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The finiteness obstruction for loop spaces

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Abstract. For finitely dominated spaces, Wall constructed a finiteness obstruction, which decides whether a space is equivalent to a finite CW-complex or not. It was conjectured that this finiteness obstruction always vanishes for quasi finite H-spaces, that are H-spaces whose homology looks like the homology of a finite CW-complex. In this paper we prove this conjecture for loop spaces. In particular, this shows that every quasi finite loop space is actually homotopy equivalent to a finite CW-complex.

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1. Introduction

A topological space is called quasi-finite, if the direct sum $\bigoplus_n H_n(X;\mathbb{Z})$ of all integral homology groups is a finitely generated abelian group. Then you can ask whether the space X is (weakly) homotopy equivalent to a finite CW-complex. For finitely dominated spaces, Wall constructed an invariant which decides this problem [29] [30]. A space X is finitely dominated, if there exist a finite CW-complex K and maps $f : K \longrightarrow X$ and $g : X \longrightarrow K$ such that the composition fg is homotopic to the identity id_X on X. One can show that finitely dominated spaces are always homotopy equivalent to CW-complexes and that each component is path connected (e.g. see [17, Section 2]). Examples of finitely dominated spaces are given by quasi-finite nilpotent spaces. Actually, for nilpotent space, both conditions, finitely dominated and quasi-finite are equivalent [15]. A space X is called nilpotent if X is homotopy equivalent to a CW-complex and if $\pi_1(X)$ acts nilpotent on the homotopy groups $\pi_*(X)$. In particular, $\pi_1(X)$ is a nilpotent group. The space X is called simple if, in addition, the action is trivial. In this case, $\pi_1(X)$ is abelian.

For a finitely dominated space X, the finiteness obstruction $\tilde{\omega}(X)$ of Wall is an element in the reduced projective class group $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of the integral group ring. And $\tilde{\omega}(X)$ vanishes if and only if X is homotopy equivalent to a finite CW-complex [29] [30]. It is interesting to investigate the question how topological

properties of spaces are reflected by properties of the finiteness obstruction. For instance, it is not known, but conjectured, that the finiteness obstruction for quasi-finite loop spaces or for H-spaces always vanishes. In this paper we will prove this conjecture for loop spaces.

A loop space L consists of a triple (L, BL, e), where L and BL are of the homotopy type of a CW-complex, BL pointed, and where $e : \Omega BL \to L$ is a homotopy equivalence between the loop space of BL and L. Examples of loop spaces are given by topological groups and their classifying spaces. Loop spaces inherit properties from the space L; e.g. L is called quasi-finite, if the space L is so. Loop spaces are always nilpotent spaces. In fact, they are simple. Thus every quasi-finite loop space is finitely dominated and the finiteness obstruction $\tilde{\omega}(L)$ is defined.

Theorem 1.1. *Let L be a quasi finite loop space. Then $0 = \tilde{\omega}(L) \in \tilde{K}_0(\mathbb{Z}[\pi_1])$ and L is homotopy equivalent to a finite CW-complex.*

For a quasi-finite nilpotent space X, Mislin proved results of this type if $\pi_1(X)$ is infinite [16] or if $\pi_1(X)$ is cyclic of prime power order [15].

The proof of our result is a consequence of the next two statements. For every fibration $F \to E \to B$, the fiber transport along loops in B establishes a homomorphism $\pi_1(B) \to [F, F]$ from the fundamental group of B into the set of homotopy classes of self maps of the fiber F [31, p. 98 ff]. The fibration is called oriented if this homomorphism is trivial.

Theorem 1.2. *Let L be a quasi-finite loop space, such that $\pi_1(L)$ is finite. Let $r := dim_{\mathbb{Q}} H^4(BL, \mathbb{Q})$. Then there exist a semi simple connected compact Lie group G and an oriented fibration*

$$G \to L \to E$$

such that the following holds:
(1) For the universal cover $\tilde{G}$ of G, we have $\tilde{G} \cong SU(2)^r$.
(2) The space E is simple and quasi-finite.

Theorem 1.3. *Let X be a finitely dominated space, such that $\pi_1(X)$ is finite. If, up to homotopy, there exists an oriented fibration*

$$G \to X \to B$$

such that $G \neq \{*\}$ is a connected compact Lie group and B a finitely dominated space, then, the finiteness obstruction $\tilde{\omega}(X)$ vanishes and X is homotopy equivalent to a finite complex.

For principal G-fibrations $X \to B$ Theorem 1.3 is proved in [22]. In the general case, the statement is known and a simple consequence of some vanishing results

of Lück for the algebraic transfer in algebraic K-theory associated to a fibration [14] (see Section 2). We couldn't find a reference for it, but believe that it is worth to be stated.

Proof of Theorem 1.1. If $\pi_1(L)$ is infinite, we can apply the above mentioned result of Mislin [16]. If $\pi(L)$ is finite, the statement follows from Theorem 1.2 and Theorem 1.3, since every quasi finite nilpotent space is finitely dominated [16]. □

Theorem 1.2 is a corollary or a weak version of the next statement.

Theorem 1.4. *Let L be a quasi-finite loop space, such that $\pi_1(L)$ is finite. Let $r := dim_{\mathbb{Q}} H^4(BL, \mathbb{Q})$. Then there exist a semi simple connected compact Lie group G, quasi-finite loop spaces M and N and a fibration*

$$E \longrightarrow BM \longrightarrow BN$$

such that the following holds:
(1) For the universal cover $\tilde{G}$ of G, we have $\tilde{G} \cong SU(2)^r$.
(2) The spaces G and M are homotopy equivalent as well as L and N.
(3) The space E is simple and quasi-finite.

The proof of Theorem 1.4 goes as follows. Since completion turns the quasi-finite loop space L into a p–compact group$(L^\wedge_p, BL^\wedge_p, e^\wedge_p)$ we can use the theory of p–compact groups to construct the above fibration at each prime (see Section 3). Localized at 0, the existence of the fibration is basically a consequence of the fact that the rationalization BL_0 is equivalent to a product of rational Eilenberg-MacLane spaces. Using the arithmetic square we glue all these data together and get the fibration claimed in Theorem 1.4.

In Section 2, we prove Theorem 1.3. The other sections are devoted to the proof of Theorem 1.4. In Section 3, we prove the p-completed version of Theorem 1.4. In Section 4 we recall some material about the arithmetic square and the genus of a space. Section 5 contains the construction of the fibration of Theorem 1.4. The analysis of the genus of those compact Lie groups appearing in Theorem 1.4 is worked out in section 6 and will complete the proof of Theorem 1.4.

We will switch between the p-adic completion of Bousfield and Kan [2] and the p-profinite completion of Sullivan [28]. But, for nilpotent spaces with mod-p homology of finite type, both constructions coincide [2].

It is a pleasure thank Wolfgang Lück for telling me that a result like Theorem 1.2 implies the vanishing of the finiteness obstruction for quasi finite loop spaces and for several helpful discussions on this subject. I am also grateful to the SFB 478 "Geometrische Strukturen in der Mathematik" at Münster for its hospitality when part of this work was done.

2. Proof of Theorem 1.3

For a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ with finitely dominated total space, base and fiber, Ehrlich [11] constructed a geometric transfer

$$p^! : K_0(\mathbb{Z}[\pi_1(B)] \longrightarrow K_0(\mathbb{Z}[\pi_1(E)]$$

for the projective class groups of the integral group rings of the fundamental groups (see also [25]). The map $p^!$ also gives a transfer for the reduced projective class groups. Ehrlich also proved a formula relating the finiteness obstruction of E and B [12], namely

$$\tilde{\omega}(E) = p^!(\tilde{\omega}(B)) + \chi(B) \cdot i_*(\tilde{\omega}(F)) \in \tilde{K}_0(\mathbb{Z}[\pi_1(E)]) \ ,$$

where $\chi(B)$ denotes the Euler characteristic of B and where $i_* : \tilde{K}_0(\mathbb{Z}[\pi_1(F)] \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi_1(E)]$ is the map induced by induction.

In [13] [14], for fibrations of the above type, Lück constructed an algebraic transfer, showed that both coincide and proved the following vanishing result.

Theorem 2.1. (Lück [14, 9.1]) *Let $G \longrightarrow E \longrightarrow B$ be an oriented fibration, such that G is a connected compact Lie group. If $\pi_1(B)$ is finite, then the algebraic transfer*

$$p^* : K_0(\mathbb{Z}[\pi_1(B)] \longrightarrow K_0(\mathbb{Z}[\pi_1(E)]$$

vanishes.

Proof of Theorem 1.3. The compact Lie group G is connected and homotopy equivalent to a finite CW-complex. Hence $\tilde{\omega}(G) = 0$. Moreover, $\pi_1(B)$ is finite since $\pi_1(X)$ is. Thus, the statement is a simple consequence of Theorem 2.1 and the above formula. □

3. Particular subgroups of p–compact groups

A p–compact group X is a loop space $X = (X, BX, e)$ such that BX is p-complete and pointed and such that X is $\mathbb{F}_p$-finite, i.e. $H^*(X; \mathbb{F}_p)$ is finite in each degree and vanishes in almost all degrees. As turned out [6] [7] [20] p–compact groups behave very much like compact Lie groups. In particular, most of the classical notions for compact Lie groups are available in this context; e.g. there exist subgroups, maximal tori and Weyl groups with the same properties and we can speak of centralizers of p–compact toral subgroups. A p–compact toral group P is a p–compact group P such that $\pi_0(P) \cong \pi_1(BP)$ is a finite p-group and such that the universal cover of BP is equivalent to an Eilenberg-MacLane space of the form $K(\mathbb{Z}_p^{\wedge n}, 2)$. For details and further notions we refer the reader to the above mentioned references and/or to the survey articles [5], [19] and [24].

In this section we want to prove the following statement, which is a p-completed version of Theorem 1.4.

Proposition 3.1. *Let X be a connected p–compact group, such that $\pi_1(X)$ is finite. Let $r := dim_{\mathbb{Q}} H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Then there exists a connected compact Lie group G and a map*

$$BG_p^\wedge \xrightarrow{f} BX$$

such that the following holds:

(1) For the universal cover $\tilde{G}$ of G, we have $\tilde{G} \cong SU(2)^r$. If p is odd, we can choose $G = SU(2)^r$.

(2) The induced map $H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \to H^4(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is an isomorphism.

(3) The homotopy fiber $X/G_p^\wedge$ of f is simple and $\mathbb{F}_p$-finite.

Before we start with the proof, we draw one corollary. For a space Y we denote by $Y\langle n\rangle$ the n-th Postnikov section and by $Y(n)$ the n-connected cover of Y. Hence these spaces fit into a fibration

$$Y(n) \to Y \to Y\langle n\rangle \ .$$

Corollary 3.2. *The composition $BG_p^\wedge \to BX \to BX\langle 4\rangle$ induces an equivalence $(BG_p^\wedge)_0 \xrightarrow{\simeq} BX\langle 4\rangle_0$.*

Proof. Since $\pi_1(G)$ and $\pi_1(X)$ are finite and since $\tilde{G} \cong SU(2)^r$, both spaces, $(BG_p^\wedge)_0$ and $BX\langle 4\rangle_0$, are Eilenberg-MacLane spaces with non vanishing 4-th homotopy group. The universal coefficient theorem shows that $H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is the dual of $H_4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ and hence, that $H_4(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong H_4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Moreover, since both spaces are rationally 3-connected, the Hurewicz map induces isomorphisms $\pi_4(BX_0) \cong \pi_4(BX) \otimes \mathbb{Q} \cong H_4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. The first isomorphism follows from [2; V, 4.1]. Hence, the composition induces an isomorphism between the homotopy groups and is therefore an equivalence. □

Proof of Proposition 3.1. First we assume that X is simply connected, hence 2-connected, and simple. In this context, simple means that the representation $\rho : W_X \to Gl(L_X \otimes \mathbb{Q})$ is irreducible. This representation is given by the action of the Weyl group W_X on the maximal torus T_X of X respectively on the p-adic lattice $L_X := H_2(BT_X; \mathbb{Z}_p^\wedge)$.

If X is 3-connected, then $H^4(BX; \mathbb{Z}_p^\wedge) = 0$ and the statement is obviously true. If $\pi_3(X) \neq 0$, then $\pi_3(X) \otimes \mathbb{Q} \cong \mathbb{Q}_p^{\wedge k}$ is isomorphic to the dual of the $\mathbb{Q}_p^\wedge$-vector space $H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong (H^4(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q})^{W_X}$. The last isomorphism follows from [6]. Every element $q \in (H^4(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q})^{W_X}$ can be interpreted as a W_X-invariant quadratic form $q : L_X \otimes \mathbb{Q} \to \mathbb{Q}_p^\wedge$. Since ρ is irreducible, every W_X-invariant quadratic form is definite and two such quadratic forms are

linearly dependent. Thus, all the groups are isomorphic to $\mathbb{Q}_p^\wedge$, generated by a definite W_X-invariant quadratic form q_X on $L_X \otimes \mathbb{Q}$. This implies that the pseudo reflection group W_X is a honest reflection group, i.e. generated by elements of order 2 fixing a hyperplane of $L_X \otimes \mathbb{Q}$.

Let $s \in W_X$ be any reflection, and let $T := ((T_X)^s)_e$ be the component of the unit of the fixed-point set of the action of s on T_X. Then, the centralizer $C := C_X(T) \subset X$ of T is a connected subgroup of X of maximal rank whose Weyl group $W_C \cong \mathbb{Z}/2$ is generated by s. All this follows from [6] [7] [20]. The p–compact group C has a finite covering , which splits into a product of a torus and a simply connected p–compact group C', and $W_{C'} \cong W_C \cong \mathbb{Z}/2$ is of order two [20]. Since $SU(2)_p^\wedge$ is the only simply connected p–compact group with Weyl group isomorphic to $\mathbb{Z}/2$ [9], we have $C' \cong SU(2)_p^\wedge$ as p–compact group. This establishes a map $BSU(2)_p^\wedge \longrightarrow BX$ respectively a homomorphism $SU(2)_p^\wedge \longrightarrow X$ of p–compact groups. This might not be a monomorphism, but the kernel $K \subset SU(2)_p^\wedge$ of this homomorphism is a finite p-group and a central subgroup of $SU(2)_p^\wedge$, actually of $SU(2)$ in the honest sense of compact Lie groups [23]. The kernel can be divided out [23] and, doing this, yields a monomorphism $(SU(2)/K)_p^\wedge \longrightarrow X$ respectively a map $B(SU(2)/K)_p^\wedge \longrightarrow BX$. The last claim follows from [6]. The center $Z(SU(2) \cong \mathbb{Z}/2$ is a 2-group. Thus, for odd primes, $K = 0$ and for $p = 2$ the quotient $G := SU(2)/K$ is a connected compact Lie group isomorphic to $SU(2)$ or $SO(3)$. In both cases, there exists a connected compact Lie group G and a fibration $X/G_p^\wedge \longrightarrow BG_p^\wedge \longrightarrow BX$ satisfying condition (1) of the statement. The homotopy fiber $X/G_p^\wedge$ is simple since the homotopy fiber of a map between simply connected spaces is. Moreover, $X/G_p^\wedge$ is $\mathbb{F}_p$-finite since $G \longrightarrow X$ is a monomorphism [6]. This establishes part (3).

Now we consider, the induced homomorphism

$$\mathbb{Q}_p^\wedge \cong H^4(BX;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \longrightarrow H^4(BG;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong \mathbb{Q}_p^\wedge \ .$$

By construction, the W_X-invariant quadratic form q_X is mapped onto a W_G-invariant quadratic form $q' \in H^4(BG;_p^\wedge) \otimes \mathbb{Q}$. If $q' = 0$, the map $BG_p^\wedge \longrightarrow BX$ induces the trivial map in reduced rational cohomology [1] and is therefore null homotopic [18]. This contradicts the fact that $X/G_p^\wedge$ is $\mathbb{F}_p$-finite and finishes the proof of the statement for simple simply connected p–compact groups.

Now let X be a connected p–compact group, such that $\pi_1(X)$ is finite. Then the universal cover $\tilde{X}$ is again a p–compact group and establishes a map $B\tilde{X} \longrightarrow BX$ which induces an isomorphism $H^*(BX;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(B\tilde{X};\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ [20]. By [8], $\tilde{X} \cong \prod_i X_i$ splits into a finite product of simply connected simple p–compact groups X_i. For each i, we choose a connected compact Lie group G_i and a monomorphism ${G_i}_p^\wedge \longrightarrow X_i$ satisfying the statement. Taking the product gives a homomorphism

$$G'^\wedge_p := \prod_i {G_i}_p^\wedge \longrightarrow \prod_i X_i \cong \tilde{X} \longrightarrow X \ ,$$

which might not be a monomorphism. But, analogously as above, the kernel K is a central subgroup of G' (in the Lie group sense) and, by dividing out the kernel, the homomorphism can be made into a monomorphism $G'^{\wedge}_p := (G'/K)^{\wedge}_p \to X$. Again, the center of G' is a 2-group. Hence, for odd primes, $G' \cong SU(2)^r$. Moreover, G is a connected compact Lie group satisfying part (1) of the statement.

By construction, all the maps establish a commutative diagram

$$\begin{array}{ccc} BG'^{\wedge}_p & \longrightarrow & B\tilde{X} \\ \downarrow & & \downarrow \\ BG^{\wedge}_p & \longrightarrow & BX \end{array}$$

where the vertical arrows induce isomorphisms in rational cohomology. The top horizontal map satisfies part (2) and so does the bottom arrow. By the same argument as above the homotopy fiber $X/G^{\wedge}_p$ of the bottom horizontal arrow is simple and $\mathbb{F}_p$-finite. □

4. The arithmetic square and genus sets

Most of the material in this section is taken from [28] and [32]. For a space X we define $X^{\wedge} := \prod_p X^{\wedge}_p$ to be the product of the p–adic completions in the sense of Bousfield and Kan [2] taken over all primes. For nilpotent spaces with mod-p homology of finite type, this is equivalent to the profinite completion of Sullivan [28] [2]. Sullivan also constructed a formal completion for spaces [28]. For a space X, the p-formal completion $X^{\flat}_p$ is defined to be the homotopy colimit $hocolim_i X_i{}^{\wedge}_p$ where X_i ranges over all finite subcomplexes of X. The formal completion $X^{\flat}$ is given by $hocolim_i X_i{}^{\wedge}$. Both completions are coaugmented functors on the homotopy category of topological spaces. That is they come with a map $X \to X^{\flat}_p$ respectively $X \to X^{\wedge}_p$. By $X^{\flat}_0$ we denote the formal completion of the rationalization of X and by $X^{\wedge}_0$ the rationalization of the profinite completion of X.

Theorem 4.1. ([Sullivan]) *Let X be a connected simple CW-complex of finite type. Then*
(1) $X^{\flat}_0 \simeq X^{\wedge}_0$.
(2) The diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{\wedge} \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X^{\flat}_0 \simeq X^{\wedge}_0 \end{array}$$

is a fiber square up to homotopy.

Proof. Actually, in [28] this is only proved for simply connected spaces, but all proofs also work in the case of simple spaces (see also [2]). □

Remark 4.2. For a simple space X, we can identify $X_0^\wedge$ and $X_0^\flat$ via the above equivalence. The homotopy groups $\pi_*(X_0^\wedge)$ are modules over $\mathbb{Q}\otimes\mathbb{Z}^\wedge$ and therefore topological groups. For simple spaces X and Y, a map $f : (X^\wedge)_0 \to (Y^\wedge)_0$ is called π_* -continuous if $\pi_*(f)$ is continuous. In the finitely generated case this is equivalent to $\pi_*(f)$ being $\mathbb{Q}\otimes\mathbb{Z}^\wedge$-linear. If f is induced from a map $X_0 \to Y_0$ by formal completion or from $X^\wedge \to Y^\wedge$ by localization then f is always π_*-continuous.

If $X_0^\wedge \simeq Y_0^\wedge$ we denote by $HEC_0^\wedge(X,Y)$ the set of homotopy classes of π_*-continuous homotopy equivalences $f : (X^\wedge)_0 \to (Y^\wedge)_0$. If $X = Y$, then we define $HEC_0^\wedge(X) := HEC_0^\wedge(X,X)$. The elements of these groups are called gluing maps.

For any element $\alpha \in HEC_0^\wedge(X,Y)$ we define $W := W(X,\alpha,Y)$ to be the homotopy pull back of

$$X_0 \to (X^\wedge)_0 \xrightarrow{\alpha} (Y^\wedge)_0 \leftarrow Y^\wedge \ .$$

Proposition 4.3. *Let X and Y are simple spaces of finite type and let $\alpha \in HEC_0^\wedge(X,Y)$. Then, for $W := W(X,\alpha,Y)$, the following holds:*
(1) $W^\wedge \simeq Y^\wedge$ and $W_0 \simeq X_0$.
(2) W is a simple space of finite type.

Proof. Part (1) is obvious. By [32; 3.7], the abelian groups $\pi_n(W)$ are finitely generated for all n and, since $\pi_n(W) \to \pi_n(W^\wedge) \cong \pi_n(X^\wedge)$ is a monomorphism, W is simple. This proves the second part of the statement. □

If $X = Y$ is a nilpotent space of finite type, the above construction gives a map $HEC_0^\wedge(X) \to G_0^\wedge(X)$ where the genus $G_0^\wedge(X)$ of X is the set of all homotopy equivalence classes of nilpotent spaces Y of finite type such that $Y^\wedge \simeq X^\wedge$ and $Y_0 \simeq X_0$. In [32; 3.8] is proved that the above assignment identifies the genus set of X with a particular double coset of $HEC_0^\wedge(X)$.

5. Construction of a particular fibration

Let L be a a quasi-finite connected loop space with finite fundamental group. Then, the rational cohomology $H^*(BL;\mathbb{Q})$ of the classifying space BL of L is a finitely generated polynomial algebra with generators $x_1,\dots,x_n$. The generators are chosen as images of non divisible integral classes, also denoted by $x_i \in H^*(BL;\mathbb{Z})$. These integral classes define a map

$$BL \xrightarrow{e} K := \prod_i K(\mathbb{Z},|x_i|) =: K\langle 4\rangle \times K(4)$$

where $K(\mathbb{Z}, |x_i|)$ is an Eilenberg-MacLane space, where $|x_i|$ denotes the degree of the class x_i (which in our case is always even) and where $K\langle 4\rangle$, as the 4-th Postnikov section, is the product of all factors of degree less than 5 and $K(4)$, as the 4-th connected cover, the product of all other factors. Rationally, this map establishes equivalences $BL_0 \to K_0 \simeq K\langle 4\rangle_0 \times K(4)_0$ and $BL_0^\wedge \simeq BL_0^\flat \to K_0^\flat \simeq K_0^\wedge \simeq K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge$.

Let $r := dim_{\mathbb{Q}} H^4(BL; \mathbb{Q}) = dim_{\mathbb{Q}_p^\wedge} H^4(BL; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Completed at every prime, L becomes a p–compact group. By Proposition 3.1 and Corollary 3.2, there exist a semi simple connected compact Lie group G, such that $\tilde{G} \cong SU(2)^r$, and, for every prime p a map $BG_p^\wedge \to BL_p^\wedge$ such that the composition $BG_p^\wedge \to BL_p^\wedge \to BL\langle 4\rangle_p^\wedge$ induces an equivalence after localizing at 0. Putting all these maps together we get a map $f : BG^\wedge \to BL^\wedge$ such that $BG_0^\wedge \to BL\langle 4\rangle_0^\wedge$ is an equivalence.

Since BG_0, $BL\langle 4\rangle_0$ and $K\langle 4\rangle_0$ are rational Eilenberg-MacLane spaces with isomorphic homotopy groups, all these spaces are homotopy equivalent. Thus, there exists a map $s : BG_0 \to BL_0$, given by a section of $BL_0 \to K\langle 4\rangle_0$ such that $BG_0 \to BL_0 \to K(4)_0$ is null homotopic. We have a diagram

$$\begin{array}{ccccccc} BG_0^\flat & \xrightarrow{s} & BL_0^\flat & \xrightarrow[\simeq]{e} & K\langle 4\rangle_0^\flat \times K(4)_0^\flat & \xrightarrow{pr_1} & BL\langle 4\rangle_0^\flat \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ BG_0^\wedge & \xrightarrow{f} & BL_0^\wedge & \xrightarrow[\simeq]{e} & K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge & \xrightarrow{pr_1} & BL\langle 4\rangle_0^\wedge \ , \end{array}$$

where the vertical arrows are the obvious homotopy equivalences, where pr_1 denotes the projection onto the first factor and where the composition of the horizontal arrows is a homotopy equivalence. The compositions $es =: (s_1, s_2)$ and $ef =: (f_1, f_2)$ can be written as two components. Confusing notations, e, f and s as well as f_i and s_i also denote the maps between the uncompleted or unlocalised spaces if they exist.

We want to change the gluing map $j : BL_0^\flat \to BL_0^\wedge$ in such a way that it lifts to the gluing map $\alpha := (pr_1 ef)^{-1} pr_1 es : BG_0^\flat \to BG_0^\wedge$. This map is a homotopy equivalence and π_*-continuous since all homotopy groups of BG and BL are finitely generated and since all maps are $\mathbb{Q} \otimes \mathbb{Z}^\wedge$-linear. As a product of Eilenberg-MacLane spaces, $BL_0^\wedge$ is an H-space. The product of the two maps

$$\beta_1 := (j|_{K\langle 4\rangle_0^\flat}, f_2 \alpha (s_1)^{-1}) : K\langle 4\rangle_0^\flat \to K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge$$

and

$$\beta_2 := j|_{K(4)_0^\flat} : K(4)_0^\flat \to K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge$$

defines a map

$$\beta := \beta_1 \cdot \beta_2 : BL_0^\flat \simeq K\langle 4\rangle_0^\flat \times K(4)_0^\flat \to K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge \simeq BL_0^\wedge \ ,$$

which, by construction, is π_*-continuous and fits into a homotopy commutative diagram

$$\begin{array}{ccccc} (L_0/G_0)^\wedge & \longrightarrow & BG_0^\flat & \xrightarrow{s} & BL_0^\flat \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ (L^\wedge/G^\wedge)_0 & \longrightarrow & BG_0^\wedge & \xrightarrow{f} & BL_0^\wedge \ . \end{array}$$

Here L_0/G_0 and $L^\wedge/G^\wedge$ denote the homotopy fibers of the map $s : BG_0 \to BL_0$ and $f : BG^\wedge \to BG^\wedge$. The maps α and β are homotopy equivalences and so is γ. Since f and s have left inverse, the map γ is also π_*-continuous. The homotopy groups of L_0/G_0 are finite dimensional rational vector spaces and the homotopy groups of $L^\wedge/G^\wedge$ finitely generated $\mathbb{Z}^\wedge$-modules.

These considerations lead to the following proposition.

Proposition 5.1. *Let L be a connected quasi-finite loop space such that $\pi_1(L)$ is finite. Let $r := dim_{\mathbb{Q}} H^4(BL;\mathbb{Q})$. Then, there exist a semi simple connected compact Lie group G, quasi-finite connected loop spaces M and N and a fibration*

$$E \to BM \to BN$$

such that the following holds:
(1) E is quasi-finite and simple.
(2) $\tilde{G} \cong SU(2)^r$.
(3) BM is in the genus of BG
(4) BN is in the genus of BL.

Proof. Using the arithmetic square, the above diagram establishes a fibration $E \to BM \to BN$. The last three properties are true by construction. The first property follows from Proposition 4.3. The loop spaces $M := \Omega BM$ and $N := \Omega BN$ are in the genus of G respectively of L and give therefore rise to quasi-finite loop spaces. This finishes the proof. □

Next we want to analyze the loop $\Omega\beta : L_0^\flat \to L_0^\wedge$ of the gluing map β.

Lemma 5.2. $\Omega\beta \simeq j : L_0^\flat \to L_0^\wedge$.

Proof. Using the H-space structure of $L_0^\wedge \simeq \Omega K\langle 4\rangle_0^\wedge \times K(4)_0^\wedge$, we can again describe the map $\Omega\beta$ as a product of two maps, We just have to multiply the loops of β_1 and β_2. Therefore, we only have to show that $\Omega\beta_2 = \Omega f_2 \alpha(s_1)^{-1}) : \Omega K\langle 4\rangle_0^\flat \to \Omega K(4)_0^\wedge$ is null homotopic. But this map fits into a commutative diagram

$$\begin{array}{ccccccc} G_0^\flat & \xrightarrow{\Omega\alpha} & G_0^\wedge & \xleftarrow{q} & \tilde{G}_0^\wedge & \longleftarrow & \tilde{G}^\wedge \\ \simeq\downarrow & & \Omega f_2\downarrow & & \Omega f_2\downarrow & & (\Omega f_2)q\downarrow \\ K'^\flat_0 & \xrightarrow{\Omega\beta_2} & K''^\wedge_0 & = & K''^\wedge_0 & \longleftarrow & K''^\wedge \end{array}$$

where $K' := \Omega K\langle 4\rangle$ and $K'' := \Omega K(4)$ are again Eilenberg-MacLane spaces. Since the first two arrows in the row are homotopy equivalences and since the map from the last column to the third column is the coaugmentation of the localization at 0, we only have to show that $\Omega f_2 q : \tilde{G}^\wedge \to K''^\wedge$ is null homotopic. The Eilenberg-MacLane space $K(4)^\wedge \simeq \prod_l K(\mathbb{Z}^\wedge, 2l)$ splits into a finite product such that $l > 2$ for all l. Since $H^*(B\tilde{G}; \mathbb{Z}^\wedge)$ is generated by classes of degree 4, each coordinate of the map $B\tilde{G}^\wedge \to K(4)^\wedge$ is given by a decomposable cohomology class x_l in $H^*(B\tilde{G}^\wedge; \mathbb{Z}^\wedge)$. The next lemma finishes the proof. □

Lemma 5.3. *Let L be a quasi-finite loop space and let $g : BL^\wedge \to K(\mathbb{Z}^\wedge, r)$ denote a decomposable cohomology class in $H^*(BL^\wedge; \mathbb{Z}^\wedge)$. Then, the map $\Omega g : L^\wedge \simeq \Omega BL^\wedge \to K(\mathbb{Z}^\wedge, r-1)$ is null homotopic.*

Proof. The adjoint of Ωg factors through $\Sigma\Omega BL^\wedge \to BL \xrightarrow{g} K(\mathbb{Z}^\wedge, r)$. Since $\Sigma\Omega BL$ is a co-H-space, all cup products vanish. Hence, the composition is null homotopic as well as Ωg. □

6. Proof of Theorem 1.4

We start with analyzing the genus of the connected compact Lie group G.

Proposition 6.1. *Let G be a semi simple connected compact Lie group, such that $\tilde{G} \cong SU(2)^r$. Then, the genus of G is rigid. That is every element of the genus of G is actually homotopy equivalent to G.*

Proof. First we prove the statement for $G = \tilde{G} \cong SU(2)^r$. Since $G_0^\wedge \simeq K((\mathbb{Q} \otimes \mathbb{Z}^\wedge)^r, 3)$ is an Eilenberg-MacLane space, the gluing map for M is given by a matrix $A \in Gl(r, \mathbb{Q} \otimes \mathbb{Z}^\wedge)$ and therefore splits into a product RC with $R \in Gl(r, \mathbb{Q})$ and $C \in Gl(r, \mathbb{Z}^\wedge)$ [32]. Since $G \cong SU(2)^r$, R can be realized by a self map $G_0 \to G_0$ and C by a self map $G^\wedge \to G^\wedge$. This implies that $M \simeq G$ [32].

Now let G be a quotient of $\tilde{G}$, and let $M \in genus(G)$. Then, $\pi_1(M) \cong \pi_1(G) =: E$ is a finite elementary abelian 2-group. The universal cover $\tilde{M}$ is contained in the genus of $\tilde{G} \simeq SU(2)^r$. Hence, $\tilde{M} \simeq \tilde{G}$ and M fits into a fibration $\tilde{G} \to M \to BE$. The 2-adic completion establishes an injection $\pi_3(\tilde{G}) \to \pi_3(\tilde{G}_2^\wedge)$ and turns the fibration into a principal fibration. Since principal fibration with

connected structure groups are simple and since self maps of $\tilde{G}$ are classified by means of homotopy groups, the above fibration is oriented and is classified by a map $BE \to BSHE(\tilde{G})$, where $SHE(\tilde{G})$ denotes the set of all homotopy equivalence classes of $\tilde{G}$ homotopic to the identity. The left action of $\tilde{G}$ on itself establishes a map $B\tilde{G} \to BSHE(\tilde{G})$ [27].

Now, we consider the diagram

$$\begin{array}{ccccc} B\tilde{G} & \longrightarrow & BSHE(\tilde{G}) & & \\ \downarrow & & \downarrow & & \\ B\tilde{G}_2^\wedge & \longrightarrow & BSHE(\tilde{G})_2^\wedge & \xrightarrow{\simeq} & BSHE(\tilde{G}_2^\wedge) \end{array}$$

Since $\tilde{G}$ is a topological group, $SHE(\tilde{G}_2^\wedge)$ is 2-complete and the map $SHE(\tilde{G})_2^1 \to SHE(\tilde{G}_2^1)$ is a homotopy equivalence [2]. Since completion and passing to classifying spaces commutes for nilpotent connected loop spaces [2], the second arrow in the bottom line is a homotopy equivalence. The homotopy fibers of both columns have uniquely 2-divisible homotopy groups. Thus, obstruction theory shows that for every map $BE \to B\tilde{G}_2^\wedge$ or $BE \to BSHE(\tilde{G}_2^\wedge)$ there exist a lift to $B\tilde{G}$ respectively to $BSHE(\tilde{G})$, unique up to homotopy. A small diagram chase shows that, up to homotopy, the classifying map $BE \to BSHE(\tilde{G})$ has a unique lift $BE \to B\tilde{G}$. Hence, $\tilde{G} \to M \to BE$ is a principal fibration. By [10], the classifying map is induced from a homomorphism $\rho : E \to \tilde{G}$. In fact, this map is a monomorphism, since the homotopy fiber M of $B\rho$ is quasi-finite [26]. All homomorphisms $E \to \tilde{G}$ are central and therefore, $M \simeq \tilde{G}/E$ is a compact Lie group.

Finally we have to show that $\tilde{G}/E \simeq G$. Using shift maps and permutation of factors, one can show, analogously as for the equivalence $SO(4) = SU(2) \times_{\mathbb{Z}/2} SU(2) \simeq SO(3) \times SU(2)$, that $\tilde{G}/E$ and G are homotopy equivalent to $SO(3)^s \times SU`(2)^{r-s}$ where s equals the rank of E. □

Now we are able to finish the proof of Theorem 1.4.

Proof of Theorem 1.4. The fibration $L \to E \to BM$ of Proposition 5.1 has almost all the properties we stated in Theorem 1.4. It is only left to show that $N \simeq L$ and that $M \simeq G$. The first assertion follows from Lemma 5.2, and the second from Proposition 6.1. This finishes the proof of Theorem 1.4 □

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