Advances on Extremal Problems in Number Theory and Combinatorics

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1. Introduction

To keep an acceptable size references not listed at the end are given by the Bibliography of the recent book [N] and/or the page number of [N].

1. Starting with solutions of extremal problems for *finite sets of numbers* under *divisibility constraints* [with L. H. Khachatrian, c.f. [PS]], then we describe the discovery of *correlation inequalities* implied by the AD-inequality [with L. H. Khachatrian, c.f. [N] P. C. Fishburn, L. Shepp, The Ahlswede-Daykin Theorem, 501–516] and conclude this section with results on extremal problems concerning *densities* of *primitive infinite sets of numbers* and related topics [with L. H. Khachatrian and A. Sárközy].
2. Divisibility properties for numbers naturally correspond to *intersection* properties for sets and thus there are also connections between methods. Especially, a new *pushing technique* for numbers lead to the *discovery* of the method of *"generated sets"*, which made it possible to establish several Intersection Theorems in Combinatorics, which are highlighted by the *Complete Intersection Theorem* [with L. H. Khachatrian; P. Frankl wrote in [F1, p. 142] "At present this conjecture appears hopelessly difficult in general"], which, as a very special case, established the *4m-Conjecture* of Erdős/Ko/Rado from 1938 [[En]; [N] C. Bey, K. Engel, Old and New Results for the Weighted t-intersection Problem via AK-Methods, 45–74; [N] G. O. H. Katona, The Cycle Method and its Limits, 129–142; [CG]; B. Bollobás wrote in "Paul Erdős - Life and Work" the foreword of [GN]: "The third problem is from the 1961 paper of Erdős, Ko and Rado; it is, in fact, the last unsolved problem of that paper." "It is widely known that vast amounts of thought and ingenuity are required in order to earn \$ 500 on an Erdős problem; even so, this problem may be far harder than its price-tag suggests."].
3. We turn now to combinatorial work, which received its incentive from Information Theory and Computer Science. We demonstrate this for the area of information storage for rewritable memories, which led over Sperner

type questions for “clouds” of antichains to *Higher Level Extremal Problems*. These problems are of one degree more complex than those usually considered: sets take the role of elements, families of sets (clouds) take the role of sets, etc. [[N] P. L. Erdős, L. A. Székely, 117–124].

4. It also led to several kinds of *Vertex-Isodiametric Theorems in the Average* [with I. Althöfer, N. Cai] and *Edge Isoperimetric Theorems* [with N. Cai] which are rate-wise optimal and proved by novel *information theoretic* approaches.
5. We continue with several basic topics (partitions, monochromatic rectangles, shadows and isoperimetry under sequence-subsequence relations, antichains: splitting, AZ-identities, dimension constraints) from Sequence Spaces, which mostly were influenced by the analysis of Communication Complexity and Unconventional Coding. The most frequent coauthors are N. Cai, Z. Zhang, L. Bassalygo and M. Pinsker.
6. Finally, we conclude with counterexamples to known conjectures and with a list of seemingly basic open problems and new conjectures.

2. Contributions to Combinatorial Number Theory

2.1. Extremal problems under divisibility constraints for finite sets

Conjecture 2.1. (Erdős 1962) *The set $E(n,k)$ of integers not exceeding n, which are divisible by one of the first k primes, has maximal cardinality among the subsets of $[n] = \{1, 2, \dots, n\}$ without $k+1$ elements, which are pairwise relatively prime.*

He caught our interest in a lecture delivered in 1992. It stimulated us to make a systematic investigation of this and related number theoretical problems. When viewed combinatorially as an extremal problem for products of chains (by the prime decomposition) with a truncation condition (caused by the property “smaller than n”) the issue is to understand whether the problem is just combinatorial in nature or does depend on the prime number distribution. The latter is the case, the conjecture was disproved, but proved for large n, in joint work [85], [101] with L. Khachatrian. Thus began a very fruitful cooperation. Immediate successes were proofs of other well-known conjectures of Erdős and Erdős/Graham (all in [E1] and [E2]).

More importantly we gained an understanding for the sensitivity towards the distribution of the primes and thus our program was very rewarding. The analysis led to the discovery of a new “pushing” method with wide applicability also in Combinatorics, where it led to the solution of several well-known problems like the 4m-conjecture, one of the oldest problems in combinatorial extremal theory (see section 3) or the isodiametric problem in Hamming spaces (see section 5). Complete proofs can be found in [En]. We describe now the results and for this adopt the following notation. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of all primes.

For two numbers $u, v \in \mathbb{N}$ we write $u \mid v$ iff u divides v, (u, v) stands for the largest common divisor of u and v, $[u, v]$ is the smallest common multiple of u and v. The numbers u and v are called coprimes, if $(u, v) = 1$. We are particularly interested in the sets

$$\mathbb{N}_s = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\} \text{ and } \mathbb{N}_s(n) = \mathbb{N}_s \cap [n] . \tag{1}$$

Erdős introduced $f(n, k, s)$ as the largest integer r for which an $A \subset \mathbb{N}_s(n)$, $|A| = r$, exists with no $k + 1$ numbers in A being coprimes. Certainly there are no $k + 1$ coprimes in the set

$$\mathbb{E}(n, k, s) = \{ u \in \mathbb{N}_s(n) : u = p_{s+i} v \text{ for some } i = 0, 1, \dots, k - 1 \} . \tag{2}$$

The case $s = 1$, in which we have $\mathbb{N}_1(n) = [n]$, gives the conjecture 2.1 above.

The papers [ESS2] and [ESS4] are centered around conjecture 2.1. Whereas it is easy to show that it is true for $k = 1$ and $k = 2$, it was proved for $k = 3$ by C. Szabó and G. Tóth 1985 and for $k = 4$ by Motzkin 1988. The popularity of this problem is documented by R. Freud in [Fr].

General Conjecture. (Erdős 1980) *$f(n, k, s) = |\mathbb{E}(n, k, s)|$ for all $n, k, s \in \mathbb{N}$.*

Theorem 2.2. (Ahlswede/Khachatrian [101]) *For every $k, s \in \mathbb{N}$ there exists an $n(k, s)$ such that for all $n \geq n(k, s)$ $\mathbb{E}(n, k, s) = f(n, k, s)$ and this optimal set is unique.*

Erdős mentions that he did not succeed in settling even the special case.

Conjecture 2.3. *$f(n, 1, s) = |\mathbb{E}(n, 1, s)|$ for all $n, s \in \mathbb{N}$.*

Whereas in [85] conjecture 2.1 was disproved for $k = 212$, surprisingly conjecture 2.3 is true.

Theorem 2.4. (Ahlswede/Khachatrian [104]) *For every $s \in \mathbb{N}$ and n $f(n, 1, s) = |\mathbb{E}(n, 1, s)|$ and the optimal configuration is unique.*

Notice that $\quad \mathbb{E}(n, 1, s) = \{ u \in \mathbb{N}_1(n) : p_s \mid u; p_1, \dots, p_{s-1} \nmid u \}$.

Studying square free numbers $\mathbb{N}^*$ one is naturally led to sets $\mathbb{E}^*(n, k, s) = \mathbb{E}(n, k, s) \cap N^*$ etc and to the function $f^*(n, k, s)$. Then

Theorem 2.4*. *For all $s, n \in \mathbb{N}$ $f^*(n, 1, s) = |\mathbb{E}^*(n, 1, s)|$.*

But even for squarefree numbers "Erdős sets" are not always optimal, that is, $f^*(n, k, 1) \neq |\mathbb{E}^*(n, k, 1)|$ can occur. Also $f^*(n, 2, s) \neq |\mathbb{E}^*(n, 2, s)|$ happens for $p_s = 101$ and $n \in [109 \cdot 113, 101 \cdot 127)$.

We generalize and analyse conjecture 2.3 first for quasi-primes in order to understand how its validity depends on the distribution of the quasi-primes and primes. Our main result is a simply structured sufficient condition on this distribution. Using sharp estimates on the prime number distribution by Rosser and Schoenfeld we show that this condition holds for $\mathbb{Q} = \{ p_s, p_{s+1}, \dots \}$, $s \geq 1$, as set of quasi-primes and thus theorem 2.4 follows.

Erdős/Graham asked for the maximal value $k = g(n)$ such that there are numbers $1 < a_1 < \cdots < a_k = n$, $(a_i, a_j) \neq 1$. Let $M(A) = \mathbb{N} \times A$ denote the set of multiples of A.

Conjecture 2.5. (After a little correction) *Let $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, $\alpha_i \geq 1$, $\alpha_i \in \mathbb{N}$ and $Q = \{q_1 < \cdots < q_r\} \subset \mathbb{P}$, then $g(n) = \max_{1 \leq j \leq r} |M(2q_1, \dots, 2q_j, q_1 \dots q_j) \cap \mathbb{N}(n)| = f(n, Q)$, say.*

We consider a more general and seemingly more natural problem by looking at sets of integers with pairwise common divisor and a factor from a specified set of primes with growth condition.

Let $Q = \{q_1, \dots, q_r\} \subset \mathbb{P}$ and let

$$g(n, Q) = \max\{|A| : A \subset \mathbb{N}(n), (a, a') \neq 1, \left(a, \prod_{i=1}^{r} q_i\right) \neq 1 (\forall a, a' \in A)\}. \tag{3}$$

Theorem 2.6. *For every $Q = \{q_1, \dots, q_r\} \subset \mathbb{P}$ and $n \geq \prod_{i=1}^r q_i$ $g(n, Q) = f(n, Q)$.*

Clearly, this implies the **Conjecture of Erdős/Graham**. For $n < \prod_{q \in Q} q$ the conclusion of theorem 2.6 does not hold.

Example 2.7. *Let $Q = \{q_1, q_2, \dots, q_{r-1}, q_r\} = \{5, 7, \dots, p_{r+1}, q_r\}$ that is $q_i = p_{i+2}$ for $i = 1, \dots, r-1$. Further $q_{r-1} = p_{r+1} > 1000$, $n = 2 \cdot 3 \cdot 11 \cdot \prod_{i=1}^{r-1} q_i$ and $\frac{n}{1000} < q_r < \frac{n}{1000}$. With Bertrand's postulate we establish the claim.*

2.2. Correlation inequalities from the AD-inequality

Next we report on new density inequalities for sets of multiples. For infinite sets $A, B \subset \mathbb{N}$ consider the set of least common multiples $[A, B] = \{[a, b] : a \in A, b \in B\}$, the set of largest common divisors $(A, B) = \{(a, b) : a \in A, b \in B\}$, the set of products $A \times B = \{a \cdot b : a \in A, b \in B\}$, and the sets of their multiples $M(A) = A \times \mathbb{N}$, $M(B)$, $M[A, B]$, $M(A, B)$, and $M(A \times B)$, resp. Our discoveries are the inequalities

$$\mathbf{d}M(A, B)\mathbf{d}M[A, B] \geq \mathbf{d}M(A) \cdot \mathbf{d}M(B) \geq \mathbf{d}M(A \times B), \tag{4}$$

where $\mathbf{d}$ denotes the asymptotic density. The first inequality is by the factor $\mathbf{d}M(A, B)$ sharper than Behrend's well-known inequality. This in turn is a generalisation of an earlier inequality of Rohrbach and Heilbronn, which settled a conjecture of Hasse concerning an identity due to Dirichlet. Our second inequality does not seem to have predecessors.

Observing the similarity to the AD inequality led to our main discovery, the inequality (with L. H. Khachatrian [102])

$$\underline{\mathbf{D}}(A, B)\underline{\mathbf{D}}[A, B] \geq \underline{\mathbf{D}}A\underline{\mathbf{D}}B,$$

where A, B are arbitrary sets of positive integers, $(A, B) = \{(a, b) : a \in A, b \in B\}$ is the set of largest common divisors, $[A, B] = \{[a, b] : a \in A, b \in B\}$ is the set of least common multiples, and $\underline{\mathbf{D}}$ denotes the lower Dirichlet density. It is much more general than the previous inequality for multiples of sets.

This is more than an analogy: AD implies this number theoretical Correlation inequality. For reasons of scaling it is important to work with Dirichlet density!

Similarly, Behrend's inequality can be obtained as *number theoretical form* of FKG, but it actually preceded it. The second inequality in (4) is the twin of the van der Berg/Kesten inequality etc.

Now AD has not only combinatorial and probabilistic correlation inequalities (see [N] Fishburn/Shepp, 501–516) as consequences, but also those above in Number Theory! We conclude with an important

Remark 2.8. *In the literature "AD-inequality" ("4-Function Theorem") refers to the inequality of [30], which holds for lattices. We emphasize that* **the much more general inequality** *of [33] makes no reference to lattices and* **should have a wider range of applications**.

2.3. Densities for primitive, prefix free, quotient, and squarefree sets

In this section we report on recent work with L. H. Khachatrian and A. Sárközy.

We begin with [150] "**On the counting function of primitive sets of integers**".

After in **1934 Besicovitch** gave an example of a set of multiples $M(B)$ *without* a density, interest arose in *primitive* sets $A \subset \mathbb{N}$: for $a, a' \in A$; $a \neq a'$ always $a \nmid a'$. Set $F(n) \triangleq$ greatest cardinality of primitive A in $[n]$ and notice that

$$F(n) = n - \left\lceil \frac{n}{2} \right\rceil \ \left(= \left(\frac{1}{2} + 0(1) \right) n \right) . \tag{5}$$

Besicovitch and Erdős 1935: $\forall\ \varepsilon > 0\ \exists$ primitive $A \subset \mathbb{N}$ with $\overline{d}(A) > \frac{1}{2} - \varepsilon$.

Behrend 1935: For $A \subset [n]$ primitive $\sum_{a \in A} \frac{1}{a} < c_1 \frac{\log n}{(\log\log n)^{1/2}}$.

Erdős 1935: A primitive $A \subset \mathbb{N}$ (finite or infinite) satisfies $\sum_{a \in A} \frac{1}{a \log a} < c_2$.

Corollary 2.9. *If $A \subset \mathbb{N}$ is primitive, then for the counting function A*

$$A(x) < \frac{x}{\log\log x \log\log\log x} \quad \textit{for } \infty \textit{ many } x\,. \tag{6}$$

How far is this upper bound from the best possible? This question is closely related to one of the **favourite problems** of Erdős (mentioned in numerous papers). Use here formulation of [ESS3]:

"The following problem seems difficult: Let $b_1 < b_2 < \dots$ be an infinite sequence of integers. What is the necessary and sufficient condition that there should exist a primitive sequence $a_1 < a_2 < \dots$ satisfying $a_n < b_n$ for every n? We must have $\sum_{i=1}^{\infty} \frac{1}{b_i \log b_i} < \infty \dots$ It is not clear whether a simple necessary and sufficient condition exists." This is followed by a lengthy discussion of the problem how large one can make $\sum_{a \le x} \frac{1}{a}$ uniformly in x for a primitive set $\{a_1 < a_2 < \dots\}$ (see also [ESS1]).

It seems to be a *more natural* (although more difficult) problem to *replace* here the sum $\sum_{a \le x} \frac{1}{a}$ by the counting function $A(x)$ i.e. to study the problem how large one can make $A(x)$ uniformly in x for a primitive set A. We provide a quite satisfactory answer by proving that (6) is best possible apart from a factor $(\log\log\log x)^{\varepsilon}$:

Theorem 2.10. *For all $\varepsilon > 0$ $\exists$ infinite primitive set $A \subset \mathbb{N}$ such that for $x > x_0(\varepsilon)$ we have $A(x) > \frac{x}{\log\log x (\log\log\log x)^{1+\varepsilon}}$.*

Our recent interest in primitive sets arose while we investigated two related new concepts "**prefix-free sets**" and "**suffix-free sets**", which are of information theoretical background.

For $a, b \in \mathbb{N}^*$ square free we write $a|_p b$ (a is prefix of b) if for primes $p_1 < \dots < p_r < p_{r+1} < \dots < p_t$ $\quad a = p_1 \dots p_r$, $b = p_1 \dots p_{r+1} \dots p_t$. Similarly, $c = p_{r+1} \dots p_t$ is suffix of b and we write $c|_s b$.

If for $A \subset \mathbb{N}^*$ there are no $a, b \in A$ with $a|_p b$ (resp. $a|_s b$) then A is said to be prefix-free (resp. suffix-free). (Both notions could be extended to non-square-free cases.) There is a close connection between prefix-freeness and primitivity: if a set $A \subset \mathbb{N}$ is primitive, then it is prefix-free.

We present first the "prefix-free analog" of (5). Let $G(n)$ denote the cardinality of the greatest prefix-free set from $\mathbb{N}^*(n)$, and let $P^+(a)$ denote the smallest prime greater than $P(a)$.

Theorem 2.11. ([N, p. 1–16])

(i) *The set $B(n) = \{b : b \in \mathbb{N}^*(n), bP^+(b) > n\}$ is prefix-free and $G(n) = |B(n)|$.*

(ii) $\lim_{n\to\infty} \frac{G(n)}{|\mathbb{N}^*(n)|} = 1$.

(iii) *For all $\varepsilon > 0$ there is an infinite prefix-free set $A \subset \mathbb{N}^*$ with $\overline{d}^*(A) > 1 - \varepsilon$.*

The "prefix-free analog" to Behrend's result reflects an interesting difference between primitive sets and prefix-free sets. Indeed, consider now instead of $G(n)$ $E(n) = \max_{\text{prefix-free } A \subset \mathbb{N}^*(n)} \sum_{a \in A} \frac{1}{a}$.

Theorem 2.12. ([N, p. 1–16]) *For every $\varepsilon > 0$ and $n > n_2(y)$, suitable, $0,2689 - \varepsilon < \frac{E(n)}{\sum_{b \in \mathbb{N}^*(n)} \frac{1}{b}} < 0,7311 + \varepsilon$. Actually, we know for every $n \in \mathbb{N}$ the unique optimal prefix-free $A \subset \mathbb{N}^*(n)$ for which $E(n)$ is assumed, but the value is hard to estimate.*

Since Erdős 1935 above uses in the proof only the prefix property of a primitive sequence, obviously for prefix-free $A \subset \mathbb{N}$ $\quad \sum_{a \in A} \frac{1}{a \log a} < c_2$ and also (6) in corollary 2.9 and theorem 2.10 "primitive" can be replaced by "prefix-free".

While the behaviour of prefix-free and primitive sets is similar as far as the maximal growth of the counting function is concerned, the behaviour of the suffix-free sets is very different. Let $H(n)$ denote the cardinality of the largest suffix-free set selected from $\mathbb{N}^*(n)$.

Theorem 2.13. ([N, p. 1–16])

(i) *The set $C(n) = \{c \in \mathbb{N}^*(n) : 2 \mid c\} \cup \{\mathbb{N}^*(n) \cap \left(\frac{n}{2}, n\right]\}$ is suffix-free and $|C(n)| = H(n)$.*

(ii) $\lim_{n\to\infty} \frac{H(n)}{|\mathbb{N}^*(n)|} = \frac{2}{3}$.

(iii) *For every $\varepsilon > 0$ there exists a suffix-free set C with $\overline{d}^* C > \frac{2}{3} - \varepsilon$.*

Finally we consider logarithmic densities of suffix-free sets. Let $K(n) = \max_{\text{suffix-free } A\subset\mathbb{N}^*(n)} \sum_{a\in A}\frac{1}{a}$.

In contrast to the case of prefix-free sets here we have a simple description of the optimal set, which yields

Theorem 2.14. ([N, p. 1–16]) $\lim_{n\to\infty} \frac{K(n)}{\sum_{a\in\mathbb{N}^*(n)}\frac{1}{a}} = \frac{31}{72}$.

2.3.1. On the quotient sequence of sequences of integers For $A \subset \mathbb{N}$, $a \in A$ let $Q_A^a \triangleq$ set of integers q such that

$$aq \in A \;\; a > 1 \;\; Q_A \triangleq \bigcup_{a\in A} Q_A^a\,. \tag{7}$$

So $Q_A =$ set of integers $q = \frac{a'}{a} > 1$ with $a, a' \in A$. Called quotient set of A.

By Behrend's and Erdős' Theorem the quotient set of a dense set A is *non-empty* (obvious).

The study of quotient sets of "dense" sets started with

Theorem 2.15. (Pomerance/Sárközy 1988) *There exist constants c_3, N_0 such that* **if** *$N \in \mathbb{N}$, $N > N_0$, $\mathcal{P}$ is a set of primes not exceeding N and* **if** *$A \subset \{1,2,\dots,N\}$ with*

$$\sum_{p\in\mathcal{P}}\frac{1}{p} > c_3 \;\; and \;\; \sum_{a\in A}\frac{1}{a} > 10\log N\left(\sum_{p\in\mathcal{P}}\frac{1}{p}\right)^{-1/2} \tag{8}$$

then *there is a $q \in Q_A$ such that $q \mid \prod_{p\in\mathcal{P}} p$.*

Here we study *density related* properties of Q_A. Our first goal is to study connections between $\overline{\delta}(A)$ and $\overline{\delta}(Q_A)$. First we thought that for all $A \subset \mathbb{N}$

$$\overline{\delta}(Q_A) \geq \overline{\delta}(A)\,. \tag{9}$$

Example 2.16. *$A = \{2m, 3m$ or $5m$ with $m \in \mathbb{N}, (m,30) = 1\}$. Then $\overline{\delta}(A) = \delta(A) = d(A) = \frac{62}{225}$ and $\overline{\delta}(Q_A) = \delta(Q_A) = d(Q_A) = \frac{4}{15} = \frac{30}{31}\overline{\delta}(A)$.*

Still, there is a connection between these densities, but they can be far apart.

Theorem 2.17. (i) *If for $A \subset \mathbb{N}$ $\overline{\delta}(A) > 0$, then $\overline{\delta}(Q_A) > 0$.*

(ii) *For all $\varepsilon > 0$, $\delta > 0$ there is a set $A \subset \mathbb{N}$ such that $\underline{\delta}(A) > 1 - \varepsilon$ however, $\overline{\delta}(Q_A) < \delta$.*

Proof. (i) Erdős/Davenport Theorem. (ii) Turan-Kubilius inequality.

Remark 2.18. *Studied also $Q_A^\infty = \bigcap_{n=1}^\infty \left(\bigcup_{a\geq n, a\in A} Q_A^a\right) =$ integers $q > 1$ with ∞-many representations $q = \frac{a'}{a}$, $a, a' \in A$.*

We conclude with [159] "On primitive sets of squarefree integers". For $A \subset \mathbb{N}$ let $S(A) = \sum_{a\in A}\frac{1}{a}$. Erdős/Sárközy/Szemerédi proved in 1967:

$$\max_{A\subset[n] \text{ primitive}} S(A) = \big(1 + o(1)\big)\frac{\log n}{(2\pi\log\log n)^{1/2}} \text{ as } n \to \infty\,. \tag{10}$$

Theorem 2.19. *Let $Q = \{q_1, q_2, \dots\} = \{p_1^{\alpha_1}, p_2^{\alpha_2}, \dots\}$ (with $p_1 < p_2 < \dots$) be a set of powers of distinct primes with $S(Q) < \infty$, then we have*

$$\max_{\substack{A\subset[n] \text{ primitive} \\ q\nmid a \text{ for } a\in A, q\in Q}} S(A) = \big(1+o(1)\big) \prod_{q\in Q}\left(1-\frac{1}{q}\right)\frac{\log n}{(2\pi\log\log n)^{1/2}} \text{ as } n\to\infty\,. \quad (11)$$

Note that here $Q = \varnothing$ is allowed and, indeed, in this special case we obtain the Theorem of Erdős/Sárközy/Szemerédi. For $Q = \{2^2, 3^2, 5^2, \dots, p^2, \dots\}$, the result was conjectured by Pomerance/Sárközy in 1988.

Another important special case is when Q consists of the primes not exceeding a fixed number K:

Corollary 2.20. *If $K \geq 2$, then we have*

$$\max_{\substack{A\subset[n] \text{ primitive} \\ (a,\prod_{p\leq K} p)=1 \text{ for all } a\in A}} S(A) = \big(1+o(1)\big) \prod_{p\leq K}\left(1-\frac{1}{p}\right)\frac{\log n}{(2\pi\log\log n)^{1/2}} \text{ as } n\to\infty\,.$$

Moreover, we can prove that if Q is finite, then Q need not consist of prime powers, it suffices to assume coprimality:

Theorem 2.19'. *Let $Q = \{q_1, \dots, q_t\}$ be a finite set of pairwise coprime positive integers: $(q_i, q_j) = 1$ for $1 \leq i < j \leq t$. Then (11) holds.*

It comes perhaps as a **surprise** that the heuristics that the density $\frac{6}{\pi^2}$ of the square free integers $\mathbb{N}^*$ should extend to the maximal primitive set, which would give

$$\max_{A\subset\mathbb{N}^*(n) \text{ primitive}} |A| = \big(1+o(1)\big)\frac{6}{\pi^2}\max_{A\in\mathbb{P}_n}|A| = \big(1+o(1)\big)\frac{3}{\pi^2}n \text{ as } n\to\infty,\ \textbf{fails}\,.$$

We will prove that this maximum is much greater, but can be estimated surprisingly well.

Theorem 2.21. *For N large we have $0.6362\frac{6}{\pi^2}N < \max_{A\subset\mathbb{N}^*(N)}|A| < 0.6366\frac{6}{\pi^2}N$.*

2.4. Cross-primitive sequences

With L. H. Khachatrian we introduced and analyzed the following concept. It is reported in [116] "Classical results on primitive and recent results on cross-primitive sequences", Paul Erdős 80, Graham/Nesetril edit. (A, B), $A, B \subset C \subset \mathbb{N}$ is *cross-primitive* in C, if $a \nmid b$; $b \nmid a$ for $a \in A, b \in B$.

Theorem 2.22. *For $M_n = \max_{(A,B) \textit{ cross-primitive in } [n]} |A||B|$*

(i) $\frac{M_n}{n^2} \leq \frac{1}{4}$.
(ii) $\lim_{n\to\infty}\frac{M_n}{n^2} = \frac{1}{4}$.

Theorem 2.23. *For $C = \mathbb{N}$*

(i) $\max_{(A,B) \textit{ cross-primitive}} \overline{d}(A)\overline{d}(B) = \frac{1}{4}$.
(ii) $\max_{(A,B) \textit{ cross-primitive}} \underline{d}(A)\cdot\underline{d}(B) = \frac{1}{16}$
$= \max_{(A,B) \textit{ cross-primitive}, d(A), d(B) \textit{ exist}} d(A)d(B)$.

3. Combinatorial Intersection Theorems

3.1. The complete intersection theorem and some extensions

In the paper [122] together with L. H. Khachatrian we were concerned with one of the oldest problems in combinatorial extremal theory. A system of sets $\mathcal{A} \subset \binom{[n]}{k}$ is called t-intersecting, if $|A_1 \cap A_2| \geq t$ for all $A_1, A_2 \in \mathcal{A}$, and $I(n,k,t)$ denotes the set of all such systems. The investigation of the function $M(n,k,t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|$, $1 \leq t \leq k \leq n$, and the structure of maximal systems was initiated by Erdős, Ko and Rado. Already in 1938 they proved and in 1961 they published

Theorem EKR. *For $1 \leq t \leq k$ and $n \geq n_0(k,t)$ (suitable) $M(n,k,t) = \binom{n-t}{k-t}$.*

This means that for large n the "naive" configuration $\mathcal{A} = \left\{A \in \binom{[n]}{k} : [1,t] \subset A\right\}$ is optimal. The smallest $n_0(k,t)$ for which this is the case has been determined by Frankl 1978 for $t \geq 15$ in [F2] and Wilson 1984 for all t in [W]: $n_0(k,t) = (k-t+1)(t+1)$. All cases are settled in

Theorem 3.1. (Complete Intersection Theorem) (Ahlswede/Khachatrian [122]) *For $1 \leq t \leq k \leq n$ with $(k-t+1)\big(2+\frac{t-1}{r+1}\big) < n < (k-t+1)\big(2+\frac{t-1}{r}\big)$ for some $r \in \{0\} \cup \mathbb{N}$ we have*

$$M(n,k,t) = |\mathcal{F}_r| = \left|\left\{F \in \binom{[n]}{k} : |F \cap [1, t+2r]| \geq t+r\right\}\right|$$

and $\mathcal{F}_r$ is —up to permutations— the unique optimum. (By convention $\frac{a}{0} = \infty$.)

For $(k-t+1)\big(2+\frac{t-1}{r+1}\big) = n$ for $r \in \mathbb{N}\{0\}$ we have $M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$ and an optimal system equals —up to permutations— either $\mathcal{F}_r$ or $\mathcal{F}_{r+1}$.

In particular, this theorem shows the validity of the **famous 4m-conjecture** of Erdős, Ko, Rado (1938), that is $M(4m,2m,2) = |\{F \in \binom{[4m]}{2m} : |F \cap [1,2m]| \geq m+1\}|$.

Remark 3.2. *The EKR Theorem is the most frequently quoted result of Erdős. In 1983 Deza and Frankl wrote a paper "The Erdős/Ko/Rado Theorem - 22 years later".*

The theorem presented and proved in [114] can be viewed as an extension or improvement of the Complete Intersection Theorem, just mentioned above. It goes considerably beyond the well-known Hilton/Milner Theorem and completely answers the question of determination of non-trivial t-intersecting families. (An $\mathcal{A} \in I(n,k,t)$ is called non-trivial if $\left|\bigcup_{A \in \mathcal{A}} A\right| < t$.)

In [137] together with H. Aydinian and L. H. Khachatrian the problem of maximal intersecting systems for direct products is considered. This problem was initiated by Frankl and arose in connection with a result of Sali. Let $n = n_1 + \cdots + n_m$, $k = k_1 + \cdots + k_m$, $[n] = [n_1] \cup [n_2] \cup \cdots \cup [n_m]$, $\mathcal{H} = \left\{F \in \binom{[n]}{k} : |F \cap [n_i]| = k_i\right.$ for $i = 1, \ldots, m\}$. For given integers t_i, $1 \leq t \leq t_i \leq k_i$, $1 \leq i \leq m$, we may say that $\mathcal{A} \subset \mathcal{H}$ is $(t_1, \dots, t_m)$-intersecting, if for every $A, B \in \mathcal{A}$ there exists an i, $1 \le i \le m$, such that $|A \cap B \cap \Omega_i| \ge t_i$ holds. Denote the set of such systems by $I(\mathcal{H}, t_1, \dots, t_m)$. The problem is to determine $\max_{\mathcal{A} \in I(\mathcal{H}, t_1, \dots, t_m)} |\mathcal{A}|$.

The case $t_1 = t_2 = \cdots = t_m = 1$ has been solved by Frankl. Here is the complete solution.

Theorem 3.3. (Ahlswede/Aydinian/Khachatrian [137]) *Let $n_i \ge k_i \ge t_i \ge 1$ for $i = 1, \dots, m$, then* $\max_{\mathcal{A} \in I(\mathcal{H}, t_1, \dots, t_m)} = \max_i \frac{M(n_i, k_i, t_i)}{\binom{n_i}{k_i}} |\mathcal{H}|$.

We emphasize that the combination of this theorem and theorem 3.1 gives an explicit value. The proof is heavily (but not only!) based on ideas and methods from [112], in particular the method of "generated sets" (c.f. [N] Bey/Engel, "Old and New Results for the Weighted t-Intersection Problem via AK-Methods", 45–74;) it takes a central role in the recent book "Sperner Theory" by K. Engel.

3.2. The diametric theorem in Hamming spaces

For $\alpha^{[n]} = \{0, 1, \dots, \alpha - 1\}^n$, endowed with the Hamming distance function d_H $\mathcal{A} \subset \alpha^{[n]}$ has a diameter d, if $\operatorname{diam}(\mathcal{A}) = \max_{a^n, b^n \in \mathcal{A}} d_H(a^n, b^n) = d$.

Given n, α, d find: $N_\alpha(n, d) = \max_{\mathcal{A} \subset \alpha^{[n]}, \operatorname{diam}(\mathcal{A}) = d} |\mathcal{A}|$ or equivalently "*find optimal anticodes*".

Previously known were the cases with pairs (d, α) of the form $(d, 2)$ [K], $(n-1, \alpha)$ [B], $\{(d, \alpha) : n \le d + \alpha - 1\}$ [FF], $\{(d, \alpha) : n \ge (\alpha - 1)^{d-1} + d\}$ [Ahlswede/Cai/Zhang [72]]. These authors conjectured the theorem below. It was conjectured in an equivalent form by Frankl/Füredi already in 1980. For $0 \le i \le \frac{d}{2}$ define $\mathcal{K}_i = \{a^n \in \alpha^{[n]} : (a_1, \dots, a_n - d + 2i)$ has at least $n - d + i$ zeros$\}$. Clearly $\mathcal{K}_i$ has diameter d.

Theorem 3.4. (Ahlswede/Khachatrian [132]) *Let r be the largest integer s.t. $n - d + 2r < \min\left\{n + 1, n - d + 2 \cdot \frac{n-d+1}{\alpha - 2}\right\}$, then $N_\alpha(n, d) = |\mathcal{K}_r|$.*

Moreover, up to permutation of $(12 \dots n)$ and permutations of the alphabet in the components the **optimal configuration is unique**, *unless $n - d > 1$, $n - d + 2\frac{n-d-1}{\alpha-2} \le n$ and $\frac{n-d-1}{\alpha-2}$ is integral, in which case we have* **two optimal configurations**: *$\mathcal{K}_{\frac{n-d-1}{\alpha-2}}$ and $\mathcal{K}_{\frac{n-d-1}{\alpha-2}-1}$.*

The result is derived from the Complete Intersection Theorem (which can also be viewed as a Diametric Theorem for d_H on the space $\binom{[n]}{k}$)) via a Comparison Lemma.

3.3. A pushing-pulling method

It came as a surprise to us that at first we did not succeed to derive Katona's Intersection Theorem for the unrestricted case, that is in the space $2^{[n]}$, by the method of "generated sets". This led us to the discovery of another method, which yields theorem 3.1 and Katona's Intersection Theorem. Subsequently we found a way to derive Katona's Theorem from theorem 3.1 via another Comparison Lemma. This is the most complicated proof, where several simple proofs exist, but it teaches something about methods, which made progress possible on the t-Intersection Problem in the truncated Boolean Lattice covering the restricted and the unrestricted intersection problem as special cases (Ahlswede/Bey/Engel/Khachatrian [ABEK]). Whereas there are vertex- and edge-isoperimetric theorems (see section 5) it went unsaid that diametric theorems are vertex-diametric theorems. We complete the story by introducing edge-diametric theorems into combinatorial extremal theory. Using again the pushing/pulling method we establish such a result for $\mathcal{V} = \{0,1\}^n$ and $\mathcal{E} = \{(a^n, b^n) : a^n, b^n \in \mathcal{V}\}$. Results and methods of this section are discussed in the survey [N, p. 45–74].

3.4. Other types of intersection problems

One type consists in looking at intersecting chains in the Boolean Lattice. It is mentioned in the context of Higher Level Extremal Problems in section 4. Another one, with origin in and interest to Computer Science, was communicated to us by R. C. Mullin in 1990 in Oberwolfach.

For a finite alphabet $[\alpha] = \{0, 1, \dots, \alpha - 1\}$ we consider the set $\alpha^{[n]}$ of words of length n and also its subsets W_α^n of words without repetition of letters, that is, $W_\alpha^n = \{x^n = (x_1, x_s, \dots, x_n) \in \alpha^{[n]} : x_s \neq x_t \text{ for } s \neq t\}$. We write $x^n \swarrow\!\searrow y^n$ if for some $s \neq t$ $x_s = y_t$. The set $F \subset \alpha^{[n]}$ is "good", if for all $x^n, y^n \in F$ $x^n \swarrow\!\searrow y^n$. Denoting the family of all good sets in W_α^n by $\mathcal{F}_\alpha^n$ the quantity of interest is $f_\alpha^n = \max\{|F| : F \in \mathcal{F}_\alpha^n\}$. Its determination constitutes an extremal problem in a (growing) class of similar problems whose prototype or historically first candidate is the intersection problem of EKR.

Clearly, it is certainly also meaningful to study $\mathcal{G}_\alpha^n$, the family of all good sets in $\alpha^{[n]}$, and the quantity $g_\alpha^n = \max\{|E| : E \in \mathcal{G}_\alpha^n\}$.

The functions f_α^n and g_α^n are rather complex. We present here results for the first two non-trivial configurations of the parameters α and n, namely the cases $n = \alpha - 1$ and $n = 3$. Also, we have a limit theorem for α tending to infinity. Specifically, we have the following results.

Theorem 3.5. (Ahlswede/Cai [91])

(i) $f_\alpha^{\alpha-1} = \frac{1}{2}|W_\alpha^{\alpha-1}| = \frac{1}{2}\alpha!$ *Moreover, we determine all optimal configurations.*
(ii) $f_\alpha^3 = f_\infty^3 = f_4^3 = 12$ *for* $\alpha \geq 4$.

Theorem 3.6. (Ahlswede/Cai [91])

(i) $g_\alpha^3 = 3\alpha + 7$ *for* $3 \leq \alpha < \infty$.
(ii) $\lim_{\alpha\to\infty} \frac{g_\alpha^n}{\binom{\alpha-1}{n-2}} = \binom{n}{2}(n-2)!$ *or, equivalently,* $g_\alpha^n = \alpha^{n-2}(\binom{n}{2} + o(1))$ *as* $\alpha \to \infty$.

4. Higher Level Extremal Problems

4.1. Coding for write-efficient rewritable memories

Imagine a tape with n cells into which we can write letters from an alphabet $\mathcal{X}$. A word $x^n = (x_1, \dots, x_n)$ stores some messages. When we want to update this record to a message represented by $y^n = (y_1, \dots, y_n)$ the per letter costs $\varphi(x_t, y_t)$ add up to $\varphi_n(x^n, y^n) = \sum_{t=1}^n \varphi(x_t, y_t)$. In order to be able to update many messages under a cost constraint D we come to the diametric problem to charaterize

$$M(\varphi_n, D) = \max\{|C| : \varphi_n(x^n, y^n) \le D \text{ for all } x^n, y^n \in C\} \tag{12}$$

for the "sum-type" cost φ_n, which also can be a distance function like the Hamming, Taxi or Lee metric, etc. These problems are discussed in section 5.

A simple, but basic, observation is that there is an advantage in having for every message i a set C_i (called "cloud") $\subset \mathcal{X}^n$ of possible representations such that for any representation $x^n \in C_i$ there exists a representation $y^n \in C_j$ with $\varphi_n(x^n, y^n) \le D$,

Example 4.1. *For* $\mathcal{X} = \{0, 1\}$, $n = 3$, $\varphi_3 = d_H$ *and* $D = 1$ *we have* $M(d_H, 1) = 2$. *On the other hand* there are 4 clouds $C_1 = \{000, 111\}$, $C_2 = \{100, 011\}$, $C_3 = \{010, 101\}$, $C_4 = \{001, 110\}$, *which can be used for updating* 4 *messages at cost* 1.

More generally we introduced together with Z. Zhang [62] *write-efficient memories* (WEM) as a new model for storing and updating information on a rewritable medium. There is a cost $\varphi\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_\infty$ assigned to changes of letters. A collection of subsets $\mathcal{C} = \{C_i : 1 \le i \le M\}$ of $\mathcal{X}^n$ is an (n, M, D) WEM code, if $C_i \cap D_j = \varnothing$ for all $i \ne j$ and if $D_{\max} = \max_{1 \le i,j \le M} \max_{x^n \in C_i} \min_{y^n \in C_j} \sum_{t=1}^n \varphi(x_t, y_t) \le D$. $D_{\max}$ is called the maximal correction cost with respect to the given cost function. The performance of a code $\mathcal{C}$ can also be measured by two parameters, namely, the maximal cost per letter $d_{\mathcal{C}} = n^{-1} D_{\max}$ and the rate of the size $r_{\mathcal{C}} = n^{-1} \log M$. The rate achievable with a maximal per letter cost d is thus $R(d) = \sup_{\mathcal{C} : d_{\mathcal{C}} \le d} r_{\mathcal{C}}$.

This is the most basic quantity (the updating capacity for maximal per letter cost d) of a WEM $(\mathcal{X}^n, \varphi^n)_{n=1}^\infty$. We give a characterization of this quantity. For this we need some definitions. For a set $\mathcal{Z}$, $\mathcal{P}(\mathcal{Z})$ denotes the set of all probability distributions on $\mathcal{Z}$. Let (X, Y) be a pair of random variables with values in $\mathcal{X} \times \mathcal{X}$ and distribution P_{XY}. We denote the (marginal) distributions of X (resp. Y) by P_X (resp. P_Y) and $H(X \mid Y)$ is the conditional entropy. Finally we need distributions

$$\mathcal{P}_d = \{P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{X}) : P_X = P_Y, \mathbb{E}\varphi(X, Y) \le d\}, \tag{13}$$

with equal marginals and an expectation of costs not greater than d and the quantity $\rho(d) = \max_{P_{XY} \in \mathcal{P}_d} H(Y \mid X)$.

Theorem 4.2. *For any* $d \ge 0$ *and* $\varphi\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ $R(d) = \rho(d)$.

4.1.1. The structure of theorem 2.2 in the Hamming case Hypergraph $(\mathcal{V},\mathcal{E}) = (\mathcal{X}^n, (S_D^n(x^n))_{x^n\in\mathcal{X}^n}$, $\kappa(\mathcal{V},\mathcal{E})$ = maximal number of colors assigned to vertices, such that every color occurs in every edge, that is ball $S_D^n(x^n)$ (General Ramsey Problem) $C_i = \{$ vertices with color $i\}$. A key tool in the spirit of [35], [36] is

Color Carrying Lemma. *For every hypergraph $\mathcal{H} = (\mathcal{V},\mathcal{E})$*

$$\kappa(\mathcal{V},\mathcal{E}) \geq (\ell n|\mathcal{V}|)^{-1} \min_{E\in\mathcal{E}} |E|\,.$$

Since in our case $|\mathcal{V}| = |\mathcal{X}^n|$ and $|E| = |S_D^n(x^n)|$ grow exponential in n, if $D = dn$, we get the optimal rate $R(d) = h(d)$, where h is the binary entropy function, —a very special case of theorem 4.2. It is remarkable that in sequence spaces with cardinality of exponential growth these Ramsey type problems can be solved at least rate-wise.

There are still issues of code *constructions*. Further, instead of worst case costs one can consider also average costs, that is, diametric problems in the average. Several interesting questions arise, if *several persons or devices use the same tape* under various conditions: multi-user memories with constraints on privacy, hierarchy and technology [92].

Notice that the two models mentioned above can be described by the side information about the content of the tape available to the writer W, before he writes a new word. The reader R remembers no previous word.

Case (W^-,R^-) amounts to diametric problems. Case (W^+,R^-) gives our model with clouds. Formally, for all (i,j): $\forall x^n\in C_i\ \exists y^n\in C_j$ with $\varphi_n(x^n,y^n)\leq D$.

4.2. From here we pass to an independent combinatorial investigation

In [119] we studied with N. Cai and Z. Zhang families of clouds $(\mathcal{A}_i)_{i=1}^M$, $\mathcal{A}_i\subset 2^{[n]}$ with

(a) *relations* "$\subset$" (comparable), "$\supset\!|\!\subset$" (incomparable), "intersecting", "disjoint".
(b) *properties* $(\forall,\exists)$, $(\exists,\forall)$, $(\forall,\forall)$, $(\exists,\exists)$.
(c) "Disjoint" clouds, "distinct" clouds. (It is erroneously stated in [N] Erdős/Székely on page 118 that we always assume disjointness.)
(d) "restricted" case $|A| = k\ \forall A\in\bigcup_{i=1}^M \mathcal{A}_i$ and "unrestricted" case.
(e) For length M bound size $\max_{1\leq i\leq M}|\mathcal{A}_i|$.

We explain now our symbolic notation in a particular case.

4.2.1. Cloud antichains

Classical: $\{A_i\}_{i=1}^N$, $A_i\subset(2^{[n]})$ is an antichain, if $A_i \not\supset\!\not\subset A_j\ \forall i\neq j$. Now sets are replaced by clouds, that is, families of sets. $(\mathcal{A}_i)_{i=1}^N$ is of

type $(\forall,\forall)$, if for all $i\neq j$ $A_i \not\supset\!\not\subset A_j$ $\forall A_i\in\mathcal{A}_i$, $\forall A_j\in\mathcal{A}_j$
type $(\exists,\forall)$, if for all $i\neq j$ $\exists A_i\in\mathcal{A}_i$ with $A_i \not\supset\!\not\subset A_j$ $\forall A_j\in\mathcal{A}_j$
type $(\forall,\exists)$, if for all $i\neq j$ $\forall A_i\in\mathcal{A}_i$ $\exists A_j\in\mathcal{A}_j$ with $A_i \not\supset\!\not\subset A_j$
type $(\exists,\exists)$, if for all $i\neq j$ $\exists A_i\in\mathcal{A}_i$, $\exists A_j\in\mathcal{A}_j$ with $A_i \not\supset\!\not\subset A_j$.

Maximal lengths $N_n(\forall,\forall)$, $N_n(\exists,\forall)$ for distinct and $M_n(\forall,\forall), M_n(\exists,\forall),\dots$ for *disjoint clouds.*

Convention: $f\sim g$ iff $\lim_{n\to\infty} f(n)g(n)^{-1}=1$. Work with L. H. Khachatrian [86] is

Theorem 4.3.

$$M_n(\exists,\forall)\sim 2^{n-1},$$

$$M_n(\forall,\exists)=\begin{cases}2 & \text{if } n=2\\ 2^{n-1}-1 & \text{if } n\geq 3\end{cases}$$

$$M_n(\exists,\exists)=\binom{n}{\lfloor\frac{n}{2}\rfloor}+\left\lfloor\frac{2^n-2-\binom{n}{\lfloor\frac{n}{2}\rfloor}}{2}\right\rfloor .$$

Theorem 4.4. (Double exponential growth) $N_n(\exists,\forall)=\binom{k}{\lfloor\frac{k}{2}\rfloor}$ *with* $k=\binom{n}{\lfloor\frac{n}{2}\rfloor}$, $N_n(\forall,\exists)\sim 2^{2^n-2}$, $N_n(\exists,\exists)\sim 2^{2^n}$.

The classical Sperner result takes the forms $M_n(\forall,\forall)=N_n(\forall,\forall)=\binom{n}{\lfloor\frac{n}{2}\rfloor}$.

For the maximal cardinality of families with the relations "comparable", "disjoint", and "intersecting" we choose the letters C, D, and I respectively. The types of problems such as $(\forall,\forall)$ etc. appear in the argument and n appears as index. In addition, in the restricted case a k appears in the argument.

4.3. Cloud-antichains (CAC) of length 2

Theorem 4.5. (Ahlswede/Zhang [68]) *A CAC* $\{\mathcal{A},\mathcal{B}\}$ *in* $2^{[n]}$ *satisfies*

(i) $|\mathcal{A}||\mathcal{B}|\leq 2^{2n-4}$. *Optimal configurations:*
  1. $\mathcal{A}=\{X\in 2^{[n]}:1\in X, 2\notin X\}$,
     $\mathcal{B}=\{X\in 2^{[n]}:1\notin X, 2\in X\}$.
  2. $\mathcal{A}=\left\{X\in 2^{[n]}:1\in X, |X|\leq\left\lfloor\frac{n-1}{2}\right\rfloor+1\right\}$,
     $\mathcal{B}=\left\{X\in 2^{[n]}:1\notin X, |X|>\left\lceil\frac{n-1}{n}\right\rceil\right\}$ *n odd.*

(ii) $\min\{|\mathcal{A}|,|\mathcal{B}|\}\leq 2^{n-2}$.

The proof uses AD-inequality. A sharper result is

Theorem 4.6. (Ahlswede/Khachatrian [107]) *For* $0\leq\alpha\leq 2^{n-1}$, *if* $f_n(\alpha)\triangleq\max\{|\mathcal{B}|:\exists\mathcal{A}$ *such that* $(\mathcal{A},\mathcal{B})$ *CAC in* $2^{[n]}$ *and* $|\mathcal{A}|=\alpha$, *then* $f_n(\alpha)=2^{n-1}+2f_{n-2}^{(\alpha)}-\alpha$.

For multisets of multiplicity k again AD and the arithmetic-geometric means inequality give $|\mathcal{A}|^{1/2}+|\mathcal{B}|^{1/2}\leq k^{n/2}$ for $(\mathcal{A},\mathcal{B})$ CAC in $k^{[n]}$.

Using results of [DKW] we provide again with L. H. Khachatrian [107] partial results for

Question I. *For every k describe all CAC* $(\mathcal{A},\mathcal{B})$ *with equality in 4.6 In the terminology of [100] this is an equality characterization problem.*

Question II. *How does* $f_{n,k}(\alpha) = \max\{|\mathcal{B}| : \exists(\mathcal{A},\mathcal{B})$ *CAC in* $k^{[n]}$, $|\mathcal{A}| = \alpha\}$ *behave asymptotically in* k, n*, and* α*?*

In generalising statements (i), (ii) in theorem 2.11 we completely answer for every k.

Question III. *What is the growth of* $g_{n,k} = \max_{(\mathcal{A},\mathcal{B})} \min_{CAC\ in\ k^{[n]}} (|\mathcal{A}|, |\mathcal{B}|)$*?*

Question IV. *What is the growth of* $S_{n,k} = \max_{(\mathcal{A},\mathcal{B})\ CAC\ in\ k^{[n]}} |\mathcal{A}||\mathcal{B}|$*?*

Question V. *What is the growth of* $a_{n,k}(\Delta) = \max\{|\mathcal{B}| : \exists\mathcal{A}$ *such that* $(\mathcal{A},\mathcal{B})$ *is CAC in* $k^{[n]}$ *and* $|\mathcal{A}| = |\mathcal{B}| + \Delta\}$ *for* $-k^n \le \Delta \le k^n$*?*

4.4. Intersection, clouds disjoint

Theorem 4.7. (Ahlswede/Cai/Zhang [95]) *In the* **restricted** *case* $k = 2$

$$I_n(\exists,\forall,2) = \begin{cases} n-1 & \text{for } n \in \mathbb{N} - \{3,5\} \\ n & \text{for } n = 3,5 \end{cases},$$

$$I_n(\forall,\exists,2) = \begin{cases} n & \text{for } n \in \mathbb{N} - \{1,2,4\} \\ n-1 & \text{for } n = 1,2,4, \end{cases}$$

$$I_n(\exists,\exists,2) \sim n^{3/2},$$

$$I_n(\forall,\forall,2) = \begin{cases} n-1 & \text{for } n \in \mathbb{N} - \{3\} \\ n & \text{for } n = 3 \end{cases} \quad \textit{(very special case of EKR)}.$$

Theorem 4.8. (Ahlswede/Cai/Zhang [119]) *In the* **unrestricted** *case*

$$I_n(\forall,\forall) = I_n(\exists,\forall) = I_n(\forall,\exists) = 2^{n-1} \quad I_n(\exists,\exists) = 2^{n-1} + 2^{n-2} - 1.$$

4.5. Disjoint, clouds disjoint

Theorem 4.9. (Ahlswede/Cai/Zhang [95]) *In the* **restricted** *case* $k = 2$

$$\lim_{n\to\infty} D_n(\exists,\forall,2)n^{-2} = \frac{1}{6},$$

$$\lim_{n\to\infty} D_n(\forall,\exists,2)n^{-2} = \lim_{n\to\infty} D_n(\exists,\exists,2)n^{-2} = \frac{1}{4}.$$

Theorem 4.10. (Alon/Sudakov (see [N, p. 123]**))** *In the* **restricted** *case*

$$\lim_{n\to\infty} \frac{D_n(\exists,\forall,k)}{\binom{n}{k}} = \frac{1}{k+1},$$

$$\lim_{n\to\infty} \frac{D_n(\forall,\exists,k)}{\binom{n}{k}} = \lim_{n\to\infty} \frac{D_n(\exists,\exists,k)}{\binom{n}{k}} = \frac{1}{2}.$$

Conjectured by Ahlswede/Cai/Zhang [119], who settled the case $k = 2$.

4.6. Key tools are results on related graph coloring problems

This and the next paragraph give results from [119]. The study of cloud families of the $(\exists,\exists)$-type naturally leads to the following coloring concept. For any graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ a coloring of type $(\exists,\exists)$ is a map $f\colon \mathcal{V}\to M_f=\{1,2,\dots,m_f\}$ such that for any two colors, say, $i,j\in M_f$, an edge $(a,b)\in\mathcal{E}$ exists with $f(a)=i$ and $f(b)=j$.

We are interested in the quantity $m(\mathcal{G})=\max\{m_f : f \text{ is } (\exists,\exists)\text{-coloring of } \mathcal{G}\}$.

Theorem 4.11. *For any graph* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *we have with* $N=2|\mathcal{E}|$ $m(\mathcal{G})\le N^{1/2}+1$.

Moreover, if $D\triangleq \max_{V\in\mathcal{V}}\deg(x)\le\left(\frac{N}{e^4\log N}\right)^{1/2}$, *then* $m(\mathcal{G})\ge\left(\frac{N}{e^4\log N}\right)^{1/2}$.

Cloud families of $(\forall,\exists)$-type lead to a coloring of $(\forall,\exists)$-type, which is a map $g\colon\mathcal{V}\to M_g=\{1,2,\dots,m_g\}$ such that for any two colors, say, $i,j\in M_g$ and for any $a\in U$ with $g(a)=i$ there is an edge $\{a,b\}\in\mathcal{E}$ with $g(b)=j$. We are interested in $m^*(g)=\max\{m_g : g \text{ is } (\forall,\exists)\text{-coloring of } \mathcal{G}\}$

Theorem 4.12. *For any graph* $\mathcal{G}$ *we have*

(i) $(\log|\mathcal{V}|)^{-1}(d+1)\le m^*(\mathcal{G})\le d+1$, *where* $d\triangleq\min_{x\in\mathcal{V}}\deg(x)$.
(ii) $m^{*\prime}(\mathcal{G})\triangleq\max|\{m^*(\mathcal{G}') : \mathcal{G}' \textit{ is subgraph of } \mathcal{G}\}|\le D+1$.

We mention that a coloring of type $(\exists,\forall)$ is a map $h\colon\mathcal{V}\to M_h=\{1,2\dots,m_h\}$ such that for any two colors $i,j\in M_h$ an $a\in\mathcal{V}$ exists with $h(a)=i$ and $h(b)=j$ for all $b\in\mathcal{N}(a)$, $b\ne a$. The quantity $m^{**}(\mathcal{G})=\max\{m_h : h \text{ is } (\exists,\forall)\text{-coloring of } \mathcal{G}\}$ is hard to analyse in general.

4.7. Asymptotic results via graph coloring

Theorem 4.13.

$$\lim_{n\to\infty}\frac{1}{n}\log C_n(\exists,\exists)=\lim_{n\to\infty}\frac{1}{n}\log D_n(\exists,\exists)=\frac{1}{2}\log 3\,.$$

Theorem 4.14.

$$\lim_{n\to\infty}\frac{1}{n}\log D_n(\exists,\exists,\varepsilon n)=\frac{1}{2}\left(h(\varepsilon)+(1-\varepsilon)h\left(\frac{\varepsilon}{1-\varepsilon}\right)\right),$$
$$\lim_{n\to\infty}\frac{1}{n}\log I_n(\exists,\exists,\varepsilon n)=h(\varepsilon)\,.$$

Theorem 4.15.

$$\lim_{n\to\infty}\frac{1}{n}\log D_n(\forall,\exists,\varepsilon n)=(1-\varepsilon)h\left(\frac{\varepsilon}{1-\varepsilon}\right),$$
$$\lim_{n\to\infty}\frac{1}{n}\log I_n(\forall,\exists,\varepsilon n)=h(\varepsilon)\,.$$

4.8. Hamming distance 1, clouds disjoint

An important relation is that of Hamming distance r for two words.

Theorem 4.16. (Ahlswede/Cai/Zhang [119]) *In the* **unrestricted** *case*

$$\lim_{n\to\infty}\frac{1}{n}\log H_n(\forall,\forall,\rho n)=\frac{1}{2}(1+h(\rho))\,.$$

Notice that $H_n(\exists,\exists,1)$ equals the maximal number of sets into which one can partition the n-cube, such that two different sets always have distance 1.

Theorem 4.17. (R. Ahlswede, S. L. Bezrukov, A. Blokhuis, K. Metsch and G. E. Moorhouse [80]) $\frac{\sqrt{2}}{2}\sqrt{n2^n} \leq H_n(\exists,\exists,1) \leq \sqrt{n2^n}+1$ *for all* n.

4.9. Another direction

Let $\mathcal{I}_n$ be the lattice of intervals in the Boolean lattice $\mathcal{L}_n$. For $\mathcal{A},\mathcal{B} \subset \mathcal{I}_n$ the pair of clouds $(\mathcal{A},\mathcal{B})$ is cross-disjoint, if $I \cap J = \phi$ for $I \in \mathcal{A}$, $J \in \mathcal{B}$. With N. Cai [109] we prove that for such pairs $|\mathcal{A}||\mathcal{B}| \leq 3^{2n-2}$ and that this bound is best possible.

Optimal pairs are up to obvious isomorphisms unique. The proof is based on a new bound on cross intersecting families in $\mathcal{L}_n$ with a weight distribution. It implies also an Intersection Theorem for multisets of Erdős and Schönheim from 1969.

Furthermore, in [115] in a canonical way we establish an AZ-identity and its consequences, the LYM-inequality and the Sperner-property. Further the Bollobás-inequality for the Boolean interval lattice turns out to be just the LYM-inequality for the Boolean lattice. We also present an Intersection Theorem for this lattice.

Perhaps more surprising is that by our approach the conjecture of P. L. Erdős, Seress, Székely [ErSS] and Füredi concerning an Erdős-Ko-Rado-type intersection property for the poset of Boolean chains could also be established. Actually we give two seemingly elegant proofs.

5. Diametric, Isoperimetric Theorems in Sequence Spaces

5.1. Introduction

Mankind believes the Isoperimetric Theorem in Euclidean 3-space "For given surfaces the ball has maximal volumes" for more than 2000 years. The discovery of the 2-dimensional analog is often attributed to Dido, the daughter of a Phoenician King. Despite strong interest in *extremal problems* and *variational principles* in physics (and also philosophy: "Best of all worlds" for Leibnitz) in modern times after the invention of calculus a proof came only in the 19-th century by Schwarz —after an incomplete geometrical proof by Steiner, showing the uniqueness but not the existence of a solution.

Replacing surface by diameter leads to (iso)-diametric Theorems. A classic is Blaschke's "Kreis and Kugel". In sequence spaces $\alpha^{[n]}$ cardinalities take the role of volumes of subsets. For some distance function d (like Hamming, Lee or Taxi metrics) surface $\Gamma_d(A)$ is the set of points in the complement of A and with distances 1 to A.

Harper's solution of the isoperimetric problem in Hamming space $(2^{[n]}, d_H)$ is mentioned in section 6. *The problem is open for* $\alpha > 2$. However, recently a **"rate-wise" optimal solution** was found with Z. Zhang for the r-th surface $\Gamma^r_{\Phi_n}(A) = \{b^n : b^n \notin A,\ \varphi_n(b^n,a^n) \leq r \text{ for some } a^n \in A\}$ with $r = \rho n$, where $\varphi_n(b^n,a^n) =$

$\sum_{t=1} \varphi(b_t, a_t)$, φ: $[\alpha] \times [\alpha] \to \mathbb{R}$ is any symmetric "sum-type" function and not just the Hamming distance: $R(\lambda,\rho) = \lim_{n\to\infty} \frac{1}{n} \max_{|A|\leq \exp\{\lambda n\}} \log |\Gamma^{\rho n}_{\varphi_n}(A)|$.

Exact solutions are not even known for the *non-binary Hamming case*.

In section 3 the Diametric Theorem in Hamming space is mentioned. For $\alpha = 2$ optimal are balls and for $\alpha > 2$ optimal are certain cartesian products of a ball and a suitable subcube (or cylinder set). Depending on the parameters this configuration can degenerate to a ball and up to isometries (with one exception of two solutions) there is only one solution.

Notice that the Complete Intersection Theorem for parameters (n, k, t) can be viewed as a Diametric Theorem on the *restricted* Hamming space $\left(\binom{[n]}{k}, d_H\right)$ for diameter $D = 2k - 2t$. Another kind of diametric theorem is for an *average diameter* constraint (in 5.2 below).

We have now gained by example an understanding of the following classification:

restricted case	—	unrestricted case,
vertex-isoperimetric	—	edge-isoperimetric,
exact solution	—	rate-wise optimal solution
worst case	—	average case
vertex-diametric	—	edge-diametric

Coauthors in this work are I. Althöfer, S. Bezrukov, N. Cai, L. H. Khachatrian, E. Yang, Z. Zhang.

5.2. Rate-wise optimal solutions for the average case (vertex)-diametric problem

Exact solutions for the worst case vertex-diametric problem have been discussed in connection with Intersection Theorems in section 3 for the Hamming distance. An earlier result with Cai and Zhang [72] concerns the Taxi (or Manhattan) metric on $\alpha^{[n]}$ and gives solutions for almost all parameters. Bollobas/Leader noticed that the missing cases are covered by an earlier result of Kleitman/Fellow. A worst case diametric theorem for edges was mentioned in section 3. The first diametric theorem for the *average* was obtained with I. Althöfer with an rate-wise optimal solution: $\mathcal{U} \subset \mathcal{X}^n$ has an *average* diameter not exceeding D, if $D_{\text{ave}} \triangleq \frac{1}{|\mathcal{U}|^2} \sum_{x^n\in\mathcal{U}} \sum_{y^n\in\mathcal{U}} \varphi_n(x^n, y^n) \leq D$. With Katona [31] already in 1978 the restricted case $k = 2$ was considered in the dual form, where the cardinality of $\mathcal{U}$ is specified and D_{ave} is minimized. An exact solution is given in the form that either $\mathcal{U}$ in lexicographic or in backwards lexicographic order is optimal. With I. Althöfer [87] we proved by entropy methods

Diametric Theorem in Average. *For the Hamming space $(\alpha^{[n]}, d_H)$ and rate $0 \leq R \leq \log\alpha$ the smallest* average *diameter per letter*

$$\frac{1}{n}\overline{d}_n(R) \triangleq \min_{A_n\subset\alpha^{[n]}, \frac{1}{n}\log|A_n|\geq R} D_{ave}(A_n), n \in \mathbb{N},$$

satisfies

$$\begin{aligned}\overline{d}(R) &\triangleq \lim_{n\to\infty} \tfrac{1}{n}\overline{d}_n(R) \\ &= \min\left[\lambda \sum\nolimits_{x,y} d_H(x,y)P(x)P(y) + (1-\lambda)d_H(x,y)P'(x)P'(y)\right],\end{aligned}$$

where "min" is taken over $\lambda \in [0,1]$*, and probability distributions on* $[\alpha]$ *with* $\lambda H(P) + (1-\lambda)H(P') \geq R$*. Here* H *stands for the entropy.*

Writing $R = h(\beta)$ for $\alpha = 2$ we get $\overline{d}(R) = 2\beta(1-\beta)$. For $\alpha = 3$ calculation shows that $P \neq P'$ occurs in the optimization.

For general cost function the result holds with d_H replaced by φ_n as shown with Cai [127]. There are also extensions to several sets with some pairwise mutual average distance (or costs) and some internal average distances all simultaneously valid are treated. These generalizations are motivated by multiuser WEM (see section 4). The proofs use a tool from Information Theory to bound the cardinality of ranges of auxiliary random variables:

Support Lemma. (Ahlswede/Körner [21]) *Let* $\mathcal{P}(\mathcal{Z})$ *= set of all PD's on finite set* $\mathcal{Z}$*, let* $f_i(j = 1,\dots,k)\colon \mathcal{P}(\mathcal{Z}) \to \mathbb{R}$ *be continuous functions, and let* μ *be a PD on* $\mathcal{P}(\mathcal{Z})$ *with Borel* σ*-algebra, then there exist elements* $P_i \in \mathcal{P}(\mathcal{Z})$ *and* $\alpha_1,\dots,\alpha_k \geq 0$*,* $\sum_{i=1}^k \alpha_i = 1$ *such that* $\int_{\mathcal{P}(\mathcal{Z})} f_j(P)\mu(dP) = \sum_{i=1}^k \alpha_i f_i(P_i)$ *for* $j = 1,2,\dots,k$*.*

5.3. Edge-isoperimetric inequalities ratewise optimal

$\Psi_i\colon 2^{\mathcal{X}_i} \to \mathbb{R}(i = 1,2)$, $\Psi_1 \times \Psi_2\colon 2^{\mathcal{X}_1\times\mathcal{X}_2} \to \mathbb{R}$ defined by

$$\Psi_1 \times \Psi_2(A) = \sum\nolimits_{x\in\mathcal{X}_2} \Psi_1(A_1(x)) + \sum\nolimits_{x\in\mathcal{X}_1} \Psi_2(A_2(x)) \text{ for } A \subset \mathcal{X}_1 \times \mathcal{X}_2$$

$\Psi^n = (((\Psi \times \Psi) \times \Psi) \times \cdots \times \Psi)$ counts inner edges.

A. (Nestedness):

For $\mathcal{X} = \{0,1,\dots,\alpha-1\}$, $k \in \mathcal{X}$, $[k] = \{0,1,\dots,k\}$, $A \subset \mathcal{X}$, $|A| = k+1$

$\varphi(A) \leq \varphi([k])$ (Satisfied by $d_H, d_L, d_M, \dots$).

B. (Submodularity):

For $A, B \subset \mathcal{X}$ $\varphi(A) + \varphi(B) \leq \varphi(A\cup B) + \varphi(A\cap B)$.

C. $\varphi(\varnothing) = 0$ and $\Delta\varphi(k) = \varphi([k]) - \varphi([k-1])$.

A pair (R,δ) is achievable, if for all $\varepsilon_1, \varepsilon_2 > 0$ there exists an $n(\varepsilon_1,\varepsilon_2)$ such that for every $n \geq n(\varepsilon_1,\varepsilon_2)$ there is an $A_n \subset \mathcal{X}^n$ with $\left|\frac{1}{n}\log|A_n| - R\right| < \varepsilon_1$ and $\frac{1}{n|A_n|}\varphi^n(A) > \delta - \varepsilon_2$. $\mathcal{R}_\varphi$ is the set of all achievable pairs (R,δ).

Theorem 5.1. (Ahlswede/Cai [125], [126]) $\mathcal{R}_\varphi = \{(H(X|U), \mathbb{E}\Delta_\varphi(X)) : X, Y$ *satisfy* $(a),(b),(c)\}$

(a) *Random variable* X *takes values in* $\mathcal{X}$ *and random variable* U *takes values in* $\mathcal{U}$*.*
(b) $|\mathcal{U}| \leq |\mathcal{X}| + 1$*.*
(c) $Pr(X = 0|U = u) \geq Pr(X = 1|U = u) \dots \quad \geq Pr(X = \alpha - 1|U = u)$*.*

Remark 5.2. *Bollobas/Leader solved the case P_k^n = powers of k-paths and C_k^n = powers of k-cycles, which are equivalent via Ψ! All trees on k vertices are equivalent.*

5.3.1. LOCAL-GLOBAL PRINCIPLE Let Ψ satisfy A, B, C. If the **Lexicographic order is exactly optimal** for edge-isoperimetry **for $n = 1, 2$**, then it is optimal *for every* n. There is related work with S. L. Bezrukov [98] and by him and his coauthors, surveyed in [N, p. 75–94].

6. Combinatorics on Sequence Spaces: Partitions, Monochromatic Rectangles, Shadows and Isoperimetry under Sequence-Subsequence Relation, Antichains Splitting, AZ-Identities, Dimension Constraints

6.1. Partitions

Consider $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set and $\mathcal{E}$ is a system of subsets of $\mathcal{V}$. For the cartesian products $\mathcal{V}^n = \prod_1^n \mathcal{V}$ and $\mathcal{E}^n = \prod_1^n \mathcal{E}$, let $\pi(n)$ denote the minimal size of a partition of $\mathcal{V}^n$ into sets that are elements of $\mathcal{E}^n$, if a partition exists at all, otherwise $\pi(n)$ is not defined. This is obviously exactly the case if it is so for $n = 1$.

Whereas the packing number $p(n)$, that is the maximal size of a system of disjoint sets from $\mathcal{E}^n$, and the covering number $c(n)$, that is the minimal number of sets from $\mathcal{E}^n$ to cover $\mathcal{V}^n$, have been studied in the literature, this seems to be not the case for the partition number $\pi(n)$.

Obviously, $c(n) \leq \pi(n) \leq p(n)$, if $c(n)$ and $\pi(n)$ are well defined. The quantity $\lim_{n\to\infty} \frac{1}{n} \log p(n)$ is Shannon's zero error capacity. Although it is known only for very few cases, a nice formula exists for $\lim_{n\to\infty}(1/n)\log c(n)$ (see [73]).

The difficulties in analyzing $\pi(n)$ are similar to those for $p(n)$. For the case of graphs with edge set $\mathcal{E}$ including all loops, we prove that $\pi(n) = \pi(1)^n$ ([83]). This result is derived from the corresponding result for complete graphs with the help of Gallai's Lemma in matching theory. Another interesting quantity is $\mu(n)$, the maximal size of a partition of $\mathcal{V}^n$ into sets that are elements of $\mathcal{E}^n$ (again only hypergraphs $(\mathcal{V}, \mathcal{E})$ with a partition are considered). We also call μ the maximal partition number. It behaves more like the packing number. Clearly, $\pi(n) \leq \mu(n) \leq p(n)$. It seems to us that an understanding of these partition problems would be a significant contribution to an understanding of the basic, and seemingly simple, notion of Cartesian products.

More generally, for hypergraphs $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$ $(1 \leq i \leq n)$, we define the product hypergraph $\mathcal{H}^n = (\mathcal{V}^n, \mathcal{E}^n) = (\prod_{i=1}^n \mathcal{V}_i, \prod_{i=1}^n \mathcal{E}_i)$. Edges of cardinality 1 are called loops. Special hypergraphs are graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined by the property $|E| \in \{1, 2\}$ for all $E \in \mathcal{E}$ and, more generally, d-uniform hypergraphs (with or without loops) that satisfy $|E| \in \{1, d\}$ for all $E \in \mathcal{E}$.

In particular, there are d-uniform hypergraphs with all loops included, that is, $\{\{v\} : v \in \mathcal{V}\} \subset \mathcal{E}$.

When the set $\binom{\mathcal{V}}{d}$ of all vertex sets of cardinality d is contained in the edge set $\mathcal{E}$, we speak of a complete d-uniform hypergraph.

We introduced the partition number $\pi(\mathcal{H})$ as the minimal size of a partition of $\mathcal{V}$ into sets that are members of $\mathcal{E}$, if a partition exists, and ∞ otherwise. When $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ $(i = 1, 2, \dots, n)$ are arbitrary finite graphs with all loops included, then we obviously have $\pi(\mathcal{G}_i) = |\mathcal{V}_i| - v(\mathcal{G}_i)$ for the partition number, where $v(\mathcal{G}_i)$ is the matching number of $\mathcal{G}_i$. A discovery of [83] is that for the hypergraph product $\mathcal{H}^n = \mathcal{G}_1 \times \dots \times \mathcal{G}_n$ $\pi(\mathcal{H}^n) = \prod_{i=1}^n \pi(\mathcal{G}_i)$.

An important step in our proof is to show the above when all $\mathcal{G}_i$s are complete. Here we establish the following generalization.

Theorem 6.1. (Ahlswede/Cai [94]) *For complete d-uniform hypergraphs with all loops $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, that is, $\mathcal{E}_i = \binom{\mathcal{V}_i}{d} \cup \{\{v\} : v \in \mathcal{V}_i\}$ $(i = 1, 2, \dots, n)$, write $|\mathcal{V}_i| = dq_i + r_i, 0 \le r_i < d$. Then for $\mathcal{H}^n = \prod_{i=1}^n \mathcal{H}_i$ satisfying $d > \prod_{i:r_i \ne 0} r_i$, we have $\pi(\mathcal{H}^n) = \prod_{i=1}^n \frac{|\mathcal{V}_i| + (d-1)r_i}{d} = \prod_{i=1}^n (q_i + r_i) = \prod_{i=1}^n \pi(\mathcal{H}_i)$. (The result above is covered by the case $d = 2$.)*

Even in the case of non-identical factors $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i \in \mathbb{N}$, with $\max_i |\mathcal{E}_i| < \infty$, the asymptotics of $c(n)$ is known [73]:

$$\lim_{n\to\infty} \frac{1}{n} \left(\log c(n) - \sum_{t=1}^{n} \log \left(\max_{q \in Prob(\mathcal{E}_t)} \min_{v \in \mathcal{E}_t} \sum_{E \in \mathcal{E}_t} 1_E(v) q_E \right)^{-1} \right) = 0\,,$$

where $Prob(\mathcal{E}_t)$ is the set of all probability distributions on $\mathcal{E}$, q_E is the indicator function of the set E.

6.2. Bounds on monochromatic rectangles

For a matrix consider the area $i \cdot j$ of an $i \times j$ minor with constant entries. This concept was introduced by Yao for estimating communication complexity. Some of our exact results and bounds, identities and inequalities, are reported by U. Tamm in [N, p. 589–602]. Interactive communication [120], a similar model, has striking phenomena and open problems. Methods from [35], [36] find application.

6.3. Shadows and isoperimetry under the sequence-subsequence relation

It has been suggested in 1988 on page 152 in project B_1 “Kombinatorik von Folgenräumen” of the SFB 343 “Diskrete Strukturen in der Mathematik” to study combinatorial extremal problems under the sequence-subsequence relation —in particular also shadow problems of the Kruskal-Katona type.

In September 94 David Daykin wrote to us that he and Danh had a counterexample to the optimality of the optimality of the B-G order. He also mentioned that they had a solution in the binary case with a very, very complicated proof (which we never have seen). Immediately thereafter a simple proof for the binary case (and also a simple counterexample to the B-G order in the general case) was given by Ahlswede/Cai [112]. Subsequently, in December 94 the former author also gave a new shorter proof. (Also published in [112], see [N, p. 75–94].)

6.3.1. Shadows of arbitrary sets under deletion of any letter For $\mathcal{X}^n = \prod_1^n \mathcal{X}$, the sequences of length n over the alphabet $\mathcal{X}$, we consider for sets $A \subset \mathcal{X}^n$ their shadow $\nabla A = \{x^{n-1} \in \mathcal{X}^{n-1} : x^{n-1}$ is subsequence of some $a^n \in A\}$. The goal is to find for given cardinalities sets of minimal cardinality of the *shadow*.

Recall the H-order of Harper: For any integer $u \in [0, 2^n]$ the u-th initial segment consists of all $x^n \in \{0,1\}^n$ with less than $n-k$ ones and all remaining elements with $n-k$ ones, whose complements are in the initial segment of the squashed order (used for instance in Kruskal-Katona).

As in the vertex isoperimetric problem in binary Hamming space it is optimal also for our shadows of sets in $\{0,1\}^n$. We use the unique binomial representation of an integer u

$$u = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \cdots + \binom{\alpha_t}{t}; n > \alpha_k > \cdots > \alpha_t \geq 1,$$

and observe that for an initial, H-order segment S with $|S| = u$

$$|\nabla S| = \binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_t - 1}{t-1} = \overset{\nabla}{G}(n,u),$$

say.

Theorem 6.2. *For every $A \subset \{0,1\}^n$ $|\nabla A| \geq \overset{\nabla}{G}(n, |A|)$ and the bound is achieved by the u-th initial segment in H-order.*

The proof is an immediate consequence of our main discovery, the

∇-Inequality. (Ahlswede/Cai [112]) *If $w_1 \leq w_0 < \overset{\nabla}{G}(n,w)$ and $w \leq w_0 + w_1$, then $\overset{\nabla}{G}(n,w) \leq \overset{\nabla}{G}(n-1,w_0) + \overset{\nabla}{G}(n-1,w_1)$.*

In [128] Ahlswede/Cai established analogous results for

6.3.2. Shadows for fixed level and specific letter

6.3.3. Shadows of arbitrary sets under insertion of any letter

6.3.4. Two isoperimetric inequalities It has been emphasized in [27] that isoperimetric inequalities in discrete metric spaces are fundamental principles in combinatorics. The goal is to minimize the union of a specified number of balls of constant radius. We speak of an isoperimetric inequality, if this minimum is assumed for a set of ball-centers, which themselves form a ball (or quasi-ball, if numbers don't permit a ball).

For any $A \subset \{0,1\}^*$ and any distance d we define (the union of balls of radius r) $\Gamma_d^r(A) = \{x^{n\prime} \in \{0,1\}^* : d(x^{n\prime}, a^n) \leq r$ for some $a^n \in A\}$.

A prototype of a discrete isoperimetric inequality is the one discovered by Harper 1966, rediscovered by Ahlswede/Gács/Körner [23], and proved again by Katona for $d = d_H$.

The optimum is achieved by the $|A|$-th initial segment $I_{|A|}$ in H-order (this is a ball of radius k, if $|A| = \sum_{j=0}^n \binom{n}{j}$).

Ahlswede/Cai [128] define two distances, θ and δ, in $\{0,1\}^* = \bigcup_{n=0}^{\infty} \mathcal{X}^n$. For $x^m, y^{m'} \in \{0,1\}^*$ $\theta(x^m, x^{m'})$ counts the minimal number of insertions and deletions which transform one word into the other. $\Delta(x^m, x^{m'})$ counts the minimal number of operations, if also exchanges of letters are allowed. Thus $\delta(x^m, x^{m'}) \leq \theta(x^m, x^{m'})$.

Theorem 6.3. *For all* $A \subset \mathcal{X}^n$ *and* $r \geq 0$

(i) $|\Gamma_\theta^r A| \geq |\Gamma_\theta^r I_A|$,
(ii) $|\Gamma_\delta^r A| \geq |\Gamma_\delta^r I_A|$.

6.4. Antichains splitting

A novel type of result was found by Ahlswede/Erdős/Graham [103]: In any *dense* finite poset $\mathcal{P}$ (e.g. in the Boolean lattice) every maximal antichain S can be partitioned into disjoint subsets S_1 and S_2, such that the union of the downset of S_1 with the upset of S_2 yields the entire poset $\mathcal{D}(S_1) \cup \mathcal{U}(S_2) = \mathcal{P}$. Here $\mathcal{P}$ is called *dense* if every non-empty open interval $\{z \in \mathcal{P} : x < z < y\}$ contains at least two elements z', z''. It is called strongly dense if there are incomparable z', z''.

For finite posets the two concepts are equivalent, but for infinite posets they do not necessarily coincide. (For example the totally ordered chain of rational numbers is dense, but not strongly dense.)

A conjecture of [103] that every countable strongly dense poset has the splitting property was disproved by Ahlswede/Khachatrian with the poset of squarefree integers in [N, p. 29–44], where the reader finds also several open questions.

6.5. AZ-identities

Ahlswede and Zhang [64] found the following identity.

Theorem 6.4. (AZ-Identity) *For every family* $\mathcal{A} \subset 2^\Omega$ *of non-empty subsets of* $\Omega = \{1, 2, \ldots, n\}$ $\sum_{Z \subset \Omega} \frac{W_\mathcal{A}(X)}{|X|\binom{n}{|X|}} = 1$, *where* $W_\mathcal{A}(X) = \left|\bigcup_{A \supset A \in \mathcal{A}} A\right|$.

We associate with every $\mathcal{E} \subset 2^\Omega$ the upset $\mathcal{U}(\mathcal{E}) = \{U \subset \Omega : U \supset E$ for some $E \in \mathcal{E}\}$ and the downset $\mathcal{D}(\mathcal{E}) = \{D \subset \Omega : D \subset E$ for some $E \in \mathcal{E}\}$. When $\mathcal{A}$ is an antichain in the poset $(2^\Omega, \supset)$, then the identity becomes $\sum_{X \in \mathcal{A}} \frac{1}{\binom{x}{|X|}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \smallsetminus \mathcal{A}} \frac{W_\mathcal{A}(X)}{|X|\binom{n}{|X|}} = 1$.

The LYM inequality is obtained by omission of the second summand, which by definition of $W_\mathcal{A}$ can also be written in the form $\sum_{X \notin \mathcal{D}(\mathcal{A})} \frac{W_\mathcal{A}(X)}{|X|\binom{n}{|X|}}$. We call this the deficiency of the inequality. More generally, in [68] the Bollobas inequality was lifted to an identity.

Theorem 6.5. $(\boldsymbol{AZ_2})$ *For two families* $\mathcal{A} = \{A_1, \ldots, A_N\}$ *and* $\mathcal{B} = \{B_1, \ldots, B_N\}$ *of subsets of* Ω *with the properties*

(a) $A_i \subset B_i$ *for* $i = 1, 2, \ldots, N$,
(b) $A_i \not\subset B_j$ *for* $i \neq j$ $\sum_{i=1}^N \frac{1}{\binom{n-|B_i \smallsetminus A_i|}{|A_i|}} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_\mathcal{A}(X)}{|X|\binom{n}{|X|}} = 1$.

In [64] it was explained that theorem AZ_1 gives immediately, what LYM does not, namely the uniqueness part in Sperner's Theorem. In [68] the uniqueness of an optimal configuration of unrelated chains of subsets due to Griggs, Stahl and Trotter 1984 [GST] was proved with the help of theorem AZ_2.

Körner and Simonyi 1990 observed the LYM-type inequality:

For $\mathcal{A} = \{A_1, \dots, A_N\}$, $\mathcal{B} = \{B_1, \dots, B_N\} \subset 2^\Omega$ with $A_i \cap B_i = \varnothing, A_i \not\subset A_j \cup B_j$, $B_i \not\subset A_j \cup B_j$ for $i \neq j$ $\sum_{i=1}^N \binom{n-|A_i|}{|B_i|}^{-1} + \binom{n-|B_i|}{|A_i|}^{-1} - \binom{n}{|A_i|+|B_i|}^{-1} \leq 1$ and they asked "Is this inequality ever tight?".

This rather modest question was a challenging test of the power of the identities above or, more precisely, of the procedure to produce new identities described in [64].

The outcome is an Ahlswede-Zhang type identity which goes considerably beyond theorem AZ_2. From a special case of this identity we derive *a full characterization* of the cases with equality even for a generalized version of the inequality above. In other words we characterize the cases with deficiency zero.

Theorem 6.6. (Ahlswede/Cai [79]) *Suppose that for a family $\mathcal{B} = \{B_1, \dots, B_N\}$ of subsets of Ω and a family $\mathcal{A}^* = \{\mathcal{A}_1, \dots, \mathcal{A}_N\}$ of subsets of 2^Ω, where $\mathcal{A}_i = \{A_i^t : t \in T_i\}$ for a finite index set T_i, we have*

(a) *$A_i^t \subset B_i$ for $t \in T_i$ and $i = 1, 2, \dots, N$,*

(b) *$A_i^t \not\subset B_j$ for $t \in T_i$ and $i \neq j$.*

Then $\sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n-|B_i - \bigcup_{t\in S} A_i^t|}{|\bigcup_{t \in S} A_i^t|}^{-1} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1.$

The specialisation $|T_i| = 1$ for $i = 1, \dots, N$ gives theorem AZ_1. Daykin and Thu 1994 presented a dual to the AZ-identity and for related identities see Thu, "Identities for Combinatorial Extremal Theory" in [112].

6.6. Extremal sets of vectors under linear dimension constraints

The paper [158] "Maximal Number of Constant Weight Vertices of the Unit n-Cube Contained in a k-Dimensional Subspace", together with H. Aydinian and L. H. Khachatrian, is the start of a *new direction* in Extremal Theory, which is indicated in the title.

We introduce and solve a seemingly basic geometrical extremal problem For the set $E(n,w) = \{x^n \in \{0,1\}^n : x^n \text{ has } w \text{ ones}\}$ of vertices of weight w in the unit cube of $\mathbb{R}^n$ we determine $M(n,k,w) \triangleq \max\{|U_k^n \cap E(n,w)| : U_k^n$ is a k-dimensional subspace of $\mathbb{R}^n\}$. We also present an extension to multi-sets and explain a connection to the (higher dimensional) Erdős-Moser problem.

The set $E(n,w)$ can also be viewed as the set in which constant weight codes are studied in Information Theory. Another interest there is in linear codes. This was a motivation for studying the interplay between two properties: constant weight and linearity. In particular we wanted to know $M(n,k,w)$.

Theorem 6.7. (Ahlswede/Aydinian/Khachatrian [158]) *For $n, k, w \in \mathbb{N}$*

(a) *$M(n,k,w) = M(n,k,n-w)$.*

(b) *For* $w \le \frac{n}{2}$ *we have*

$$M(n,k,w) = \begin{cases} \binom{k}{w}, & \text{if (i) } 2w \le k \\ \binom{2(k-w)}{k-w} 2^{2w-k}, & \text{if (ii) } k < 2w < 2(k-1) \\ 2^{k-1}, & \text{if (iii) } k-1 \le w\,. \end{cases}$$

The sets giving the claimed values of $M(n,k,w)$ *in the three cases are*

(i) $S_1 = E(k,w) \times \{0\}^{n-k}$.
(ii) $S_2 = E(2(k-w), k-w) \times \{10,01\}^{2w-k} \times \{0\}^{n-2w}$.
(iii) $S_3 = \{10,01\}^{k-1} \times \{1\}^{w-k+1} \times \{0\}^{n-k-w+1}$.

A key tool in the proof is an extremal problem for families of w-element sets involving antichain properties for certain restrictions.

Lemma 6.8. *Let* $X = X_1 \dot\cup \ldots \dot\cup X_s$ *with* $|X_i| = n_i$ *for* $i = 1, \ldots, s$ *and let* $\mathcal{A} \subset \binom{X}{w}$ *be a family with the following property:*

(P) *for any* $A, B \in \mathcal{A}$ *and* $j = 1, \ldots, s$ $E \triangleq A \cap \left(\bigcup_{i=1}^{j} X_i\right) \ne B \cap \left(\bigcup_{i=1}^{j} X_i\right) \triangleq F$ *implies that* E *and* F *are incomparable (form an antichain). Then*

$$g(n_1, \ldots, n_s, w) \triangleq \max\left\{|\mathcal{A}| : \mathcal{A} \subset \binom{X}{w}, \mathcal{A} \text{ has property (P)}\right\}$$
$$= \max_{\sum_{i=1}^{s} w_i = w} \prod_{i=1}^{s} \binom{n_i}{w_i}\,.$$

For $s = 1$ we get Sperner's result. Evaluation of the max gives the formulas in the theorem.

7. Counterexamples, Conjectures and Problems

7.1. Counterexamples

7.1.1. Frankl/Pach Conjecture for uniform, dense families A family $\mathcal{F} \subset \binom{[n]}{\ell}$ is called ℓ-dense, if there exists an $F \in \mathcal{F}$, such that $|F \cap F_1 : F_1 \in \mathcal{F}| = 2^{\ell}$.

Frankl/Pach [FP] conjectured that every $\mathcal{F} \subset \binom{[n]}{\ell}$ with $|\mathcal{F}| > \binom{n-1}{\ell-1}$ is ℓ-dense.

Ahlswede/Khachatrian [129] provide a counterexample to this conjecture by the construction of a set $\mathcal{F} \subset \binom{[n]}{\ell}$, $|\mathcal{F}| = \binom{n-1}{\ell-1} + \binom{n-4}{\ell-3}$, which is not ℓ-dense.

7.1.2. Kleitman's conjecture In [27] Ahlswede/Katona considered the following "Excess-problems": For $\mathcal{A} \subset \binom{[n]}{k}$ let $I(\mathcal{A}) = |\{(A_1, A_2) \in \mathcal{A}^2 : A_1 \cap A_2 \ne \phi\}|$, $G(\mathcal{A}) = |\{(A_1, A_2) \in \mathcal{A}^2 : |A_1 \cap A_2| \ge k-1\}|$. Determine $f(M) = \max_{|\mathcal{A}|=M} I(\mathcal{A})$ or $g(M) = \max_{|\mathcal{A}|=M} G(\mathcal{A})$.

The problems are the same for $k = 2$ and here these authors described two configurations, quasi-ball and quasi-star, one of which is always optimal. For $k \ge 3$ none of the problems is solved. The second has also been called Kleitman-West Problem. Ahlswede/Cai [147] disproved a conjecture of Kleitman for this problem.

7.1.3. Conjecture of R. Aharoni from 1991 concerning matching theory for infinite families of finite sets The conjecture states that in any such family $\mathcal{A}$ there exists a subfamily $\mathcal{B}$ of disjoint sets (called strong maximal matching) such that no substitution of k of these sets by more than k sets from $\mathcal{A}$ results again in a subfamily of disjoint sets.

The counterexample of Ahlswede/Khachatrian [105] is the family $\mathcal{A}$ of those subsets of $\mathbb{N}$, whose cardinality equals its minimal element (in canonical order).

7.1.4. Conjectured sharpening of EKR by Ahlswede/Cai/Zhang The equation $I_n(\exists, \forall, k) = \binom{n-1}{k-1})$ for $n \geq 3k$ was shown by Ahlswede/Alon/P. L. Erdős/ Ruszinko/ Székely (see [N, p. 117–124]) to hold for $k = 2, 3$ and to be false for $k \geq 8$.

7.2. Open problems

Problem 7.1. *Prove or disprove Aharoni's Conjecture for families of sets of* bounded *cardinality k. For $k = 2$ it has been proved by Aharoni.*

Problem 7.2. *Exact solutions of isodiametric problems for general sumtype functions are hard to obtain. Solve the case of the Lee-metric already mentioned in [72].*

Problem 7.3. (Classification of D-perfect codes) *We learnt in section 5 that optimal anticodes need not be balls. This led to the new concept of D-perfect codes. Continue the classification by Ahlswede/Aydinian/Khachatrian [118].*

Problem 7.4. (Optimal rate for codes with localized errors) *The optimal rate R of codes over the alphabet $\{0, 1, \ldots, q-1\}$ with localized $t = \tau \cdot n$ errors does not exceed the Hamming bound $1 - h_q(\tau) - \tau \log_q(q-1)$: In 1987 Bassalygo, Gelfand, Pinsker showed that for $q = 2$ the bound is optimal. A series of investigations [75], [77], [90], [96], [97] gave in particular for $q \geq 2$.*

Theorem 7.5. (Ahlswede/Bassalygo/Pinsker [151]) *Let $0 < \tau < 1/2 - \frac{q-2}{2q(2q-3)}$, then for any $\varepsilon > 0$ the Hamming bound can be achieved in rate up to ε with codes correcting τn localized errors.*

Prove or disprove: Not only for $q = 2$, but for all $q \geq 2$ this is true for $0 < \tau < 1/2$.

Problem 7.6. Equality characterisation *for AD constitutes by itself a rich area in combinatorial extremal theory. Less demanding are equality characterisation problems for the consequences of AD. Aharoni/Holzman did this for Marica-Schönheim and Beck for another special case of AD (c.f. [100]). Already in 1979 Daykin/Kleitman/West investigated Kleitman's inequality. Ahlswede/Khachatrian [100] completed these investigations.*

Problem 7.7. *Determine $H_n(\exists, \exists, k, r)$. We believe that* $\lim_{n\to\infty} H_n(\exists, \exists, 3, 2) n^{-2} = \frac{1}{\sqrt{2}}$.

Problem 7.8. *Determine* $H_n(\forall, \exists, k, r)$. *We believe that* $\lim_{n\to\infty} H_n(\forall, \exists, 3, 2) = 2$.

There are **many problems** in Ahlswede/Zhang [60] and Ahlswede/Ye/Zhang [63] on "Creating order". (See also P. Vanroose [N, p. 603–614] and in U. Tamm [N, p. 589–602].)

7.3. Conjectures

Conjecture 7.9. (Ahlswede/Khachatrian; also Erdős) *In theorem 2.2 in section 2 one can choose for every* k $n(k) = cp_k^2$ *for suitable constant* c. *Presently we have only* $n(k) = \prod_{p\le (p_{c_1 k})} p_{c_2 k}$.

The theory of Communication Complexity led via Yao's monochromatic rectangles for special functions to the following

Conjecture 7.10. $\lim_{n\to\infty} I_n(\forall, \exists, 3)\binom{n}{2}^{-1} = \frac{5}{4}$, $\lim_{n\to\infty} I_n(\exists, \exists, k)\binom{n}{k-1}^{-1} = 1$ *[119].*

Conjecture 7.11. *For* $a^n = (a_1, \dots, a_n)$, $b^n = (b_1, \dots, b_n)$ *define* $m_n(a^n, b^n) = \sum_{t=1}^n a_t \wedge b_t$ *and* $M(\delta, n) = \max\{|A||B| : A, B \subset \{0,1\}^n, M_n(a^n, b^n) = \delta$, *for all* $a^n \in A, b^n \in B\}$, *Ahlswede [57] guesses that* $M(\delta, n) = \max_{0\le m\le n-\delta} 2^m \binom{n-m}{\delta}$.

Conjecture 7.12. *The pair* (A, B) *with* $A, B \subset \{0, 1, \dots, \alpha - 1\}^n$ *is an* (n, δ) **constant distance code pair**, *if* d_H $d(a, b) = \delta$ *for all* $a \in A, b \in B$. *Let* $S_\alpha(n, \delta)$ *be the set of those code pairs, let* $M_\alpha(n, \delta) = \max\{|A||B| : (A, B) \in S_\alpha(n, \delta)\}$, *and let for* $\alpha \ge 4$ $F_\alpha(n, \delta) = \max_{\delta_1+\delta_2=\delta} \left(\frac{\lceil\alpha\rceil}{2} \frac{\alpha}{\lfloor 2\rfloor}\right)^{\delta_1} \binom{n-\delta}{\delta_2}(\alpha - 1)^{\delta_2}$.

Theorem 7.13. (Ahlswede [57]) *For* $n \in \mathbb{N}$, $0 \le \delta \le n$ $M_\alpha(n, \delta) = F_\alpha(n, \delta)$ *for* $\alpha = 4, 5$.

Prove $M_\alpha(n, \delta) = F_\alpha(n, \delta)$ for all $\alpha \ge 6$ (300 US $ are offered for first solver).

Conjecture 7.14. *($\frac{3}{4}$-Conjecture for fix-free codes by Ahlswede/Balkenhol/Khachatrian [118].) A (variable length) code is fix-free if no codeword is a prefix or a suffix of any other. A database construction by a fix-free code is instantaneously decodable from both sides. Prove or disprove: For numbers* $\ell_1, \dots, \ell_N$ *satisfying* $\sum_{i=1}^n 2^{-\ell_i} < \frac{3}{4}$ *a fix-free code of word lengths* $\ell_1, \dots, \ell_N$ *exists. If true, this bound is the best possible.*

Further Conjectures can be found in Ahlswede [56].

References

[N] "Numbers, Information and Complexity, Special volume in honour of R. Ahlswede on the occasion of his 60th birthday"; I. Althöfer, N. Cai, G. Dueck, L. Khachatrian, M. S. Pinsker, A. Sárközy, I. Wegener, Z. Zhang editors. Kluwer Acad. Publ., Boston, Dordrecht, London 2000.

[ABEK] R. Ahlswede, C. Bey, K. Engel and L. Khachatrian, "The t-intersection problem in the truncated boolean lattice", submitted to European Journal of Combinatorics.

[B] C. Bérge, "Nombres de coloration de l'hypergraphe h-parti complet", Hypergraph Seminar (Proc. First Working Sem., Ohio State Univ., Columbus, Ohio, 1972; dedicated to Arnold Ross), pp. 13–20. Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974.

[CG] F. Chung and R. Graham, "Erdös on graphs. His legacy of unsolved problems", A K Peters, Ltd., Wellesley, MA, 1998.

[DKW] D. E. Daykin, D. J. Kleitman and D. B. West, "On the number of meets between two subsets of a lattice", J. Comb. Theory, Ser. A 26, 135–155, 1979.

[E1] P. Erdős, "Problems and results in combinatorial number theory", ch. 12 in a Survey of Combinatorial Theory, J. N. Srivastava et al. (eds) 1973.

[E2] P. Erdős, "A survey of problems in combinatorial number theory", Ann. Discrete Math. 6, 89–115, 1980.

[En] K. Engel, "Sperner Theory", Cambridge University Press 1997.

[ESS1] P. Erdős, A. Sárközy and E. Szemerédi, "On a theorem of Behrend", J. Austral. Math. Soc. 7, 9–16, 1967.

[ESS2] P. Erdős, A. Sárközy and E. Szemerédi, "On some extremal properties of sequences of integers", Annales Univ. Sci. Budapest, Eőtvös 12, 131–135, 1969.

[ESS3] P. Erdős, A. Sárközy and E. Szemerédi, "On divisibility properties of sequences of integers", Number Theory (Colloq., Jänos Bolyai Math. Soc., Debrecen, 1968) pp. 35–49, North-Holland, Amsterdam , 1970.

[ESS4] P. Erdős, A. Sárközy and E. Szemerédi, "On some extremal properties of sequences of integers II", Publications Math. 27, 117–125, 1980.

[ErSS] P. L. Erdős, A. Seress and L. A. Székely, "On intersecting chains in Boolean algebras". Combinatorics, geometry and probability (Cambridge, 1993), 299–304, Cambridge Univ. Press, Cambridge, 1997.

[F1] P. Frankl, "Intersection theorems for finite sets and geometric applications", Proceedings of the International Congress of Mathematicians, Berkeley, USA, 1419–1430, 1986.

[F2] P. Frankl, "The Erdős-Ko-Rado Theorem is true for $n = ckt$", Combinatorics (Proc. Fifth Hungarian Colloq. Keszthely, 1976), Vol. I, pp. 365–375, Colloq. Math. Soc. Janos Bolyai, 18, North-Holland, Amsterdam-New York, 1978.

[FF] P. Frankl and Z. Füredi, "The Erdős-Ko-Rado theorem for integer sequences", SIAM J. Algebraic Discrete Methods 1, no. 4, 376–381, 1980.

[FP] P. Frankl and J. Pach, "On disjointly representable sets", Combinat. 4, no. 1, 39–45, 1984.

[Fr] R. Freud, "Paul Erdős, 80 — A Personal Account", Period. Math. Hungar. 26(2), 87–93, 1993.

[GN] R. L. Graham and J. Nesetril Eds., "The Mathematics of Paul Erdős I", Springer-Verlag, 1997.

[GST] J. R. Griggs, J. Stahl and W. T. Trotter, "A Sperner theorem on unrelated chains of subsets", J. Combin. Theory Ser. A 36, no. 1, 124–127, 1984.

[K] D. J. Kleitman, "On a combinatorial conjecture of Erdős", J. Combinatorial Theory 1, 209–214, 1966.

[PS] C. Pomerance and A. Sárközy, "Combinatorial Number Theory", pp. 967–1018, in Handbook of Combinatorics, edited by R. L. Graham, M. Grötschel and L. Lovász. Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995.

[W] R. M. Wilson, "The exact bound in the Erdős-Ko-Rado theorem", Combinatorica 4, no. 2-3, 247–257, 1984.

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