SPDES driven by Poisson Random Measures and their numerical Approximation

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Outline

- SPDEs - a short Example
- Simulating a Lévy walk
- Simulating an space time Lévy noise;
- Simulating an SPDEs driven by a Poisson Random Measure
A typical Example

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.

The Equation:

\[
\begin{cases}
    \frac{du(t,\xi)}{dt} = \Delta u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi))\dot{L}(t,\xi) \\
    \quad + f(u(t,\xi)), \quad \xi \in \mathcal{O}, \ t > 0; \\
    u(0,\xi) = u_0(\xi) \quad \xi \in \mathcal{O}; \\
    u(t,\xi) = u(t,\xi) = 0, \quad t \geq 0, \ \xi \in \partial \mathcal{O};
\end{cases}
\]

where $u_0 \in L^p(\mathcal{O})$, $p \geq 1$, $g$ a certain mapping and $L(t)$ is a space time Lévy noise defined over a probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ specified later.

Problem: To find an approximation of the stochastic process

\[ u : \Omega \times [0, \infty) \times \mathcal{O} \longrightarrow \mathbb{R} \]

where $u(t,\xi) = \begin{cases} 
    u(t,\xi), & \xi \in \mathcal{O}, \\
    0, & \xi \in \partial \mathcal{O}.
\end{cases}$

(see e.g. St. Lupert Bié, Albeverio, Wu and Zhang, Applebaum, Brzezniak and Hausenblas, . . .)
Motivations

SPDEs versus PDEs

- Change of the dynamical behaviour of the system, e.g.
  - White noise eliminates instability [Crauel (2000)]
  - Stochastic Resonance [Schimansky-Geier, ...]
  - Uniqueness of the solution (Lecture notes of St. Flour by Flandoli)
  - Uniqueness of the Invariant Measures, etc. [Flandoli, Mattingly, Rozovskii, Shirykan and Kuksin, ...]

- Better description of the real world system
  - E.g. in thin-film models, SPDEs leads to a better description of data’s gained by experiments [Grüne, Mecke, Rauscher (2006)]
Motivations

Lévy processes versus Wiener processes

- Systems with jumps have another dynamical behaviour [see Imkeller and Pavlyukevich (2006)]
- In Climatology, e.g. the time in which the temperature is changing is short compared to the time scale [see Imkeller and Pavlyukevich (2006)]
- Finance Mathematics (Eberlein and his coworkers)
- In case of semilinear SPDEs driven by a Lévy process the conditions of the diffusion term $g$ can be weakened.
The Lévy Process \( L \)

**Definition 1** Let \( E \) be a Banach space. A stochastic process 
\( L = \{L(t) : 0 \leq t < \infty \} \) is an \( E \)-valued Lévy process over \((\Omega; \mathcal{F}; \mathbb{P})\) if the following conditions are satisfied:

- \( L(0) = 0 \);
- \( L \) has independent and stationary increments;
- \( L \) is stochastically continuous, i.e. for \( \phi \) continuous and bounded, the function \( t \mapsto \mathbb{E}\phi(L(t)) \) is continuous on \( \mathbb{R}^+ \);
- \( L \) has a.s. càdlàg paths;
The Lévy Process \( L \)

\( E \) denotes a separable Banach space and \( E' \) the dual on \( E \). The **Fourier Transform of** \( L \) is given by the Lévy - Hinchin - Formula (see Linde (1986)):

\[
\mathbb{E} \ e^{i\langle L(1), a \rangle} = \exp \left\{ i\langle y, a \rangle \lambda + \int_{E} \left( e^{i\lambda\langle y, a \rangle} - 1 - i\lambda y 1_{\{|y| \leq 1\}} \right) \nu(dy) \right\},
\]

where \( a \in E', \ y \in E \) and \( \nu : \mathcal{B}(E) \to \mathbb{R}^+ \) is a Lévy measure.
The Lévy Process

In what follows $E$ denotes a separable Banach space, $\mathcal{B}(E)$ denotes the Borel-$\sigma$ algebra on $E$ and $E'$ the dual on $E$.

**Definition 2** (see Linde (1986), Section 5.4) A $\sigma$–finite symmetric Borel-measure $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is called a Lévy measure if $\nu(\{0\}) = 0$ and the function

$$E' \ni a \mapsto \exp \left( \int_E \left( \cos(\langle x, a \rangle) - 1 \right) \nu(dx) \right) \in \mathbb{C}$$

is a characteristic function of a certain Radon measure on $E$. An arbitrary $\sigma$-finite Borel measure $\nu$ is a Lévy measure if its symmetrization $\nu + \nu^-$ is a symmetric Lévy measure.

\[^a\nu(A) = \nu(-A) \text{ for all } A \in \mathcal{B}(E)\]
Remark 1 Let $L$ be a real valued Lévy process over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and let $A \in \mathcal{B}(\mathbb{R})$. Defining the counting measure

$$N(t, A) = \# \{ s \in (0, t] : \Delta L(s) = L(s) - L(s-) \in A \} \in \mathbb{N} \cup \{\infty\}$$

one can show, that for any $A \in \mathcal{B}(\mathbb{R})$

- $N(t, A)$ is a random variable over $(\Omega; \mathcal{F}; \mathbb{P})$;
- $N(t, A) \sim \text{Poisson}(t\nu(A))$ and $N(t, \emptyset) = 0$;
- (independently scattered) For any disjoint sets $A_1, \ldots, A_n$, the random variables $N(t, A_1), \ldots, N(t, A_n)$ are mutually independent;
Poisson Random Measure

Definition 3 Let \((Z, \mathcal{Z})\) be a measurable space.

A Poisson random measure \(\eta\) on \((Z, \mathcal{Z})\) over \((\Omega, \mathcal{F}, \mathbb{P})\) is a measurable function \(\eta : (\Omega, \mathcal{F}) \rightarrow (\mathcal{M}_I(Z \times \mathbb{R}_+), \mathcal{M}_I(Z \times \mathbb{R}_+))\), such that

(i) for each \(B \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+)\), \(\eta(B) := i_B \circ \eta : \Omega \rightarrow \bar{\mathbb{N}}\) is a Poisson random variable with parameter \(\mathbb{E}\eta(B)\);

(ii) \(\eta\) is independently scattered, i.e. if the sets \(B_j \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+), j = 1, \cdots, n\), are disjoint, then the random variables \(\eta(B_j), j = 1, \cdots, n\), are independent;

(iii) for each \(U \in \mathcal{Z}\), the \(\bar{\mathbb{N}}\)-valued process \((N(t, U))_{t \geq 0}\) defined by

\[
N(t, U) := \eta(U \times (0, t]], \ t \geq 0
\]

is \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted and its increments are independent of the past, i.e. if \(t > s \geq 0\), then \(N(t, U) - N(s, U) = \eta(U \times (s, t])\) is independent of

\[
\mathcal{F}_s = \{\text{collection of all events happened before time } s\}.
\]
Remark 2  *The mapping*

\[ S \ni A \mapsto \nu(A) := \mathbb{E} \eta(A \times [0, 1]) \in \mathbb{R} \]

is a measure on \((Z, \mathcal{Z})\).

Assume \(Z\) is a separable Banach space. Then the integral

\[ L(t) := \int_0^t \int_Z z \tilde{\eta}(dz, ds) \]

is well defined, iff \(\nu\) is a Lévy measure.
Definition 4 (Blumenthal - Getoor - index) For $\alpha \in [0, 2]$ and $\beta \geq 0$ we call an $\mathbb{R}$-valued Lévy process of type $(\alpha, \beta)$, iff

$$\alpha = \inf \left\{ \tilde{\alpha} \geq 0 : \lim_{x \to 0} \nu((x, \infty)) x^{\tilde{\alpha}} < \infty \right\}$$

and

$$\lim_{x \to 0} \nu((-\infty, -x)) x^{\tilde{\alpha}} < \infty$$

$$\beta = \inf \left\{ \tilde{\beta} \geq 0 : \int_{\{|x|>1\}} |x|^{\tilde{\beta}} \nu(dx) < \infty \right\}.$$
The Numerical Approximation of a Lévy walk
The Numerical Approximation of a Lévy walk

Let $\eta$ be a Poisson random measure on $\mathbb{R}$ with intensity $\nu$, i.e. let
$L = \{L(t) : 0 \leq t < \infty\}$ be the corresponding Lévy process with characteristic
measure $\nu$ defined by

$$[0, \infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}} z \tilde{\eta}(dz, ds) \in \mathbb{R}. $$
The Numerical Approximation of a Lévy walk

Let $\eta$ be a Poisson random measure on $\mathbb{R}$ with intensity $\nu$, i.e. let $L = \{L(t) : 0 \leq t < \infty\}$ be the corresponding Lévy process with characteristic measure $\nu$ defined by

$$[0, \infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}} z \tilde{\eta}(dz, ds) \in \mathbb{R}.$$ 

Question:
The Numerical Approximation of a Lévy walk

Let \( \eta \) be a Poisson random measure on \( \mathbb{R} \) with intensity \( \nu \), i.e. let
\[ L = \{L(t) : 0 \leq t < \infty \} \]
be the corresponding Lévy process with characteristic measure \( \nu \) defined by
\[
[0, \infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}} z \tilde{\eta}(dz, ds) \in \mathbb{R}.
\]

**Question:**
Given the characteristic measure of a Lévy process, how to simulate the Lévy walk?

That means, how to simulate
\[
(\Delta^0 L, \Delta^1 L, \Delta^2 L, \ldots, \Delta^k L, \ldots),
\]
where
\[
\Delta^k L := L((k + 1)\tau) - L(k\tau), \quad k \in \mathbb{N}.
\]
The Numerical Approximation of a Lévy walk

Different approaches:
The Numerical Approximation of a Lévy walk

Different approaches:

- Direct generation: in case of stable, exponential, gamma, geometric and negative binomial distribution.
- Generation from compound Poisson Process (Rubenthaler (2003), Amussen and Rósinski, Fornier (2010))
- Generation with shot noise representation (Bondesson (1980), Rosinski (2001), Kawai and Imai (2007))
- Approximation of the density function by series (Eberlein, . . .)
The Numerical Approximation of a Lévy walk

Generation from compound Poisson Process

• cutting off the small jumps at $\epsilon > 0$,

• simulating an exponential distributed random variable $\tilde{T}$ with parameter $\nu([\epsilon, \infty))$,

• simulating $Z = L(\tilde{T}) - L(\tilde{T}^-)$, i.e. simulating a random variable $Z$ with distribution $\nu_\epsilon$ defined by

$$\mathcal{B}(\mathbb{R}) \ni A \mapsto \nu_\epsilon(A) := \frac{\nu(A \cap (-\infty, -\epsilon] \cap [\epsilon, \infty))}{\nu((-\infty, -\epsilon] \cap [\epsilon, \infty))};$$

• optional: simulating the small jumps by a Wiener process.
The Numerical Approximation of a Lévy walk

Generation with shot noise representation

Assume $U$ is a random variable on $[0, T]$ the Lévy measure can be written as

$$\nu(B) = \int_0^\infty \mathbb{P}(H(r, U) \in B) \, dr, \quad B \in \mathcal{B}(\mathbb{R}).$$

- $\{E_k : k \in \mathbb{N}\}$ sequence of iid exponential random variables with unit mean;
- $\{\Gamma_k : k \in \mathbb{N}\}$ sequence of standard Poisson arrivals time, in particular $\{\Gamma_k - \Gamma_{k-1} : k \in \mathbb{N}\}$ are exponential distributed with certain parameter;
- $\{U_k : k \in \mathbb{N}\}$ sequence of iid uniform distributed random variables on $[0, 1]$.
The Numerical Approximation

Our suggestions:
(Hausenblas and Marchis (2007), Hausenblas, Prohl and Dunst (2010))

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_L L_{\epsilon}$ which approximates $\Delta_k L_{\epsilon}$;
- Approximating the small jumps by a Wiener process (optional);
The Numerical Approximation of a Lévy walk

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_{\tau}^k L_\epsilon$ which approximates $\Delta_{\tau}^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);
Figure 1: Cutting of the small jumps
The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}_{T}^{k}L_{\epsilon}$ which approximates $\Delta_{T}^{k}L_{\epsilon}$;
- Approximating the small jumps by a Wiener process (optional);
Approximating the characteristic measure by a discrete measure

Approximate the characteristic measure $\nu^{ce}$ by a characteristic measure $\hat{\nu}^{ce}$ having the following form:

- Let $D_1, \ldots, D_J$ be disjoint sets with $\bigcup_{j=1}^{J} D_j \subset (-\infty, -\epsilon] \cup [\epsilon, \infty)$;

- let $c_1, \ldots, c_J$ be points belonging to $\mathbb{R}$ such that $c_j \in D_j$, $j = 1 \ldots J$;

- $\nu^{ce}$ is approximated as

$$\hat{\nu}^{ce} := \sum_{j=1}^{J} \nu(D_j) \delta_{c_j}.$$ 

**Example 1**

- Put $D_j = [x_j, x_{j+1})$, $j = 1, \ldots, J/2 - 1$,

  $D_{J/2} = [x_{J/2}, \infty)$, $D_{J/2+j} = [-x_j, -x_{j+1})$, $j = 1, \ldots, J/2 - 1$, such that $\nu(B_j) < \gamma$;

- Put $D_J = [-x_{J-1}, -\infty)$, where $x_0 = \epsilon < x_1 < \cdots < x_{J/2} < \infty$;

- Put $c_j := x_j$. 

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The Numerical Approximation of a Lévy walk

\[ \hat{\nu}^{ce} := \sum_{j=1}^{J} \nu(D_j) \delta_{c_j} \]

Figure 2: Discretization of the Lévy measure
The Numerical Approximation

Different Discretization:

- **equally weighted mesh**: Let $D_1 = [\epsilon, x_2)$, $D_J = [x_J, \infty)$, and $D_i = [x_i, x_{i+1})$ such that $\nu(D_i) = \nu(D_j)$ for $1 \leq i, j \leq J \sim O(\epsilon^{-1})$.

- **equally spaced mesh**: Let $D_1 = [\epsilon, x_2)$, $D_J = [x_J, \infty)$, and $D_i = [x_i, x_{i+1})$ such that $\lambda(D_i) = \lambda(D_j)$ for $1 \leq i, j \leq J \sim O(\epsilon^{-1})$.

- **stretched mesh**: Let $D_1 = [\epsilon^2, x_2)$, $D_J = [x_J, \infty)$, and choose $\epsilon_j := \lambda(D_j)$ according to

  $\epsilon_j := \begin{cases} 
  j \epsilon^2 & \text{if } x_j < 1, \\
  j \gamma \epsilon & \text{if } x_j \geq 1,
  \end{cases}$

for some $0 < \gamma < 1$.

(Here, one can fit $2$ and $\gamma$ according to $\alpha$ and $\beta$)
The Numerical Approximation

The Errors: In general, the error in $L^p$-mean is given by

$$Err_\epsilon := \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} |\xi|^p (\nu - \hat{\nu}^\epsilon)(d\xi).$$

In particular, we have for the

- equally weighted mesh: $Err_\epsilon \leq C \epsilon^{\frac{1-\alpha}{\beta}} (\beta-p)$ for $0 \leq \alpha < 1$,
- equally spaced mesh: $Err_\epsilon \leq C \max(\epsilon^{p-\alpha}, \epsilon)$
- stretched mesh: $Err_\epsilon \leq C \max(\epsilon^{2(p-\alpha)}, \epsilon^{1-2\gamma}, \epsilon^{\gamma(\beta-p)})$
The Numerical Approximation

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}^k_T L_\epsilon$ which approximates $\Delta^k_T L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional);
Generating a random variable $\hat{\Delta}_{\tau}^{k}L_{\epsilon}$

Remember

- (Independently scattered property)

The family

$$\{\eta(D_j \times [t_{k-1}, t_k)) : j = 1, \ldots, n, k = 1, \ldots, K\}$$

of random variables over $(\Omega, \mathcal{F}, \mathbb{P})$ is mutually independent;

- for any $D \times I \in \{D_j \times [t_{k-1}, t_k) : j = 1, \ldots, n, k = 1, \ldots, K\}$ the random variable

$$\eta(D \times I)$$

is Poisson distributed with Parameter $\nu(D) \cdot \lambda(I)$. 
Generating a random variable $\hat{\Delta}^k_{\tau}L_\epsilon$

At each time step $k$ generate a family $\{q^1_k, \ldots, q^J_k\}$ of independent random variables, where we have for each $j = 1, \ldots, J$

$$\mathbb{P}(q^j_k = l) = \exp(-\tau \nu(D_j)) \frac{\tau^l \nu(D_j)^l}{l!}, \quad l \geq 0.$$  

The approximation $\hat{\Delta}^k_{\tau}L$ will be given as follows.

$$\hat{\Delta}^k_{\tau}L_\epsilon := \sum_{j=1}^{J} \left( q^j_k - \tau \nu(D_j) \right) c_j, \quad k \geq 0.$$  

The Lévy random walk is approximated by

$$\left( \hat{\Delta}^0_{\tau}L_\epsilon, \hat{\Delta}^1_{\tau}L_\epsilon, \hat{\Delta}^2_{\tau}L_\epsilon, \ldots, \hat{\Delta}^k_{\tau}L_\epsilon, \ldots \right).$$
Simulation des Lévy walks

- 3D histogram of total number of jumps over time and jump size.
- Line graph showing increments over time.
- 3D histogram of episodic jumps and spring height.
- Line graph showing episodic jumps over time.
At each time step $k \geq 1$ is it sufficient to simulate a family $\{q_{k1}, \ldots, q_{kJ_k}\}$ of independent random variables, where

\[
\mathbb{P}\left(q_{jk}^j = 0\right) = \exp(-\tau \nu(D_j)) \quad j = 1, \ldots, k
\]

and

\[
\mathbb{P}\left(q_{jk}^j = 1\right) = 1 - \exp(-\tau \nu(D_j)) \quad j = 1, \ldots, k.
\]

The approximation is now given as follows.

\[
\hat{\Delta}_\tau^k L_\epsilon := \sum_{j=1}^{J} \left(q_{jk}^j - \tau \nu(D_j)\right) c_j, \quad k \geq 0.
\]

The Lévy random walk is approximated by

\[
\left(\hat{\Delta}^0_{\tau} L_\epsilon, \hat{\Delta}^1_{\tau} L_\epsilon, \hat{\Delta}^2_{\tau} L_\epsilon, \ldots, \hat{\Delta}^k_{\tau} L_\epsilon, \ldots\right).
\]
The Numerical Approximation of a Lévy walk

- Cutting off the small jumps;
- Approximating the characteristic measure by a discrete measure;
- Generating a random variable $\hat{\Delta}^k L_\epsilon$ which approximates $\Delta^k L_\epsilon$;
- Approximating the small jumps by a Wiener process (optional).
Approximating the small jumps (optional);

At each time step $k$ generate a Gaussian random variables $\Delta^k T W$, where

$$\mathcal{L}(\Delta^k T W_\epsilon) = \sigma(\epsilon) \mathcal{N}(0, \tau)$$

and

$$\sigma(\epsilon) := \sqrt{\int^{\epsilon}_{-\epsilon} \zeta^2 \nu(d\zeta)}.$$

The Lévy random walk is approximated by

$$\left( \hat{\Delta}^0 T L_\epsilon + \Delta^0 T W_\epsilon, \hat{\Delta}^1 T L_\epsilon + \Delta^1 T W_\epsilon, \hat{\Delta}^2 T L_\epsilon + \Delta^2 T W_\epsilon, \ldots, \hat{\Delta}^k T L_\epsilon + \Delta^k T W_\epsilon, \ldots \right).$$
Simulation des Lévy walks

Assume, that the intensity is of type $\alpha$, $\alpha \in (0, 2]$, the time-step size $\tau > 0$ and that the truncation parameter $\epsilon > 0$ satisfies $\epsilon \leq C \tau^{1/\alpha}$. Then for all $p > 0$

$$\mathbb{E} \left| \Delta^k_L^\epsilon - \Delta^k W^\epsilon \right|^p \leq C \tau^{\frac{p}{\alpha}} \epsilon^{p - \frac{p}{2} \alpha}, \quad k \in \mathbb{N}$$

where $\Delta^k_L^\epsilon := \Delta^k_L - \Delta^k L^\epsilon$. 
Main ingredients of the proof:

- For any random variable $X$, we have

$$
\mathbb{E}|X|^p \leq C \int_0^\infty \mathbb{P}(|X| \geq x) x^{p-1} dx;
$$
Simulation des Lévy walks

Main ingredients of the proof:

- For any random variable $X$, we have

$$
\mathbb{E}|X|^p \leq C \int_0^\infty \mathbb{P}(|X| \geq x) \ x^{p-1} \ dx;
$$

- For any random variable $X$ with Fourier transform $\hat{X}$, we have

$$
\mathbb{P}(|X| \geq x) \leq \frac{x}{2} \int_{-\frac{2}{x}}^{\frac{2}{x}} \left( 1 - \mathcal{R} \left( \hat{X}(\lambda) \right) \right) \ d\lambda.
$$
Simulation des Lévy walks

Main ingredients of the proof:

- For any random variable $X$, we have

$$\mathbb{E} |X|^p \leq C \int_0^\infty \mathbb{P} (|X| \geq x) x^{p-1} \, dx;$$

- For any random variable $X$ with Fourier transform $\hat{X}$, we have

$$\mathbb{P} (|X| \geq x) \leq \frac{x}{2} \int_{-\frac{2}{x}}^{\frac{2}{x}} \left( 1 - \Re \left( \hat{X}(\lambda) \right) \right) \, d\lambda.$$

- Therefore we have to give an estimate of

$$\mathbb{E} e^{i\lambda (\Delta^k_{\tau} L_{\epsilon}^c - \Delta^k_{\tau} W)} = \mathbb{E} e^{i\lambda \Delta^k_{\tau} L_{\epsilon}^c} \mathbb{E} e^{-i\lambda \Delta^k_{\tau} W}$$

$$= \exp \left( \tau \int_{\mathbb{R}} (e^{iy\lambda} - 1 - iy\lambda) \nu_{\epsilon}(dy) - \tau\sigma^2(\epsilon)\lambda^2 \right). \quad (-6)$$
Simulation des Lévy walks

Main ingredients of the proof:

- For any random variable \( X \), we have
  \[
  \mathbb{E}|X|^p \leq C \int_0^{\infty} \mathbb{P}(|X| \geq x) x^{p-1} \, dx;
  \]

- For any random variable \( X \) with Fourier transform \( \hat{X} \), we have
  \[
  \mathbb{P}(|X| \geq x) \leq \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left( 1 - \Re \left( \hat{X}(\lambda) \right) \right) d\lambda.
  \]

- Therefore we have to give an estimate of
  \[
  \mathbb{E}e^{i\lambda(\Delta^k_{\tau}L^c_\epsilon - \Delta^k_{\tau}W)} = \mathbb{E}e^{i\lambda\Delta^k_{\tau}L^c_\epsilon} \mathbb{E}e^{-i\lambda\Delta^k_{\tau}W}
  \]
  \[
  = \exp \left( \tau \int_{\mathbb{R}} (e^{iy\lambda} - 1 - iy\lambda) \nu_\epsilon(dy) - \tau \sigma^2(\epsilon)\lambda^2 \right).
  \]  
  \hspace{2cm} (-9)

- Taylor expansion of the RHS
Let us recall the Definition of a Gaussian white noise (Dalang 2002):

**Definition 4** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and \((E, \mathcal{E}, \sigma)\) a measure space. Then a **Gaussian white noise on** \(E\) **based on** \(\sigma\) **is a measurable mapping**

\[
W : (\Omega, \mathcal{F}) \rightarrow (M(E), \mathcal{M}(E))
\]

- **for any** \(A \in \mathcal{E}\), \(W(A)\) **is a Gaussian random variable with mean** 0 **and variance** \(\sigma(A)\), **provided** \(\sigma(A) < \infty\);

- **for any** \(A, B \in \mathcal{E}\), **if** \(A\) **and** \(B\) **are disjoint**, then the random variables \(W(A)\) **and** \(W(B)\) **are independent and** \(W(A \cup B) = W(A) + W(B)\).

\(M(E)\) **denotes the set of all measures from** \(\mathcal{B}(E)\) **into** \(\mathbb{R}\), **i.e.**

\[
M(E) := \{\nu : \mathcal{B}(E) \rightarrow \mathbb{R}\}
\]

and \(\mathcal{M}(E)\) **is the** \(\sigma\)-**field on** \(M(E)\) **generated by functions** \(i_B : M(E) \ni \eta \mapsto \eta(B) \in \mathbb{R}, B \in \mathcal{E}\).
Put

- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $E = \mathcal{O} \times [0, \infty)$,
- $\mathcal{E} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty))$
- $\sigma = \lambda_{d+1}$.

The measure valued process

\[ t \mapsto W(\cdot \times [0, t)) \]

defines a space time Gaussian white noise.

$\lambda_{d+1}$ denotes the Lebesgue measure in $\mathbb{R}^d$. 
Space - Time - White - Noise
Definition 4 Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \((E, \mathcal{E}, \sigma)\) be a measurable space. Then the \textit{Lévy white noise on} \(E\) \textit{based on} \(\sigma\) \textit{with characteristic jump size measure} \(\nu \in \mathcal{L}(\mathbb{R})\) \textit{is a measurable mapping}

\[ L : (\Omega, \mathcal{F}) \rightarrow (\mathcal{M}(E), \mathcal{M}(E)) \]

such that

- for any \(A \in \mathcal{E}\), \(L(A)\) is an infinite divisible random variables with characteristic exponent
  \[ e^{i\theta L(A)} = \exp\left(\sigma(A) \int_{\mathbb{R}} \left(1 - e^{i\theta x} - i \sin(\theta x)\right) \nu(dx)\right), \]
  provided \(\sigma(A) < \infty\).

- for any \(A, B \in \mathcal{E}\), if \(A\) and \(B\) are disjoint, then the random variables \(L(A)\) and \(L(B)\) are independent and
  \[ L(A \cup B) = L(A) + L(B). \]

\(\mathcal{M}(E)\) denotes the set of all measures from \(\mathcal{B}(E)\) into \(\mathbb{R}\),

i.e. \(\mathcal{M}(E) := \{\nu : \mathcal{B}(E) \rightarrow \mathbb{R}\}\) and \(\mathcal{M}(E)\) is the \(\sigma\)-field on \(\mathcal{M}(E)\) generated by functions \(i_B : \mathcal{M}(E) \ni \eta \mapsto \eta(B) \in \mathbb{R}, B \in \mathcal{E}\).
Again put

- \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain with smooth boundary.
- \( E = \mathcal{O} \times [0, \infty) \),
- \( \mathcal{E} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty)) \)
- \( \sigma = \lambda_{d+1} \).

The measure valued process

\[
t \mapsto L(\cdot \times [0, t))
\]

defines a space time Lévy white noise, for more details we refer to Peszat and Zabczyk (2007), or Brzezniak and Hausenblas (2010).
The Approximation of a space time Lévy noise

Let $\mathcal{O}$ be a bounded domain and let $\mathcal{T}$ be a subdivision with

- a set of element domains $\mathcal{K} = \{K_i : i = 1 \ldots, l\}$,
- a set of nodals variable $\mathcal{N} = \{N_i : i = 1 \ldots, l\}$ and
- a set of space functions $\mathcal{P} = \{\phi_i : i = 1, \ldots, l\}$
- a Voronoi decomposition $\mathcal{J} = \{J_i : i = 1, \ldots, l\}$ induced by $\mathcal{N}$

Let $\mathcal{L} = \{L^i : i = 1, \ldots l\}$ be a family of independent Lévy processes where $L^i$ has characteristic $\rho_i \nu$, $\rho_i = \lambda(J_i)/h^d$, $i = 1, \ldots, l$. The measure valued process $\tilde{\mathcal{L}} = \{\tilde{\mathcal{L}}(t) : t \geq 0\}$ given by

$$\tilde{\mathcal{L}}(t) : \mathcal{B}(\mathcal{O}) \ni A \mapsto \sum_{i=1}^{N} \rho_i \int_{A} \phi_i(x) \, dx \, L^i(t), \quad t \geq 0,$$

is an approximation in space of the space time Lévy noise.
Space - Time - White - Noise

\[ N_i \]
Space - Time - White - Noise
Space - Time - Poissonian Noise
Our Example

The Equation:

\[
\begin{cases}
\frac{du(t, \xi)}{dt} = \Delta u(t, \xi) + \alpha \nabla u(t, \xi) + g(u(t, \xi)) \dot{L}(t) \\
+ f(u(t, \xi)), \quad \xi \in \mathcal{O}, \ t > 0; \\
u(0, \xi) = u_0(\xi) \quad \xi \in \mathcal{O}; \\
u(t, \xi) = 0, \quad t \geq 0, \ \xi \in \partial \mathcal{O};
\end{cases}
\]

where \(u_0 \in L^p(0, 1), \ p \geq 1, \) \(g\) a certain mapping and \(L(t)\) is a Lévy process taking values in a certain Banach space \(Z\).

A mild solution of Equation \((\ast)\) is an adapted Banach-valued càdlàg process \(u = \{u(t) : t \in [0, T]\}\) such that for \(t \geq 0\) we have a.s.

\[
u(t) = S(t)u_0 + \int_0^t S(t - s) f(u(s)) \, dt + \int_0^t \int_Z S(t - s)g(u(s); z) \, \tilde{\eta}(ds, dz)
\]

\(\{S(t) : t > 0\}\) denotes the semigroup generated by the Laplace operator \(\Delta\) with the given boundary conditions.
The Numerical Approximation
Numeric of SPDEs

The implicit Euler scheme. Here

\[
\frac{u_n(t + \tau_n) - u_n(t)}{\tau_n} \approx A u_n(t + \tau_n) \\
f(u_n(t)) + g(u_n(t)) \Delta_n L(t),
\]

Again, if \( v_n^k \) denotes the approximation of \( u_n(k\tau_n) \), then

\[
\begin{align*}
    v_n^{k+1} &= (1 - \tau_n A_n)^{-1} v_n^k + \tau_n f(v_n^k) \\
            &\quad + \sqrt{\tau_n} g(u_n(t)) \xi_k^n,
\end{align*}
\]

\[
v_n^0 = x.
\]

Between the points \( k\tau_n \) and \( (k + 1)\tau_n \) the solution is linear interpolated.
Theorem 1  (E.H. 2008)

• Let \( u = \{ u(t) : 0 \leq t < \infty \} \) be the mild solution of an SPDE with space time Lévy noise

• Let \( \hat{u} = \{ \hat{u}_k : k \in \mathbb{N} \} \) be the approximation of \( u \)
  - Discretization in space: affine finite Elements with length \( h \);
  - Discretization in time: implicit Euler scheme with time step \( \tau_h \).

Then, the error is given by

\[
\left[ \mathbb{E} \left\| u(k \tau_h) - \hat{u}_h^k \right\|_{L^p(\mathcal{O})}^p \right]^{\frac{1}{p}} \leq C_0 \tau_h^\alpha + C_1 h^\delta.
\]

where \( \alpha < \frac{2}{p} - d + \frac{d}{p} \) and \( \delta < \left( 2 + \frac{d}{2} - \frac{d}{p} \right) \left( \frac{2}{p} - d + \frac{d}{p} \right) \).
Theorem 2 (EH, 2005 EJP) Assume that there exist some $\delta_g < \frac{1}{p}$ and $\delta_f, \delta_I < 1$ such that

- $u_0$ satisfies $\mathbb{E}|(-A)^{-\delta_I}u_0|^p < \infty$;
- $(-A)^{-\delta_f}f : E \to E$ is Lipschitz continuous;
- $(-A)^{-\delta_g}g : E \to L(Z, E)$ satisfies
  \[ \int_Z |(-A)^{-\delta_g}[g(x, z) - g(y, z)]|^p\nu(dz) \leq C|x - y|^p_E, \quad x, y \in E. \]

Then, there exists a unique mild solution to Problem (1), such that for any $T > 0$

\[ \int_0^T \mathbb{E}|u(s)|_E^p \, ds < \infty, \]

and $(-A)^{-\gamma}u \in L^0(\Omega; \mathbb{D}([0, T]; E))$, where $\gamma > \frac{1}{p}$. 
The Numerical Approximation

(Dunst, Prohl, E.H. (2010)) Assume in addition that the space time Lévy white noise is approximated as described before. Then the following holds.

- For any $M \in \mathbb{N}$ there exist constants $C_1$ and $C_2$ such that

$$\sup_{0 \leq m \leq M} \mathbb{E} \left\| v_h^m - \hat{v}_{\epsilon_h}^m \right\|_{L^p(\mathcal{O})}^p \leq C_1 \epsilon_h^{p-\alpha}$$

$$+ |\Xi| \frac{2}{p} \frac{d}{p} d + \frac{d}{p} C_2 \int_{\zeta \in \mathbb{R} \setminus (-\epsilon_h, \epsilon_h)} |\zeta|^p d(\hat{\nu}^{\epsilon_h} - \nu^{\epsilon_h})(d\zeta).$$

- If the stability condition ($\star$) is satisfied, then there exist constants $C_1$ and $C_2$ such that for any $\theta > 0$, and $\epsilon_h$ with $\epsilon_h \sim \tau_h^\theta$,

$$\sup_{0 \leq m \leq M} \mathbb{E} \left\| v_h^m - \hat{v}_{\epsilon_h}^w,m \right\|_{L^p(\mathcal{O})}^p \leq C_1 \tau_h^{\frac{2}{3}} \epsilon_h^{\frac{p(3-\alpha)}{3}}$$

$$+ C_2 \int_{\zeta \in \mathbb{R} \setminus (-\epsilon_h, \epsilon_h)} |\zeta|^p d(\hat{\nu}^{\epsilon_h,c} - \nu^{\epsilon_h})(d\zeta).$$
The Numerical Approximation

The stability condition (⋆):

There exists a $\gamma < \frac{1}{p}$ such that

$$h^{d+2\gamma+d\frac{p}{2}-dp} \tau_h^p \sigma(\epsilon_h)^p \sim h^d \tau_h, \quad h \in (0, 1].$$

(-9)
Numerical Experiments
Numerical Experiments
Numerical Experiments
the End