14. Questions on §5.4 and §5.5

Let G be a group. Let $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ denote the *augmentation* ring homomorphism, defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g$ for $a_g \in \mathbb{Z}$. Let $A = \Delta(G) = \ker(\varepsilon)$, the augmentation ideal of the group algebra $\mathbb{Z}G$.

Question 14.1. Let $C = [G,G]$, the commutator subgroup of G, recalled and discussed in Question 1.5.

(1) Using a result from the notes prove that $(g - 1_G \mid g \in G \setminus \{1_G\})$ is a $\mathbb{Z}$-module basis of A.

(2) Recall that C is normal in G. Explain why, for any $a, b \in \mathbb{Z}$ and any $g, h \in G$, we have the equality of cosets $h^a g^b C = g^b h^a C$. Hence explain why the assignment

$$\theta'\colon A \to G/C, \quad \textstyle\sum_{h\in G} a_h h \mapsto \prod_{h\in G} h^{a_h} C$$

gives a well-defined homomorphism of groups. Prove that $\theta'((g-1)(h-1))$ is the trivial coset for all $g, h \in G$, and hence explain why θ' induces a map $\theta\colon A/A^2 \to G/C$.

(3) Working in $\mathbb{Z}G$ prove that if $x, y \in G$ then $xy - yx = (x-1)(y-1) - (y-1)(x-1) \in A^2$. Prove by induction on $n \geq 1$ that if $g_1, \ldots, g_n \in G$ then the following equality holds

$$g_1 \ldots g_n - 1 = (g_1 - 1) + g_1(g_2 - 1) + \cdots + g_1 \ldots g_{n-1}(g_n - 1).$$

Using this, and that $x^{-1}y^{-1}xy - 1 = x^{-1}y^{-1}(xy - yx)$ for $x, y \in G$, prove that $g \in C$ implies $g - 1 \in A^2$ and explain why $\varphi(gC) = g - 1 + A^2$ defines a homomorphism of groups $\varphi\colon G/C \to A/A^2$. Prove that $\theta\varphi = \mathrm{id}_{G/C}$ and $\varphi\theta = \mathrm{id}_{A/A^2}$ and hence that $A/A^2 \cong G/C$ as abelian groups.

Question 14.2. Let M be a left module over $\mathbb{Z}G$.

(1) Recall that if R is a ring and if I is an ideal of R then $R/I \otimes_R X \cong X/IX$ for any left R-module X. Hence, using a result from the notes, prove that $H_0(G; M) \cong M/AM$.

(2) Recalling that ε denotes the augmentation, prove that the map $\mathrm{id}_Z \otimes \varepsilon\colon \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}$, which is defined by $1 \otimes \sum_{g\in G} a_g g \mapsto 1 \otimes \sum_{g\in G} a_g$, is an isomorphism. Considering a long exact sequence associated to $0 \to A \to \mathbb{Z}G \to \mathbb{Z} \to 0$, prove that $H_1(G; \mathbb{Z}) \cong H_0(G; A)$.

(3) Using everything so far show that $H_1(G; \mathbb{Z}) \cong G/C$. Hence prove that $|H_1(S_n, \mathbb{Z})| = 2$ where S_n is the symmetric group with $n > 1$. Explain how this is consistent with the Corollary on page 94.

Question 14.3. Check that the bar resolution for G is a complex, that is, check that the composition of consecutive differentials is 0.