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A Sylowlike theorem for integral group rings of finite solvable groups

By

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1. Introduction. For a finite group G and a commutative ring R we denote by

$$RG = \left\{ \sum_{g \in G} r_g \cdot g \right\}$$

the *group ring* of G over R. This group ring is an *augmented algebra* with

$$\textit{augmentation}\ \varepsilon\colon\ RG \to R,\ \sum_{g \in G} r_g \cdot g \to \sum_{g \in G} r_g.$$

By $V(RG)$ we denote the units in RG, which have augmentation 1. The group of units in RG is then the product of the units in R and $V(RG)$.

A subgroup H of $V(RG)$ with $|H| = |G|$ is called a *group basis*, provided the elements of H are linearly independent. This latter condition is automatic, provided no rational prime divisor of $|H|$ is a unit in R [1]. If H is a group basis, then $RG = RH$ as augmented algebras and conversely.

The object of this note is to prove the following

Theorem 1. *Let G be a finite solvable group, and let H be a group basis of $\mathbb{Z}G$ with Sylow p-subgroup P. Then there exists a unit $a \in \mathbb{Q}G$ such that aPa^{-1} is a Sylow p-subgroup of G.*

Remark 1. For solvable groups it was conjectured by Hans Zassenhaus [12, 11] that for any finite subgroup U of $V(\mathbb{Z}G)$ there exists $a \in \mathbb{Q}G$ with $aUa^{-1} \subset G$.

It is known that for a solvable group G, the Sylow p-subgroups of different group bases in $\mathbb{Z}G$ are isomorphic; however, the above result gives information about the embedding of these Sylow p-subgroups into $\mathbb{Z}G$.

The isomorphism of the Sylow p-subgroups is an immediate consequence of the following more general result: ($\hat{\mathbb{Z}}_p$ stands for the complete ring of p-adic integers.)

Theorem 2 ([9]). *Let G be a finite group such that the generalized Fitting subgroup $F^*(G)$ is a p-group* [1]*). Then a group basis H of $\mathbb{Z}G$ is conjugate by a unit in $\hat{\mathbb{Z}}_pG$ to a subgroup of G.*

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[1]) This is to say that G has a normal p-subgroup N with the centralizier $C_G(N) \subset N$ or that the generalized p'-core $O_{p^*}(G)$ is trivial [2, 3].

We shall state next a more general result, which does not only apply to solvable groups, and of which Theorem 1 is a special case – as will become transparent later on. For this we have to introduce some more *notation.*

Definition 1. Let G be a finite group.

1. $\pi(G)$ is the set of rational prime divisors of $|G|$.
2. For the rational prime p, the group $O_{p'}(G)$ is the largest normal subgroup of G with order relatively prime to p.
3. $O_{p^*}(G)$ is the generalized p'-core of G [4, Ch. X, Paragraph 14].
4. Let π be a finite set of rational primes. We call a finite group G *π-constrained*, if for each $q \in \pi$ there exists a rational prime p such that $O_{p^*}(G/O_{p'}(G)) = 1$ and q does not divide $|O_{p'}(G)|$.

Remark 2. Note that in the above definition, the prime q need not be different from p. Therefore a p-constrained group G is also π-constrained for $\pi = \pi(G)\setminus\pi(O_{p'}(G))$. Clearly a finite solvable group is π-constrained for every set of primes π. However, there are many insolvable groups which are π-constrained for some set π (e.g. every Frobenius group is π-constrained for a suitable set π). It is not true though, that a π-constrained group G is p-constrained for every $p \in \pi$ [2, 3].

We can now state the result, which we shall prove here:

Theorem 3. *Let G be a finite π-constrained group, and let H be a group basis in $\mathbb{Z}G$. For each $p \in \pi$ and P a Sylow p-subgroup of H, there exists a unit $a \in \mathbb{Q}G$ with aPa^{-1} a Sylow p-subgroup of G.*

2. Connection with the Zassenhaus conjecture. Let us return to a weak form of the Zassenhaus conjecture (cf. Remark 1):

Conjecture 1 (Zassenhaus [12, 11]). *Let G be a finite group. If H is a group basis in $\mathbb{Z}G$, then H is conjugate in $\mathbb{Q}G$ to G; i.e. there exists a unit $a \in \mathbb{Q}G$ such that $aHa^{-1} = G$.*

Remark 3. It was shown in [7] that the above conjecture is true for finite nilpotent groups. However, in [8] a metabelian group was constructed, which is a counterexample to the above Zassenhaus conjecture.

It is convenient, to rephrase the Zassenhaus conjecture in terms of *isomorphisms over class sums.*

Definition 2. Let G be a finite group.

1. A *class sum* in $\mathbb{Z}G$ is an element of the form

$$CS_G(g) = \sum_{x \in G/C_G(g)} {}^x g;$$

i.e. the sum of the different conjugate elements of g.

2. Let H be a group basis in $\mathbb{Z}G$. Then there is a *class sum correspondence* [1]: For every $h \in H$ there exists an element $\gamma(h) \in G$, such that $CS_H(h) = CS_G(\gamma(h))$ *in* $\mathbb{Z}G$. Note that $\gamma(h)$ is only determined up to conjugacy. Since the conjugacy class of h and $\gamma(h)$ must have

the same cardinality – use the augmentation – the map γ can be extended to a

bijection $\gamma\colon G \to H$.

We shall call such a map a *class sum correspondence*. Note that γ is in general not unique and is in general not a homomorphism of groups; however, it sends p-power elements of G to p-power elements of H; it even preserves the order of the elements [6].

3. This class sum correspondence induces a correspondence between the normal subgroups of G and H, essentially since a normal subgroup is a union of conjugacy classes, cf. e.g. [10].

4. Let H be a group basis in $\mathbb{Z}G$. An isomorphism $\varrho\colon H \to G$ is called an *isomorphism over the class sums* provided the induced automorphism – note $\mathbb{Z}G = \mathbb{Z}H$ – which we shall also denote by ϱ

$$\varrho\colon \mathbb{Z}H \to \mathbb{Z}G,\ \sum_{h\in H} r_h \cdot h \to \sum_{h\in H} r_h \cdot \varrho(h)$$

has the property $\varrho(CS_H(h)) = CS_G(\gamma(h))$.

We can now reformulate the Zassenhaus conjecture – using the theorem of Skolem-Noether:

Proposition 1. *The Zassenhaus conjecture is equivalent to the statement that for each group basis H of $\mathbb{Z}G$ there exists an isomorphism*

$$\varrho\colon H \to G$$

– this means that the isomorphism problem has a positive answer – which is an isomorphism over the class sums; with other words the above bijection

$$\gamma\colon H \to G$$

can be chosen to be a group isomorphism.

Remark 4. We shall collect here some observations:

1. Theorem 2 thus states, that in case $F^*(G)$ is a p-group, then for every group basis H there exists an isomorphism over the class sums.

2. In our Theorems 1, 3 we are not dealing with the group basis, but rather with a subgroup of a group basis H. Thus we are looking for an extension of Proposition 1 to a subgroup U of the group basis H (cf. Remark 1).

3. The obvious extension would be to require that the bijection γ in the Definition 2,2 could be chosen in such a way that it is a group isomorphism when restricted to U.

Theorem 4. *Let G be a finite group and let U be a finite subgroup of $V(\mathbb{C}G)$. Denote by L an algebraic number field such that $U \subset LG$. Then the following statements are equivalent.*

1. *There exists a unit $a \in LG$ with $aUa^{-1} \subset G$.*
2. *There exists a group basis H of $\mathbb{C}G$, and there exists a bijection*

$$\varrho\colon H \to G,$$

such that

$$\varrho_{|U}: U \to \varrho(U)$$

is a class sum preserving group isomorphism; i.e.

$$CS_H(u) = CS_G(\varrho(u))$$

for every $u \in U$. *Moreover,*

$$|CS_H(u)| = |CS_G(\varrho(u))|,$$

here $|CS_G(g)| = |G : C_G(g)|$ *denotes the number of elements conjugate to* g.

3. The Proofs.

Proof of Theorem 4. (1) ⇒ (2): If we take $H = a^{-1}Ga$, then the conjugation by a is the desired map ϱ.

(2) ⇒ (1): Let $L \subset K$ be an algebraic number field, which is a splitting field for G and choose a simple Wedderburn component A of $KG = KH$.

Via the projection onto A we obtain two representations of U, denoted by ϕ_U and $\phi_{\varrho(U)}$ resp., where ϕ_U is the representation of $U \subset H$ and $\phi_{\varrho(U)}$ is the representation of U induced from ϱ.

We shall show that the characters for U of ϕ_U and $\phi_{\varrho(U)}$ coincide. In fact, by assumption $CS_H(u) = CS_G(\varrho(u))$ and so we have for the trace of ϕ_U and $\phi_{\varrho(U)}$ resp. with $l = |CS_H(u)| = |CS_G(\varrho(u))|$:

$$\begin{aligned}\mathrm{tr}_{\phi_U}(u) &= l^{-1} \cdot (l \cdot \mathrm{tr}_{\phi_U}(u)) = l^{-1} \cdot (\mathrm{tr}_{\phi_U}(CS_H(u)) = l^{-1} \cdot (\mathrm{tr}_{\phi_\varrho(U)}(CS_G(\varrho(u)) \\ &= \mathrm{tr}_{\phi_\varrho(U)}(\varrho(u)).\end{aligned}$$

This holds for every $u \in U$, and since the characters determine a representation up to isomorphism (conjugacy), we conclude, that ϕ_U and $\phi_{\varrho(U)}$ are conjugate in A. Since this can be done for every simple Wedderburn component of KG, we conclude that there exists $b \in KG$ such that $bUb^{-1} = \varrho(U)$.

It remains to show that this conjugation can already be achieved in LG. We shall be using bimodules to reach this goal:

We consider $M = LG$ as $L(U \times G)$-bimodule, by letting U act in its natural way on M from the left and G acts on the right by its natural action. ${}^{\varrho}M$ has the same right action as M, but the left action is twisted by ϱ:

$$u \cdot_{\varrho} m = \varrho(u) \cdot m$$

Since U and $\varrho(U)$ are conjugate in KG, the bimodules

$$K \otimes_L M \quad \text{and} \quad K \otimes_L {}^{\varrho}M$$

are isomorphic. Invoking the Noether-Deuring theorem, we conclude that the bimodules M and ${}^{\varrho}M$ must be isomorphic. Let

$$M \to {}^{\varrho}M$$

be an isomorphism of $L(U \times G)$-bimodules. We put $a = \tau(1)$. Then a is a unit in LG and moreover,

$$\varrho(u) \cdot a = a \cdot u$$

for every $u \in U$. q.e.d.

The *proof of Theorem* 3 will now follow from Theorem 4, if we can show

Proposition 2. *Let G be a finite π-constrained group for π a finite set of rational primes. H is a group basis in $\mathbb{Z}G$.*

For $q \in \pi$ there exists by Definition 1,4 a prime p such that

$$O_{p^*}(G/O_{p'}(G)) = 1.$$

Let S be a Sylow q-subgroup of H. Then there exists a class sum correspondence

$$\varrho \colon H \to G$$

such that

$$\varrho_S \colon S \to \varrho(S)$$

is a group isomorphism.

Proof. Let

$$\kappa \colon \mathbb{Z}G \to \mathbb{Z}G/O_{p'}(G)$$

be the augmented ring homomorphism induced from reduction modulo $O_{p'}(G)$.

Since G is π-constrained, q does not divide $|O_{p'}(G)|$, and so

$$\kappa_{|S} \text{ injects } S \text{ into } \mathbb{Z}G/O_{p'}(G).$$

By the choice of p, we may apply Theorem 2, to conclude that the Zassenhaus conjecture holds for $\mathbb{Z}G/O_{p'}(G)$, and so there exists a class sum correspondence in

$$\mathbb{Z}(\kappa(G)) = \mathbb{Z}(\kappa(H)),$$

inducing an isomorphism of groups

$$\bar{\varrho} \colon \kappa(H) \to \kappa(G).$$

With the correspondence of normal subgroups (Definition 2,3) we conclude that

$$\ker(\kappa_{|H}) = O_{p'}(H)$$

and that

$$|O_{p'}(H)| = |O_{p'}(G)|.$$

Thus we can find a Sylow q-subgroup of G, say, T such that

$$\bar{\varrho}_{|\kappa(S)} \colon \kappa(S) \to \kappa(T)$$

is a group isomorphism.

Summarizing, we have now constructed a group isomorphism

$$\varrho_S =: \kappa_{|T}^{-1} \circ \bar{\varrho} \circ \kappa_{|S}$$

from S to T.

Claim 1. *Let now*

$$\gamma \colon H \to G$$

be a class sum correspondence (Definition 2,2). Then

$$CS_G(\gamma(s)) = CS_G \quad {}_S\, s \quad .$$

Proof of the claim. Because of the class sum correspondence γ, there exists for every $s \in S$ an element $t \in T$ such that

$$CS_H(s) = CS_G(t)$$

– note that γ sends q-power elements to q-power elements (Definition 2,2).

On the other hand, $\bar{\varrho}$ induces the class sum correspondence on $\mathbb{Z}G/O_{p'}(G)$, and so we must have

$$CS_{G/O_{p'}(G)}(\kappa(t)) = CS_{G/O_{p'}(G)}(\bar{\varrho} \circ \kappa(s)).$$

Thus t is conjugate in G to a q-power element of the form $w \cdot \varrho_S(s)$ for some $w \in O_{p'}(G)$. Note that we still have freedom in choosing t in its conjugacy class. Thus we can assume that t is such that $\kappa(t) = \kappa(\varrho(s))$. In $O_{p'}(G) \cdot T$ the element $w \cdot \varrho_S(s)$ is – by Sylow's theorem – conjugate by an element $w_1 \in O_{p'}(G)$ to an element $t_1 \in T$. But then $\kappa(t) = \kappa(t_1)$ and so we must have $t = t_1$, since $\kappa_{|T}$ is injective.

Consequently $\varrho_S(s)$ and t are conjugate.

This proves the claim and also finishes the proof of Proposition 2, and hence completes the proof of Theorem 3 and consequently of Theorem 1.

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