, n$, the ideal $\langle x_1, \ldots, x_i \rangle$ in R is a prime ideal.
(2) Prove that the sequence $x_1, \ldots, x_n \in R$ is regular for R.
(3) Prove that $\mathrm{Kdim}(R) = n$.

Question 13.2. Let R be a commutative ring and $n > 0$.

(1) Let $a_1, \ldots, a_n \in R$, let $I = \langle a_1, \ldots, a_n \rangle$, the ideal generated by these elements, and let $d > 0$. Explain why every element $r \in I^d$ defines a homogeneous polynomial $f_d(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ of degree d such that $r = f_d(x_1, \ldots, x_n)$. State Hilbert's *basis theorem* from Algebra II.

From now on assume R is a noetherian ring.

(2) Prove that there exists $t > 0$ and $g_1, \ldots, g_t \in R[x_1, \ldots, x_n]$ such that $f_{t+1} = g_1 f_1 + \cdots + g_t f_t$. Hence prove that there exists $h_1, \ldots, h_t \in R[x_1, \ldots, x_n]$, where each h_d is homogeneous of degree $t + 1 - d$, such that $f_{t+1} = h_1 f_1 + \cdots + h_t f_t$. Hint: define h_d to be the degree $t + 1 - d$ part of g_d.
(3) Let I be an ideal of R and let $r \in \bigcap_{d>0} I^d$. Prove that $r \in \langle r \rangle I$.

From now on, for simplicity, assume R is a local noetherian ring with maximal ideal $\mathfrak{m}$.

(4) Prove *Krull's intersection theorem*, that says that $\bigcap_{d>0} J^d = 0$ for any ideal J such that $J \subseteq \mathfrak{m}$.

Question 13.3. Let R be a commutative ring and let M be an R-module. Let $r, s \in R$ where r is regular for M and if s is regular for M/rM.

(1) Prove that r is regular for M/sM.
(2) Prove that if $sm = 0$ for some $m \in M$ then for each $d > 0$ we have $m = r^d m_d$ for some $m_d \in M$.

From now on assume that R is local noetherian with maximal ideal $\mathfrak{m}$, and assume M is finitely generated.

(3) Prove that if $r, s \in \mathfrak{m}$ then s is regular for M and r is regular for M/sM.
(4) Prove that if $x_1, \ldots, x_n \in R$ is a regular sequence for M and if σ is a permutation of $\{1, \ldots, n\}$ then the permuted sequence $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ is regular for M.

Question 13.4. Let $R = \mathbb{C}[x, y, z]$.

(1) Prove that $(x, y(1-x), z(1-x))$ is a regular sequence for R considered as a module over itself.
(2) Prove that $(y(1-x), z(1-x), x)$ is not a regular sequence for R.