Spectral Analysis of Nonrelativistic Quantum Electrodynamics

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Abstract. I review the research results on spectral properties of atoms and molecules coupled to the quantized electromagnetic field or on simplified models of such systems obtained during the past decade. My main focus is on the results I have obtained in collaboration with **Jürg Fröhlich** and **Israel Michael Sigal** [8, 9, 10, 11, 12, 13].

1. Introduction

In this lecture I review the progress achieved during the past decade on the mathematical description of quantum mechanical matter interacting with the quantized radiation field. My main focus will be on the results I have obtained in collaboration with **Jürg Fröhlich** and **Israel Michael Sigal** [8, 9, 10, 11, 12, 13].

1.1. Basic notions of quantum mechanics

I start by recalling some basic mathematical notions of quantum mechanics. The states of a quantum mechanical system to be described are vectors in a separable Hilbert space, $\mathcal{H}$. The dynamics on $\mathcal{H}$ is generated by the selfadjoint Hamiltonian operator, H. That is, given an initial state $\psi(0) = \psi_0 \in \mathcal{H}$ at time $t = 0$, the state at time $t > 0$ is given by $\psi(t) = \exp[-itH]\psi_0$. The corresponding differential equation fulfilled by $\psi(t)$ is Schrödinger's equation,

$$i \frac{d\psi(t)}{dt} \;=\; H\,\psi(t)\,. \tag{1}$$

Stone's theorem [43] states that the selfadjointness of H is equivalent for $t \mapsto \exp[-itH]$ to be a strongly continuous one-parameter unitary group. Thus selfadjointness of the Hamiltonian is the crucial property for the existence of quantum mechanical dynamics.

As a first example, I describe the above notions for a single, nonrelativistic electron moving in a potential $V\colon \mathbb{R}^3 \to \mathbb{R}$. The Hilbert space of states and the Hamiltonian are, in this case,

$$\mathcal{H}_{el} \;:=\; L^2(\mathbb{R}^3 \times \mathbb{Z}_2)\,, \qquad H_{el} \;=\; -\Delta_x + V(x)\,, \tag{2}$$

Key words and phrases. Renormalization Group, Spectrum, Resonances, Fock space, QED..

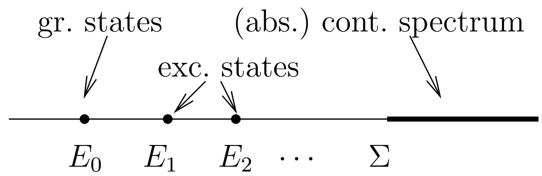

FIGURE 1. The spectrum of H_{el}

where Δ_x is the Laplacian on $\mathbb{R}^3$, and the potential $V(x)$ acts as a multiplication operator, $[V\psi](x,\sigma) := V(x)\psi(x,\sigma)$. Moreover, the $\mathbb{Z}_2$ factor in the definition of $\mathcal{H}_{el}$ accounts for the spin of the electron. Under the assumption that $V \in L^2 \cap L^\infty(\mathbb{R}^3;\mathbb{R})$, the Hamiltonian H_{el} is selfadjoint on the standard Sobolev space, $\mathrm{H}^2(\mathbb{R}^3 \times \mathbb{Z}_2) \subseteq L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, the domain $\mathrm{dom}(-\Delta_x)$ of selfadjointness of the Laplacian. If $\lim_{|x|\to\infty} V(x) = 0$, and if $\|(V)_-\|_{L^{3/2}}$ is not too small, then H_{el} has the following standard spectrum [42], see Fig. 1:

- Below 0, the spectrum is purely discrete, i.e., it consists only of isolated eigenvalues, $E_0 < E_1 < \cdots < 0$, each E_j being of finite multiplicity $n_j < \infty$. Thus there is an orthonormal basis of the corresponding spectral subspace of eigenvectors, $\{\varphi_{j,\alpha}\}_{\alpha=1,\ldots,n_j}$, i.e., $H_{el}\varphi_{j,\alpha} = E_j\varphi_{j,\alpha}$ and $\langle\varphi_{i,\alpha}|\varphi_{j,\beta}\rangle = \delta_{i,j}\,\delta_{\alpha,\beta}$. If there are infinitely many eigenvalues, they accumulate at 0.
- The positive half-axis supports the purely absolutely continuous spectrum.
- The singular continuous spectrum is empty.

$$\sigma(H_{el}) \;=\; \sigma_{\mathrm{disc}}(H_{el}) \cup \sigma_{\mathrm{ac}}(H_{el})\,, \tag{3}$$
$$\sigma_{\mathrm{disc}}(H_{el}) \;=\; \{E_0, E_1, E_2, \ldots\} \subseteq (-\infty, 0)\,, \tag{4}$$
$$\sigma_{\mathrm{ac}}(H_{el}) \;=\; [0,\infty)\,. \tag{5}$$

A typical potential to bear in mind is $V(x) := -|x|^{-1}$. Then $H_{el} = -\Delta_x - |x|^{-1}$ is the Hamiltonian of a *hydrogen atom*. It is actually not more difficult to include more than one electron in the model. The structure (3)-(5) of the spectrum of $\sigma(H_{el})$ would not change, qualitatively. I summarize the assumptions and definitions made in the following hypothesis.

Hypothesis 1.1. *Let $\mathcal{H}_{el} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, and assume that $V \in L^2 \cap L^\infty(\mathbb{R}^3;\mathbb{R})$ and $\lim_{|\vec{x}|\to\infty} V(x) = 0$. Let $H_{el} = -\Delta_x + V(x)$ be the corresponding selfadjoint, semibounded on the domain $\mathrm{H}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, and assume that H_{el} as (at least) one negative eigenvalue,*

$$E_0 \;=\; \inf \sigma(H_{el}) \;<\; 0 \;=\; \inf \sigma_{\mathrm{ess}}(H_{el})\,. \tag{6}$$

In my second example, for $N \in \mathbb{N}$, and real numbers $E_0 < E_1 < \cdots < E_N$, the Hilbert space of states is finite dimensional and the Hamiltonian is a diagonal $N \times N$ matrix,

$$\mathcal{H}_{el} \;:=\; \mathbb{C}^N\,, \qquad H_{el} \;=\; \mathrm{diag}\big[E_0, E_1, \ldots, E_N\big]\,. \tag{7}$$

While in itself this second example is trivial, it is of some importance as a model for the dynamics of the Schrödinger operator $-\Delta_x + V(x)$, restricted to the spectral subspace corresponding to (some part of) its discrete spectrum (implicitly assuming that the electron is spinless and that $n_j = 1$, for all j). Indeed, with this interpretation in mind and in the context of radiation theory, the 2×2 matrix $\mathrm{diag}[E_0, E_1]$ is also referred to in the physics literature as a *two-level atom*. I summarize the assumptions and definitions made in the following hypothesis.

Hypothesis 1.2. *Let $\mathcal{H}_{el} := \mathbb{C}^N$, for some $N \in \mathbb{N}$, and assume that $H_{el} = \mathrm{diag}[E_0, E_1, \ldots, E_N]$, for some real numbers $E_0 < E_1 < \ldots < E_N$.*

My third example is the standard model for the quantized radiation field in quantum field theory. The Hilbert space carrying the field is a Fock space,

$$\mathcal{F} \; := \; \mathcal{F}_b[L^2(\mathbb{R}^3 \times \mathbb{Z}_2)] \; := \; \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)} \,, \tag{8}$$

where $\mathcal{F}^{(n)}$ is the state space of all n-photon states, the so-called n-photon sector. The space of no photons, $\mathcal{F}^{(0)}$, is one-dimensional, and the vacuum vector, Ω, is a unit vector in $\mathcal{F}^{(0)} := \mathbb{C}\,\Omega$. For $n \geq 1$, The n-photon sector is the subspace of the n-fold tensor product of $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ which consists of all totally symmetric vectors (= wave functions),

$$\begin{aligned} \mathcal{F}^{(n)} \; &:= \; \Big\{ \psi_n \in L^2\big[(\mathbb{R}^3 \times \mathbb{Z}_2)^n\big] \;\Big|\; \forall \pi \in \mathcal{S}_n : \\ &\qquad\qquad \psi_n(k_{\pi(1)}, k_{\pi(2)}, \ldots, k_{\pi(n)}) \; = \; \psi_n(k_1, k_2, \ldots, k_n) \Big\} \\ &\subseteq \; \bigotimes_{j=1}^{n} L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \,, \end{aligned} \tag{9}$$

where $k_j := (\vec{k}_j, \lambda_j) \in \mathbb{R}^3 \times \mathbb{Z}_2$ indicates that $\psi_n \in \mathcal{F}^{(n)}$ is given in momentum representation (Fourier transform). The symmetry of the wave functions accounts for the fact that photons are indistinguishable particles obeying Bose-Einstein statistics.

The Hamiltonian on $\mathcal{F}$ representing the energy of the free photon field is given by

$$H_f \;\; := \;\; \bigoplus_{n=0}^{\infty} H_f^{(n)} \,, \tag{10}$$

$$\big[H_f^{(n)}\, \psi_n\big](k_1, \ldots, k_n) \;\; := \;\; \big(\omega(k_1) + \ldots + \omega(k_n)\big)\, \psi_n(k_1, \ldots, k_n) \,, \tag{11}$$

for suitable $\psi_n \in \mathcal{F}^{(n)}$, and $H_f \Omega := 0$. Here, $\omega(k) := |\vec{k}| = \sqrt{\vec{k}^2 + m^2}\,|_{m=0}$ is the *photon dispersion law*, in accordance with the principles of special relativity. From the explicit form of H_f it is clear that

$$\sigma(H_f) \; = \; [0, \infty) \,, \quad \sigma_{\mathrm{pp}}(H_f) \; = \; \{0\} \,, \quad \sigma_{\mathrm{ac}}(H_f) \; = \; (0, \infty) \,. \tag{12}$$

Note that $H_f^{(1)} = \sqrt{-\Delta_x}$. Further note that H_f leaves the n-photon sector invariant. The Hamiltonians from physics to be discussed do not have this invariance, however, and the representations (8)-(11) of $\mathcal{F}$ and H_f is rather cumbersome for those models.

It is more convenient instead to express $\mathcal{F}$ and H_f in terms of creation and anihilation operators. Given $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, the creation operator $a^*(f)$ and the annihilation operator $a(f)$ are defined by $a(f)\,\Omega := 0$ and, for $n \geq 1$, by

$$\begin{aligned} a(f) \quad &: \quad \mathcal{F}^{(n)} \to \mathcal{F}^{(n-1)}\,, \qquad (13)\\ \big[a(f)\,\psi_n\big](k_1,\ldots,k_{n-1}) \quad &:= \quad \sqrt{n}\int dk\,\overline{f(k)}\,\psi_n(k_1,\ldots,k_{n-1},k)\,,\\ a^*(f) \quad &: \quad \mathcal{F}^{(n-1)} \to \mathcal{F}^{(n)}\,, \qquad (14)\\ \big[a^*(f)\,\psi_{n-1}\big](k_1,\ldots,k_n) \quad &:= \quad \frac{\sqrt{n}}{n!}\sum_{\pi\in\mathcal{S}_n} f(k_{\pi(1)})\,\psi_{n-1}(k_{\pi(2)},\ldots,k_{\pi(n)})\,, \end{aligned}$$

and then extended to (a dense domain in) $\mathcal{F}$ by linearity and continuity. Note that $\big(a(f)\big)^* = a^*(f)$. The important feature of the creation and anihilation operators is that they represent the canonical commutation relations (CCR),

$$\forall f, g \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2): \quad [a(f)\,,\,a(g)] \;=\; [a^*(f)\,,\,a^*(g)] \;=\; 0\,, \qquad (15)$$

$$[a(f)\,,\,a^*(g)] \;=\; \langle f|g\rangle\,\mathbf{1}_{\mathcal{F}}\,. \qquad (16)$$

Here, $[A,B] := AB - BA$ on a suitable domain. Note that $f \mapsto a^*(f)$ is linear and $f \mapsto a(f)$ is antilinear in f. Hence, I may consider these maps as *operator-valued distributions* with formal distribution kernels $a^*(k)$ and $a(k)$, respectively. Bearing this interpretation in mind, one writes

$$a^*(f) \;=:\; \int dk\,f(k)\,a^*(k)\,, \qquad a(f) \;=:\; \int dk\,\overline{f(k)}\,a(k)\,. \qquad (17)$$

I remark that $a(k)$ is a densely defined operator, but not closable, while $a^*(k)$ is not even densely defined, because, e.g., $\Omega \notin \mathrm{dom}(a^*(k))$. In the sense of operator-valued distributions, i.e., with smearing by suitable test functions understood, I may rewrite the CCR as

$$\forall k, k' \in \mathbb{R}^3 \times \mathbb{Z}_2: \quad [a(k)\,,\,a(k')] \;=\; [a^*(k)\,,\,a^*(k')] \;=\; 0\,, \qquad (18)$$

$$[a(k)\,,\,a^*(k')] \;=\; \delta_{\lambda,\lambda'}\,\delta(\vec{k}-\vec{k}')\,\mathbf{1}_{\mathcal{F}}\,. \qquad (19)$$

By means of creation and annihilation operators, I rewrite

$$\mathcal{F}^{(n)} \;=\; \overline{\mathrm{span}\Big\{a^*(f_1)\cdots a^*(f_n)\Omega \;\Big|\; f_1,\ldots,f_n \in L^2(\mathbb{R}^3\times\mathbb{Z}_2)\Big\}}\,, \qquad (20)$$

$$H_f \;=\; \int dk\,\omega(k)\,a^*(k)a(k)\,. \qquad (21)$$

As a fourth example, I describe a system consisting of an electron in an atom and the quantized radiation field. The appropriate Hilbert space for this description is

$$\mathcal{H} \;:=\; \mathcal{H}_{el} \otimes \mathcal{F}\,. \qquad (22)$$

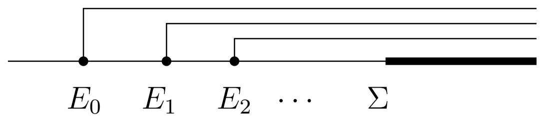

FIGURE 2. The Spectrum of $H_0 = H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes H_f$

In the trivial case that the electron and the photon field do not interact, the Hamiltonian is given by

$$H_0 \ := \ H_{el} \otimes \mathbf{1}_f \ + \ \mathbf{1}_{el} \otimes H_f \, . \tag{23}$$

My ultimate goal is the study of an *interacting* electron-photon system. To develop sensible questions to be answered for such a system, however, it is instructive to first discuss the spectral properties of H_0. I note the general fact [42] that, for a sum of two selfadjoint operators as in (23), I have

$$\sigma(H_0) \ = \ \overline{\sigma(H_{el}) \ + \ \sigma(H_f)} \, . \tag{24}$$

and the spectral measure of H_0 is simply the product measure of the spectral measures of H_{el} and H_f,

$$\mu_{\varphi\otimes\psi}(H_0 \, , \ \lambda + \mu) \ = \ \mu_\varphi(H_{el} \, , \ \lambda) \otimes \mu_\psi(H_f \, , \ \eta) \, . \tag{25}$$

As a result, E_j is still an eigenvalue of H_0 with multiplicity n_j and corresponding eigenvectors $\{\varphi_{j,\alpha} \otimes \Omega\}_{\alpha=1,\ldots,n_j}$. Note, however, that E_j are not isolated anymore. The lowest eigenvalue, the *ground state energy*, $\inf \sigma(H_0) = E_0$, is located at the bottom of $\sigma_{\mathrm{ac}}(H_0) = [E_0, \infty)$, and the higher eigenvalues, the *excited energies*, E_j, $j \geq 1$, are now embedded in continuous spectrum, see fig. 2.

I now turn to the main object of study, the interacting electron-photon Hamiltonian,

$$H_g \ := \ H_0 \ + \ g \, W \, , \tag{26}$$

acting on $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$, as in (22). For the Hamiltonian H_g, I now formulate important tasks which have been addressed and/or even completed during the past decade.

(0.) **Models and Selfadjointness.** To give criteria for W ensuring that H_g defines a selfadjoint, semibounded Hamiltonian and general enough to include the most important applications for H_g in physics.
→ See hypothesis 2.1 and corollary 2.3, below.

(1.) **Binding.** To specify conditions under which the Hamiltonian H_g has a ground state, i.e., under which $E_0(g) := \inf \sigma(H_g)$ is an eigenvalue.
→ See theorem 3.1, below.

(2.) **Resonances.** To develop an appropriate framework for a theory of resonances of H_g, to apply this theory to H_g, and to prove that the embedded excited energies turn into resonances with corresponding metastable states of finite life-time.
→ See theorems 4.3 and 4.5, below.

(3.) **Scattering Theory.** To derive continuous spectrum and scattering theory. To develop tools for the study of the asymptotic behaviour of e^{itH_g}, as $t \to \pm\infty$, like positive commutator estimates. Ultimately, to prove asymptotic completeness of scattering of such systems.
$\to$ See theorem 5.3, below.

(4.) **Positive Temperatures.** To study the systems under consideration for non-zero temperature, given that the Hamiltonian and its spectral properties describe the dynamics of the system at zero temperature.
$\to$ See theorem 6.1, below.

(5.) **Feshbach Renormalization Map.** To develop a renormalization group that allows for a direct analysis of the spectral properties of H_g and L_g.
$\to$ See theorems 7.2, below.

In the remaining sections 2–7, I discuss the topics (0.)–(5.) of the list above. Besides the papers mentioned or discussed below, there are many important contributions which cannot be discuss here but should, nevertheless, be mentioned: [2, 3, 4, 5, 1, 18, 19, 20, 21, 25, 26, 27, 38, 45, 46]

2. Models and Selfadjointness

2.1. Modelling the interaction

According to *first principles* in physics, the physically correct coupling of an electron to the electromagnetic field is the *minimal coupling*. Writing the Schrödinger operator $H_{el}=-\Delta_x+V(x)$ in Eq. (2) as $H_{el}=(\vec{\sigma}\cdot i\vec{\nabla}_x)^2+V(x)$ ($\vec{\sigma}=(\sigma^{(x)},\sigma^{(y)},\sigma^{(z)})$ being the three Pauli matrices), it amounts to replacing the momentum operator $-i\vec{\nabla}_x$ by $-i\vec{\nabla}_x - 2\pi^{1/2}\alpha^{3/2}\vec{A}(\alpha x)$ (to accommodate for gauge invariance),

$$H_\alpha \ := \ \Big[\vec{\sigma}\cdot\big(-i\vec{\nabla}_{\vec{x}} - 2\pi^{1/2}\alpha^{3/2}\vec{A}(\alpha\vec{x}_j)\big)\Big]^2 \ + \ V_c(x)\otimes\mathbf{1}_f \ + \ \mathbf{1}_{el}\otimes H_f\,, \qquad (27)$$

where $\alpha \sim 1/137$ is the fine structure constant, and $\vec{A}(\vec{x})$ denotes the quantized vector potential of the transverse modes of the electromagnetic field in the Coulomb gauge, i.e.,

$$\vec{A}(\vec{x}) \ := \ \int dk\ \vec{G}_{\vec{x}}(k)\otimes a^*(k) \ + \ \overline{\vec{G}_{\vec{x}}(k)}\otimes a(k)\,, \qquad (28)$$

with *coupling function*

$$\vec{G}_{\vec{x}}(\vec{k},\lambda) \ := \ \frac{\sqrt{2}\ \kappa(|\vec{k}|/K)}{\sqrt{\pi\, K^3\,\omega(\vec{k})}}\ \exp[-i\vec{k}\cdot\vec{x}]\ \vec{\varepsilon}_\lambda(\vec{k})\,, \qquad (29)$$

where $\vec{\varepsilon}_\lambda(\vec{k})$, $\lambda=1,2$, are photon polarization vectors satisfying

$$\vec{\varepsilon}_\lambda(\vec{k})^*\cdot\vec{\varepsilon}_\mu(\vec{k}) \ = \ \delta_{\lambda\mu}\,, \qquad \vec{k}\cdot\vec{\varepsilon}_\lambda(\vec{k}) \ = \ 0\,, \qquad \text{for } \lambda,\mu=1,2. \qquad (30)$$

Furthermore, κ is an entire function of rapid decrease on the real line, e.g., $\kappa(r) := \exp(-r^4)$. Hence, the factor $\kappa(|\vec{k}|/K)$ in (28) cuts off the vector potential in the ultraviolet domain, $|\vec{k}| \gg K$. It is artificial in the sense that physical principles actually imply that $\kappa \equiv 1$. With $\kappa \equiv 1$, however, $\|\vec{G}_x\|_{L^2}$ would diverge at $|\vec{k}| = \infty$, which, in turn, would imply that $\Omega \notin \mathrm{dom}(\vec{A}(x))$, for any $x \in \mathbb{R}^3 \times \mathbb{Z}_2$. In order to give a meaning to $\vec{A}(x)$ as a densely defined operator, I thus have to regularize $\vec{G}_x$ at (a preferably large) momentum scale $K \gg 1$. Indeed, by choosing κ to be a sufficiently rapidly decreasing, analytic function obeying $\kappa(0) = 1$, it is ensured that $\vec{G}_x \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, uniformly in $x \in \mathbb{R}^3 \times \mathbb{Z}_2$. The function $\kappa(|\cdot|/K)$ is called *ultraviolet cutoff*, and the construction of the limit $K \to \infty$, of this regularization is one of the open problems in nonrelativistic quantum electrodynamics.

I return to Eqn. (27), which I write as

$$H_g \;=\; H_0 \;+\; W_g\,, \tag{31}$$

where H_0 is defined in (23), and I obtain

$$\begin{aligned} W_g + C_{no} \;\;=\;\; & 4\pi^{1/2}\alpha^{3/2}\vec{A}(\alpha\vec{x}_j)\cdot(i\vec{\nabla}_{\vec{x}}) \;+\; 2\pi\alpha^3\vec{A}^2(\alpha\vec{x}) \\ & + 2\pi^{1/2}\alpha^{5/2}\vec{\sigma}\cdot\big(\vec{\nabla}\wedge\vec{A}\big)(\alpha\vec{x}) \end{aligned} \tag{32}$$

from expanding the square in (27). Note that W_g contains terms linear and quadratic in the creation and annihilation operators, $a^*(k)$, $a(k)$. Hence, I may write

$$W_g \;=\; gW_{1,0} \;+\; gW_{0,1} \;+\; gW_{2,0} \;+\; g^2W_{1,1} \;+\; g^2W_{0,2}\,, \tag{33}$$

where $W_{1,0}$ and $W_{0,1}$ are linear in $a^*(k)$ and $a(k)$,

$$W_{1,0} \;:=\; \int dk\; w_{1,0}(k)\otimes a^*(k)\,, \qquad W_{0,1} \;:=\; \int dk\; w_{0,1}(k)\otimes a(k)\,, \tag{34}$$

and $W_{2,0}$, $W_{1,1}$ and $W_{0,2}$ are quadratic in $a^*(k)$ and $a(k)$,

$$W_{2,0} \;\;:=\;\; \int dk\,dk'\; w_{2,0}(k,k')\otimes a^*(k)a^*(k')\,, \tag{35}$$

$$W_{1,1} \;\;:=\;\; \int dk\,dk'\; w_{1,1}(k,k')\otimes a^*(k)a(k')\,, \tag{36}$$

$$W_{0,2} \;\;:=\;\; \int dk\,dk'\; w_{0,2}(k,k')\otimes a(k)a(k')\,. \tag{37}$$

The tensor products in (34)-(37) indicate that I consider the *coupling functions* $w_{m,n}$ as functions on $(\mathbb{R}^3 \times \mathbb{Z}^2)^{m+n}$ with values in the operators on $\mathcal{H}_{el}$.

Comparing (34)-(37) to (32), I find that

$$w_{1,0}(k) \;=\; w_{0,1}(k)^* \;:=\; 2i\,\vec{G}_{\vec{x}}(k)\cdot\vec{\nabla}_{\vec{x}} \;+\; \vec{\sigma}\cdot(\vec{B}_{\vec{x}}(k)\,, \tag{38}$$

where the magnetic field $\vec{B}_{\vec{x}}(k)$ corresponds to the term $2\pi^{1/2}\alpha^{5/2}\vec{\sigma}\cdot(\vec{\nabla}\wedge\vec{A})(\alpha\vec{x})$ in W_g,

$$\vec{B}_{\vec{x}}(k) \;:=\; \frac{\alpha\sqrt{2}\,\kappa(|\vec{k}|/K)}{i\sqrt{\pi\,K^3\,\omega(\vec{k})}}\;\exp[-i\alpha\vec{k}\cdot\vec{x}]\,\Big(\vec{k}\wedge\vec{\varepsilon}_\lambda(\vec{k})\Big)\,. \tag{39}$$

Furthermore,

$$w_{2,0}(k_1,k_2) \;=\; w_{0,2}(k_1,k_2)^* \;:=\; \vec{G}_{\vec{x}}(k_1)\cdot\vec{G}_{\vec{x}}(k_2)\,, \tag{40}$$

$$w_{1,1}(k_1,k_2) \;:=\; \vec{G}_{\vec{x}}(k_1)^*\cdot\vec{G}_{\vec{x}}(k_2) \;+\; \vec{G}_{\vec{x}}(k_1)\cdot\vec{G}_{\vec{x}}(k_2)^*\,. \tag{41}$$

The constant C_{no} in (32) equals $\|\vec{G}_{\vec{x}}\|^2_{L^2}$, which is independent of $\vec{x}$. It results from *normal-ordering* one term contributing to $W_{1,1}$,

$$a\big(\vec{G}_{\vec{x}}\big)\,a^*\big(\vec{G}_{\vec{x}}\big) \;=\; a^*\big(\vec{G}_{\vec{x}}\big)\,a\big(\vec{G}_{\vec{x}}\big) \;+\; \big\|\vec{G}_{\vec{x}}\big\|^2_{L^2}\,\mathbf{1}\,. \tag{42}$$

Note that the finiteness of C_{no} is due to the introduction of the ultraviolet cutoff.

The next observation to be made is that the coupling functions $w_{m,n}$ obey the following bounds, pointwise in $k,k' \in \mathbb{R}^3\times\mathbb{Z}_2$,

$$\big\|w_{1,0}(k)\,(-\Delta_{\vec{x}}+\mathbf{1})^{-1/2}\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;+\; \big\|w_{0,1}(k)\,(-\Delta_{\vec{x}}+\mathbf{1})^{-1/2}\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;\le\; J(k)\,, \tag{43}$$

$$\big\|w_{2,0}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})}+\big\|w_{1,1}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})}+\big\|w_{0,2}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;\le\; J(k)\,J(k')\,, \tag{44}$$

with

$$J(k) \;:=\; \frac{4\,\kappa(|\vec{k}|/K)}{\omega(k)^{1/2}}\,, \tag{45}$$

and I note for later reference that, for any $0\le\beta<2$,

$$\int\Big(1+\omega(k)^{-\beta}\Big)\,|J(k)|^2\,dk \;<\; \infty\,. \tag{46}$$

I use this example as a guideline for the following hypothesis on the form of the interaction W_g.

Hypothesis 2.1. *The interaction be of the form*

$$W_g \;=\; gW_{1,0} \;+\; gW_{0,1} \;+\; gW_{2,0} \;+\; g^2W_{1,1} \;+\; g^2W_{0,2}\,, \tag{47}$$

where

$$W_{1,0} \;:=\; \int dk\; w_{1,0}(k)\otimes a^*(k)\,, \qquad W_{0,1} \;:=\; \int dk\; w_{0,1}(k)\otimes a(k)\,, \tag{48}$$

$$W_{2,0} \;:=\; \int dk\,dk'\; w_{2,0}(k,k')\otimes a^*(k)a^*(k')\,, \tag{49}$$

$$W_{1,1} \;:=\; \int dk\,dk'\; w_{1,1}(k,k')\otimes a^*(k)a(k')\,, \tag{50}$$

$$W_{0,2} \;:=\; \int dk\,dk'\; w_{0,2}(k,k')\otimes a(k)a(k')\,. \tag{51}$$

The coupling functions $w_{m,n}$ are functions on $(\mathbb{R}^3\times\mathbb{Z}^2)^{m+n}$ with values in the operators on $\mathcal{H}_{el}$ obeying $w_{m,n}=w^*_{n,m}$. Moreover, there is a measurable function $J:\mathbb{R}^3\times\mathbb{Z}_2\to\mathbb{R}^+_0$ such that

$$\big\|w_{1,0}(k)\,(-\Delta_{\vec{x}}+\mathbf{1})^{-1/2}\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;+\; \big\|w_{0,1}(k)\,(-\Delta_{\vec{x}}+\mathbf{1})^{-1/2}\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;\le\; J(k)\,, \tag{52}$$

$$\big\|w_{2,0}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})}+\big\|w_{1,1}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})}+\big\|w_{0,2}(k,k')\big\|_{\mathcal{B}(\mathcal{H}_{el})} \;\le\; J(k)\,J(k')\,, \tag{53}$$

for all $k,k'\in\mathbb{R}^3\times\mathbb{Z}_2$.

2.2. Relative bounds and selfadjointness

In this section I present the results on task (0.) in the list above, establishing the existence of the Hamiltonian H_g by deriving it from a semibounded quadratic form under minimal conditions. Furthermore, I give a criterion that ensures the stability of the domain of definition for H_g, i.e., $\mathrm{dom}(H_g) = \mathrm{dom}(H_0)$. The arguments are based on Kato perturbation theory and variations of the following simple estimate. Namely, given f such that $f/\sqrt{\omega} \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ and $\psi \in \mathrm{dom}(H_f)$, I observe that

$$\begin{aligned} \|a(f)\,\psi\| &\leq \int |f(k)|\,\|a(k)\psi\|\,dk \\ &\leq \Big(\int \frac{|f(k)|^2\,dk}{\omega(k)}\Big)^{1/2} \Big(\int \omega(k)\,\|a(k)\psi\|^2\,dk\Big)^{1/2} \\ &= \|\omega^{-1/2} f\|_{L^2}^2 \cdot \big\|H_f^{1/2}\psi\big\|\,, \end{aligned} \tag{54}$$

hence, for any $\rho > 0$,

$$\big\|a(f)\,(H_f + \rho)^{-1/2}\big\| \;\leq\; \|\omega^{-1/2} f\|_{L^2}^2\,. \tag{55}$$

The following lemma derives from (55).

Lemma 2.2. *Assume hypotheses 1.1 or 1.2 and 2.1.*

(i) *If $\omega^{-1}J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$ then $W_{m,n}$ defines a quadratic form on $Q(H_0)$, and I have that*

$$\big\|(H_0 + i)^{-1/2}\,W_{m,n}\,(H_0 + i)^{-1/2}\big\| \;\leq\; C(V)\,\big\|\omega^{-1}\,J^2\big\|_{L^1}\,, \tag{56}$$

for all $1 \leq m + n \leq 2$, where $C(V) < \infty$ is a constant depending on the potential V.

(ii) *If $(1 + \omega^{-1})J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$ then $W_{m,n}$ and $W_{m,n}^*$ define bounded operators on* $\mathrm{dom}(H_0)$, *and*

$$\big\|W_{m,n}^{\#}\,(H_0 + i)^{-1}\big\| \;\leq\; C(V)\,\big\|(1 + \omega^{-1})\,J^2\big\|_{L^1}\,, \tag{57}$$

where $W_{m,n}^{\#}$ is $W_{m,n}$ or $W_{m,n}^*$, and the constant $C(V) < \infty$ depends only on V.

Now, standard Kato perturbation theory implies that

Corollary 2.3. *Assume hypotheses 1.1 or 1.2 and 2.1.*

(i) *If $\omega^{-1}J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$ and $|g| > 0$ is sufficiently small then H_g defines a symmetric, semibounded quadratic form on $Q(H_0)$, and hence the corresponding selfadjoint operator is essentially selfadjoint on* $\mathrm{dom}(H_0)$.

(ii) *If $(1 + \omega^{-1})J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$ and $|g| > 0$ is sufficiently small then H_g is a semibounded, selfadjoint operator on* $\mathrm{dom}(H_g) = \mathrm{dom}(H_0)$.

The proofs for these basic statements can be found in many papers on this subject, e.g., [19, 10, 5]

3. Binding

In this section I focus on the bottom of the spectrum, $E_0(g) := \inf \sigma(H_g)$ of the interacting Hamiltonian H_g. Besides hypotheses 1.1 or 1.2 and 2.1, I will now assume that $(1+\omega^{-1})J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$. Then corollary 2.3(ii) insures that $H_g = H_g^*$ on $\mathrm{dom}(H_0)$ and that $E_0(g) > -\infty$. Furthermore, from the discussion of the spectral properties of H_0 in section 1, I know that $E_0(0) = E_0$ is an eigenvalue. Indeed, the corresponding eigenspace is spanned by $\{\varphi_{0,\alpha} \otimes \Omega\}_{\alpha=1,\ldots,n_0}$.

The question of stability of this eigenvalue under perturbation now arises.

Theorem 3.1. *Assume hypotheses 1.1 or 1.2 and 2.1. Furthermore assume $W_{2,0} = W_{1,1} = W_{0,2} = 0$, $(1+\omega^{-2})J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$. There exists a constant $C(V) < \infty$ such that if $2\alpha := |E_0| - C(V)\big\|(1+\omega^{-2})J^2\big\|_{L^1} g^2 > 0$ then $E_0(g)$ is an eigenvalue with corresponding eigenvector, $\Psi_0(g) \in \mathcal{H}$. Moreover,*

$$\big\| e^{\alpha|x|} \otimes N_f \Psi_0(g) \big\| \; < \; \infty \, . \tag{58}$$

Theorem 3.1 states that $\inf \sigma(H_g)$ is an eigenvalue and that the corresponding eigenfunction is exponentially localized about the origin. The physical interpretation of this statement is that the atom or molecule under consideration does not dissolve by switching on the interaction of the electron and the electromagnetic field. In fact, the spatial localization of the atom or molecule is continuous in $g \to 0$.

A first existence result for a ground state in the framework of hypotheses 1.1 and 2.1, i.e., an eigenvalue at the bottom of the spectrum, was derived in [20], and another important result in the context of the Spin-Boson model was given in [45]. In the form stated above, theorem 3.1 was proved under hypotheses 1.1 and 2.1 in [10] and under hypotheses 1.2 and 2.1 in [5]. The strategies of the proof in [10] and in [5] are similar, and they are both building on ideas given in [20]. The range of validity w.r.t. g was further enlarged in [46], and in [23] it was finally shown that no restriction on the magnitude of g is necessary, whatsoever.

Statements about uniqueness of the ground state, i.e., about the non-degeneracy of $E_0(g)$ as an eigenvalue, were given in [10, 27].

I outline the strategy of the proof of theorem 3.1 as in [10].

- First, the coupling functions $w_{0,1}(k) = w_{1,0}(k)^*$, are replaced by $\chi[\omega(k) \geq m]\, w_{0,1}(k)$ and $\chi[\omega(k) \geq m]\, w_{1,0}(k)$, respectively, where $m > 0$ is interpreted to be a "photon mass". The resulting Hamiltonian is denoted $H_g^{(m)}$.
- By a suitable additional discretization, one shows that $E_0^{(m)}(g) + m = \inf \sigma_{\mathrm{ess}}(H_g^{(m)})$ where $E_0^{(m)}(g) := \inf \sigma(H_g^{(m)})$. Hence, $E_0^{(m)}(g)$ is an eigenvalue of finite multiplicity. Denote by $\Psi_0^{(m)}(g)$ a normalized eigenfunction, $H_g^{(m)}\, \Psi_0^{(m)}(g) = E_0^{(m)}(g)\, \Psi_0^{(m)}(g)$.
- From a simple norm bound follows the convergence $H_g^{(m)} \to H_g^{(0)} = H_g$ in norm-resolvent sense, as $m \to 0$. In particular, $\lim_{m\to 0} E_0^{(m)}(g) = E_0(g)$, and, possibly after passing to a subsequence, $w - \lim_{m\to 0} \Psi_0^{(m)}(g) =: \Psi$ is a ground state of H_g: $H_g\, \Psi = E_0(g)\, \Psi$.

- The key step in the proof is to show that $\Psi \neq 0$. At this point, Agmon estimates for the localization in the x-variable and *soft-photon bounds* insure that the sequence $\Psi_0^{(m)}(g)$ is compact and hence $\Psi \neq 0$.

4. Resonances

The notion of *resonances* discussed here is based on the analytic continuation of resolvent matrix elements by means of complex deformations (here: dilatations). More precisely, a resonance is a singularity of the function

$$F_{\varphi,\psi}(z) \;:=\; \langle\varphi|\,(H-z)^{-1}\psi\rangle\,, \tag{59}$$

analytically continued from $z := \lambda + i\varepsilon \in \mathbb{C}^+$, $\lambda > E_0(g)$, across the real axis onto the second Riemann sheet in $\mathbb{C}^-$. Note that $\lambda \in \sigma_{\mathrm{ess}}(H_g)$, so such an analytic continuation cannot be expected to exists for all $\varphi, \psi \in \mathcal{H}$. Rather, the goal is to construct the analytic continuation of $F_{\varphi,\psi}$, for φ, ψ contained in a natural dense set $\mathcal{D}$.

The set $\mathcal{D}$ is not unique, but it is characterized by a maximality requirement: Denoting by $\mathcal{A}(\varphi,\psi)$ the domain of analyticity of $F_{\varphi,\psi}$, the intersection $\bigcap_{\varphi,\psi\in\mathcal{D}} \mathcal{A}(\varphi,\psi)$ should be the largest possible set under the requirement that $\mathcal{D} \subseteq \mathcal{H}$ be dense.

Our construction of the analytic continuation of $F_{\varphi,\psi}$ goes through complex dilatation [42, 15]. For $\theta \in \mathbb{R}$ and $\psi_n \in \mathcal{F}^{(n)}$, I define a unitary dilatation operator by

$$\big[U_\theta^{(n)}\,\psi_n\big](k_1,\ldots,k_n) \;:=\; e^{-\frac{3\theta}{2}(|\vec{k}_1|+\cdots+|\vec{k}_n|)}\,\psi_n(e^{-\theta}k_1,\ldots,e^{-\theta}k_n)\,, \tag{60}$$

where $e^{-\theta}k = e^{-\theta}(\vec{k},\lambda) := (e^{-\theta}\vec{k},\lambda)$. Furthermore, $U_\theta^{(0)}\Omega := \Omega$. Then, the unitary dilatation U_θ on $\mathcal{H}$ is defined by

$$U_\theta \;:=\; \mathbf{1}_{el}\,\otimes\,\bigoplus_{n=0}^{\infty} U_\theta^{(n)}\,. \tag{61}$$

As in the introduction, it is instructive to discuss the action of U_θ on H_0 before applying it to H_g. I remark that $U_\theta \mathbf{1}_{el}\otimes H_f U_\theta^{-1} = e^{-\theta}\mathbf{1}_{el}\otimes H_f$ and hence

$$H_0(\theta) \;:=\; U_\theta\,H_0\,U_\theta^{-1} \;=\; H_{el}\otimes\mathbf{1}_f \;+\; e^{-\theta}\,\mathbf{1}_{el}\otimes H_f\,. \tag{62}$$

Observe that $H_0(\theta)$ extends from $\theta \in \mathbb{R}$ to an analytic family of type A [41] on the strip $\theta \in S_\pi := \{\theta| -\pi < \mathrm{Im}(\theta) < \pi\}$, i.e., the Banach space-valued map

$$S_\pi \;\ni\; \theta \;\mapsto\; H_0(\theta)\,\big(H_0+i\big)^{-1} \;\in\; \mathcal{B}(\mathcal{H}) \tag{63}$$

is analytic. Note that, for $\theta \notin \mathbb{R}$, $H_0(\theta)$ is not selfadjoint. Yet, $H_0(\theta)$ is a normal operator, even for $\theta \notin \mathbb{R}$. Thus, the discussion of the spectral properties of $H_0(\theta)$ is as simple as the one for H_0. Namely,

$$\sigma[H_0(\theta)] \;=\; \overline{\sigma(H_{el})\;+\;e^{-\theta}\sigma(H_f)} \;=\; \sigma(H_{el})\;+\;e^{-\theta}\mathbb{R}_0^+\,. \tag{64}$$

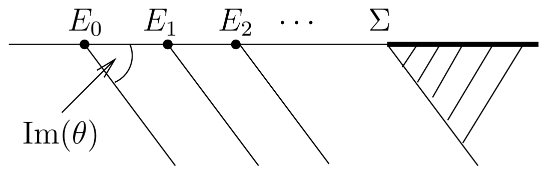

FIGURE 3. The spectrum of $H_0(i\vartheta)$, with $\vartheta > 0$.

For $j=0,1,\ldots$, the real numbers E_j are eigenvalues of $H_0(\theta)$ at the tips of branches of continuous spectrum, the corresponding eigenvectors remain unchanged (see fig. 3).

I now construct the analytic continuation of $F_{\varphi,\psi}$, for $g=0$. I define $\mathcal{D}$ to be the set of *dilatation analytic vectors*, i.e., those vectors $\psi \in \mathcal{H}$, for which the Hilbert space-valued map

$$S_\pi \ni \theta \mapsto \psi_\theta := U_\theta \psi \in \mathcal{H} \tag{65}$$

is analytic. Then, for any $z \in \mathbb{C}^+$ and $\theta = i\vartheta$, $0 < \vartheta < \pi/2$,

$$F_{\varphi,\psi}(z) = \big\langle \varphi \big| (H_0 - z)^{-1} \psi \big\rangle = \big\langle \varphi_{\bar\theta} \big| (H_0(\theta) - z)^{-1} \psi_\theta \big\rangle , \tag{66}$$

by analytic continuation in θ. (Note that φ continues to $\varphi_{\bar\theta}$ because of the antilinearity of $\varphi \mapsto \langle\varphi|\psi\rangle$.) I now obtain the desired analytic continuation of $z \mapsto F_{\varphi,\psi}(z)$ into $\mathbb{C}^- \setminus \sigma[H_0(\theta)]$ by continuing the right side of (66) in z. From this point of view, the complex dilatation in θ defines a projection of the Riemann surface associated to $F_{\varphi,\psi}(z)$ onto the complex plane different from the one obtained for $\theta = 0$. The branches of continuous spectra appear as branch cuts associated to the chosen projection, and the eigenvalues coincide with the branch points of these cuts. Their position is independent of the chosen projection, i.e., invariant under (local) variations of the deformation parameter θ.

The construction of the analytic continuation of $F_{\varphi,\psi}(z)$ for $g > 0$ is similar to the one for $g = 0$, in principle. I recall from hypothesis 2.1 that $W_g = \sum_{1\le m+n\le 2} g^{m+n} W_{m,n}$ and that the coupling functions $w_{m,n}$ in $W_{m,n}$ are functions on $(\mathbb{R}^3 \times \mathbb{Z}^2)^{m+n}$ with values in the operators on $\mathcal{H}_{el}$. For the existence of resonances, I shall employ the following additional assumption.

Hypothesis 4.1. *There exists $0 < \theta_0 < \pi/2$ such that, for all $k \in \mathbb{R}^3 \times \mathbb{Z}_2$ and all $1 \le m+n \le 2$, the Banach space-valued maps*

$$D_{\theta_0} \ni \theta \mapsto w_{m,n}(e^{-\theta}k) \big(\Delta_x + \mathbf{1}\big)^{-1+\frac{m+n}{2}} \in \mathcal{B}(\mathcal{H}_{el}) \tag{67}$$

are analytic, where $D_{\theta_0} := \{|\theta| < \theta_0\} \subseteq \mathbb{C}^2$. Moreover, there is a measurable function $J\colon \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{R}_0^+$ such that

$$\sup_{|\theta|<\theta_0} \big\| w_{1,0}(e^{-\theta}k)\,(-\Delta_{\vec x} + \mathbf{1})^{-1/2} \big\|_{\mathcal{B}(\mathcal{H}_{el})} + \sup_{|\theta|<\theta_0} \big\| w_{0,1}(e^{-\theta}k)\,(-\Delta_{\vec x} + \mathbf{1})^{-1/2} \big\|_{\mathcal{B}(\mathcal{H}_{el})} \le J(k)\,, \tag{68}$$

$$\sup_{|\theta|<\theta_0} \big\| w_{2,0}(e^{-\theta}k, e^{-\theta}k') \big\|_{\mathcal{B}(\mathcal{H}_{el})} + \sup_{|\theta|<\theta_0} \big\| w_{1,1}(e^{-\theta}k, e^{-\theta}k') \big\|_{\mathcal{B}(\mathcal{H}_{el})} + \sup_{|\theta|<\theta_0} \big\| w_{0,2}(e^{-\theta}k, e^{-\theta}k') \big\|_{\mathcal{B}(\mathcal{H}_{el})} \leq J(k)\, J(k') , \quad (69)$$

for all $k, k' \in \mathbb{R}^3 \times \mathbb{Z}_2$*.*

For $j = 1, 2, \ldots,$ *let* $\{\varphi_{j,\alpha}\}_{\alpha=1,\ldots,n_j} \subseteq \mathcal{H}_{el}$ *be an orthonormal basis of eigenfunctions of* H_{el} *corresponding to the eigenvalue* E_j*. Then the following genericity assumption on the coupling function* $w_{1,0}$ *is assumed to hold, for all* $j \geq 1$ *and* $1 \leq \alpha \leq n_j$*,*

$$\sum_{i=0}^{j} \sum_{\beta=1}^{n_i} \Big| \mathrm{supp}\big\{ \langle \varphi_{i,\beta} |\, w_{1,0}(\,\cdot\,)\, \psi_{el,j,\alpha} \rangle \big\} \Big| > 0 . \quad (70)$$

I remark that the pointwise analyticity assumed in hypothesis 4.1 is slightly stronger than what it necessary.

Furthermore, I remark that hypothesis 4.1 does not hold for the coupling functions of the physical example in (38), (40), and (41) because the dilatation operator U_θ only acts on the photon Fock space and leaves the electron variable unchanged, and consequently, the factor $\exp[i\alpha \vec{k} \cdot \vec{x}]$ in $\vec{G}_{\vec{x}}(k)$, defined in (29), is turned into $\exp[\vartheta\, \alpha\, \vec{k} \cdot \vec{x}]$ which is exponentially growing, as $\vec{k} \cdot \vec{x}$ becomes large. This may be avoided by dilating both the electron and the photon variables, for then $\exp[i\alpha \vec{k} \cdot \vec{x}]$ is simply invariant under dilatation. The price to pay is that I have to require dilation analyticity of the potential V in H_{el} and that $H_{el}(\theta) := U_\theta H_{el} U_\theta^{-1}$ is not selfadoint anymore, if $\theta \notin \mathbb{R}$. The latter makes certain estimates on norms of resolvents of $H_{el}(\theta)$ slightly more complicated than for the selfadjoint case, $\theta = 0$. This has been carried out in [12].

Lemma 4.2. *Assume hypotheses 1.1 or 1.2 and 2.1 and 4.1. If* $(1 + \omega^{-1}) J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$ *then* $H_g(\theta)$ *defines a analytic family of type A, i.e., the Banach space-valued map*

$$D_{\theta_0} \ni \theta \mapsto H_g(\theta) \big(H_0 + i \big)^{-1} \in \mathcal{B}(\mathcal{H}) \quad (71)$$

is analytic.

Lemma 4.2 insures that, for all $\varphi, \psi \in \mathcal{D}$, for any $z \in \mathbb{C}^+$, and $\theta = i\vartheta$, with $0 < \vartheta < \theta_0$,

$$F_{\varphi,\psi}(z) = \big\langle \varphi \big| \, (H_g - z)^{-1} \psi \big\rangle = \big\langle \varphi_{\bar\theta} \big| \, (H_g(\theta) - z)^{-1} \psi_\theta \big\rangle , \quad (72)$$

by analytic continuation in θ. So, as for H_0, I can analytically continue in z from $\mathbb{C}^+$ to $\mathbb{C}^- \setminus \sigma[H_g(\theta)]$.

Theorem 4.3. *Assume hypotheses 1.1 or 1.2 and 2.1 and 4.1. Furthermore, assume that* $\theta = i\vartheta$*, where* $\vartheta > 0$ *is small but fixed, and that* $(1 + \omega^{-\beta}) J^2 \in L^1(\mathbb{R}^3 \times \mathbb{Z}_2)$*, for some* $\beta > 1$*. Then, for each* $j \geq 1$*, there exist constants,* $\Gamma_j > 0$ *and* $C_j < \infty$*, such that, for* $g > 0$ *sufficiently small,*

$$\Big[E_{j-1} + C_j g \, , \, E_{j+1} - C_j g \Big] + i \Big(-g^2 \, \Gamma_j \, , \, \infty \Big) \subseteq \rho[H_g(\theta)] := \mathbb{C} \setminus \sigma[H_g(\theta)] . \quad (73)$$

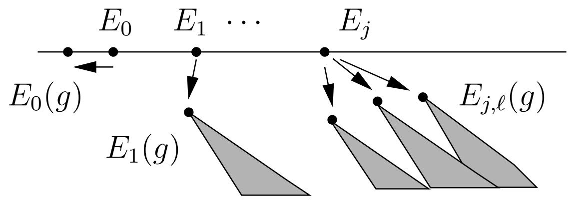

FIGURE 4. The spectrum of $H_g(i\vartheta)$, with $\vartheta > 0$, up to $\mathcal{O}(g^{2+\varepsilon})$-neighbourhoods, for some $\varepsilon > 0$.

Moreover, the spectrum is located in $\mathcal{O}(g^{2+\varepsilon})$-neighbourhoods of the comet-shaped regions depicted in fig. 4, for some $\varepsilon > 0$.

Theorem 4.3 has the important consequence that the spectrum of H_g in the interval $[E_0(g) + Cg\,,\, \Sigma - Cg]$ is purely absolutely continuous (see, e.g., [15]).

Under more stringent conditions on the coupling functions it is possible to derive more precise information about the nature of the resonances than what is given in theorem 4.3. The comet-shaped regions (see fig. 4) are only a rough description of their location. The additional assumption that allows for a more precise statement is as follows.

Hypothesis 4.4. *Assume hypothesis 4.1. For some $\mu > 0$, the function $J: \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{R}_0^+$ obeys the following additional bound,*

$$\sup_{k \in \mathbb{R}^3 \times \mathbb{Z}_2} \Big\{ \omega(k)^{\frac{1-\mu}{2}} \, J(k) \Big\} \;<\; \infty\,. \tag{74}$$

A renormalization group analysis, as described below in section 7, reveals that the singularities of $F_{\varphi,\psi}$, are actually confined in cuspidal domains whose tip is an eigenvalue of $H_g(\theta)$, see fig. 5.

Theorem 4.5. *Assume hypotheses 1.1 or 1.2 and 2.1, 4.1, and 4.4. Furthermore, assume that $\theta = i\vartheta$, where $\vartheta > 0$ is small but fixed. Then, for each $j \geq 1$ and $g > 0$ sufficiently small, $H_g(\theta)$ possesses complex eigenvalues, $E_{j,\alpha}(g) = E_j + \mathcal{O}(g) \in \mathbb{C}^-$, with corresponding eigenvectors $\Psi_{j,\alpha}(g) = \varphi_{j,\alpha} \otimes \Omega + \mathcal{O}(g) \in \mathcal{H}$, where $\alpha = 1, \ldots, n_j$. The spectrum of H_g is locally (in a disk of radius $g^{2-\varepsilon}$ about E_j, where $\varepsilon > 0$) contained in the cuspidal domains*

$$E_j(g) \;+\; e^{-i\vartheta} \left\{ a + ib \;\middle|\; a \geq 0\,,\; |b| \leq C a^{1+\mu/4} \right\}, \tag{75}$$

see fig. 5. Moreover, the eigenvalues $E_{j,\alpha}(g)$ and the corresponding eigenvectors $\Psi_{j,\alpha}(g)$ are obtained from a series expansion in (fractional) powers of g which is determined by the iterated application of the Feshbach renormalization map.

I remark that the same assertion holds for $j = 0$ and $\vartheta = 0$, in which case $E_0(g) = E_0 - \mathcal{O}(g) = \inf \sigma(H_g) \in \mathbb{R}$, is the perturbed ground state energy, and $\Psi_0(g)$ is the corresponding ground state. Under certain genericity assumptions similar to (70) the ground state will be unique, even if the multiplicity n_0 of the

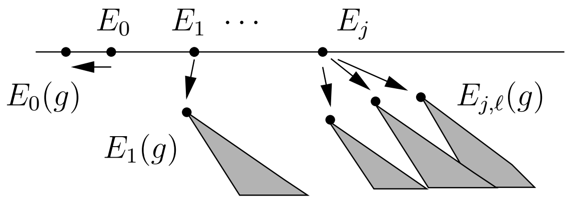

FIGURE 5. The spectrum of $H_g(i\vartheta)$, with $\vartheta > 0$.

unperturbed ground state energy E_0 is 2 or even higher. The degeneracy of E_0, however, is not lifted in second but in higher order in g.

I further remark that, comparing (63) to the physical coupling function $\vec{G}_{\vec{x}}$ in (29), I observe that hypothesis 4.4 is not fulfilled. Indeed, $\mu = 0$ in this case, and theorem 4.5 does not apply. This should not cause any disappointment because, for $\mu = 0$, there are several counterexamples to the existence of a ground state $\Psi_0(g) \in \mathcal{H}$ known [19, 45, 1]. In these counterexamples it is shown that, if at all, the ground state of H_g is a density matrix in a different representation of Fock space, not unitarily equivalent to our original Fock space.

5. Scattering Theory

The subject of scattering theory is the asymptotics of the time evolution, e^{-itH}, as $t \to \infty$. One of the central mathematical goals of scattering theory is to prove asymptotic completeness. The most general results on asymptotic completeness for models of the type discussed here have been obtained in [22, 16, 17], essentially under two additional assumptions.

Hypothesis 5.1. *The photon field is* massive*, i.e., the photon dispersion* $\omega(k) := |\vec{k}|$ *has been replaced by*

$$\omega_m(k) \ := \ \sqrt{\vec{k}^2 \, + \, m^2} \ , \tag{76}$$

for some arbitrary but fixed $m > 0$*, and*

Hypothesis 5.2. *The particle system is* confined*, that is, either* $\lim_{|x|\to\infty} V(x) = \infty$*, or*

$$\big\| (|x| + 1)^{1+\mu} \, w_{1,0}(k) \, (-\Delta_{\vec{x}} + \mathbf{1})^{-1/2} \big\|_{\mathcal{B}(\mathcal{H}_{el})} \ \le \ J(k) \, , \tag{77}$$

for some $\mu > 0$*.*

It is additionally assumed that $w_{0,1} = w_{1,0}^*$, for H_g to be selfadjoint, and $w_{0,2} = w_{1,1} = w_{2,0} = 0$, for simplicity.

To formulate asymptotic completeness for the type of models discussed here, I first introduce the *asymptotic creation* and *annihilation operators*. For given $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, they are defined by

$$a_\pm^\#(f) \ := \ s - \lim_{t \to \pm\infty} \Big\{ e^{-itH_g} \, e^{itH_0} \, a^\#(f) \, e^{-itH_0} \, e^{itH_g} \Big\} \, , \tag{78}$$

where $a^{\#} = a$ or $= a^*$. Note that these operators act on the full space $\mathcal{H}$, rather than on $\mathcal{F}$. In [28, 29, 30] these operators were shown to exist, and this is the first time that the positivity of the mass $m > 0$ enters. The asymptotic creation and annihilation operators play the same role for H_g as the usual creation and annihilation operators do for H_0, namely

$$e^{-itH}\, a_\pm^{\#}(f)\, e^{itH} \;=\; a_\pm^{\#}\big(e^{it\omega} f\big)\,. \tag{79}$$

It is easy to see that the asymptotic creation and annihilation operators yield another representation of the canonical commutation relations. It is, however, less clear, which vectors in $\mathcal{H}$ replace the vacuum vector, i.e., which vectors are contained in

$$\mathcal{K}_\pm \;:=\; \Big\{ \psi \in \mathcal{H} \;\Big|\; a_\pm(f)\,\psi = 0\,,\ \forall f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \Big\}\,. \tag{80}$$

Observe that $\mathcal{K}_\pm$ contain all bound states of H_g, for if $H_g\psi = E\psi$ then

$$\big\| e^{-itH_g}\, e^{itH_0}\, a(f)\, e^{-itH_0}\, e^{itH_g}\, \psi \big\| \;=\; \big\| a\big(e^{-it\omega} f\big)\,\psi \big\| \;\to\; 0\,, \quad \text{as } t \to \infty, \tag{81}$$

because $e^{-it\omega} f \to 0$, weakly in L^2. Consequently,

$$\mathcal{H}_{pp}(H_g) \;\subseteq\; \mathcal{K}_+ \cap \mathcal{K}_-\,, \tag{82}$$

where $\mathcal{H}_{pp}(H_g)$ is the subspace corresponding to the pure point spectrum of H_g, i.e., onto all its eigenvectors. *Asymptotic completeness* is the statement that these three subspaces are all equal, in fact. The following result can be found in [17].

Theorem 5.3. *Assume hypotheses 1.1 or 1.2 and 2.1, 5.1, and 5.2. Then*

$$\mathcal{H}_{pp}(H_g) \;=\; \mathcal{K}_+ \;=\; \mathcal{K}_-\,. \tag{83}$$

As a consequence of this theorem, there exists a unitary operators $J_\pm : \mathcal{H} \to \mathcal{H}_{pp}(H_g) \otimes \mathcal{F}$ such that $a_\pm^{\#}(f) = J\, a^{\#}(f)\, J^*$. By theorem 3.1, I know that $\mathcal{H}_{pp}(H_g) \neq 0$, since $E_0(g)$ is an eigenvalue. The general belief is that $E_0(g)$ is simple, with corresponding eigenvector $\Psi_0(g)$, and that H_g has no other eigenvalues, i.e., $\mathcal{H}_{pp}(H_g) = \mathbb{C} \cdot \Psi_0(g)$. If that was the case then $JH_gJ^* = \int dk\, \omega(k)\, a_\pm^*(k) a_\pm(k)$.

I remark that one of the basic inputs for the proof of asymptotic completeness is a positive commutator- or Mourre estimate [39, 40], and such an estimate is indeed derived and applied in [22, 16, 17] to prove propagation estimates. The typical form of these estimates is

$$\chi_\Delta(H_g)\, i\big[H_g\,,\, A\big]\, \chi_\Delta(H_g) \;\geq\; \mu\, \chi_\Delta^2(H_g) \;+\; K\,, \tag{84}$$

where $\Delta \subseteq \mathbb{R}$ is a Borel set, A is a suitable observable, a customary choice being the dilatation generator, $\mu > 0$ is a strictly positive number, and K is a compact operator. Here is another point where positivity of the mass enters, as it guarantees that H_f is relatively bounded by N_f, the number operator on $\mathcal{F}$, and vice versa.

Positive commutator estimates, like (84), are interesting in their own right, for example, because they imply that in Δ, the spectrum of H_g is purely absolutely continuous. A variety of positive commutator estimates for the models discussed here were derived in [31, 33, 32, 44, 22, 14].

6. Positive Temperatures

The dynamics e^{-itH_g} generated by the Hamiltonian H_g on the state space $\mathcal{H}$ of the quantum mechanical system under consideration is the appropriate description for systems at zero temperature, $T = 0$. At positive temperature, $T > 0$, however, it is necessary to pass to a description in which the dynamics is generated by the Liouvillian, L_g, which acts on the tensor product $\mathcal{H} \otimes \mathcal{H}$ of two copies of $\mathcal{H}$. I briefly motivate this change and sketch the resulting mathematical objects, below.

It is well-known that the Gibbs state of a finite quantum mechanical system, with Hamiltonian H and at inverse temperature $\beta := (kT)^{-1}$ is given by $\rho := \mathrm{Tr}\{\,\cdot\, e^{-\beta H}\}\big(\mathrm{Tr}\{e^{-\beta H}\}\big)^{-1}$, i.e., for a given observable $A = A^* \in \mathcal{B}(\mathcal{H})$, its expectation value is $\mathrm{Tr}\{A\, e^{-\beta H}\}\big(\mathrm{Tr}\{e^{-\beta H}\}\big)^{-1}$. The important point here is that I assumed the system to be *finite* or *confined*, meaning that $\mathrm{Tr}\{e^{-\beta H}\} < \infty$. Indeed, confining an infinite quantum system to a large but finite box $\Lambda \subseteq \mathbb{R}^3$ (with periodic b.c., say), I turn the continuous spectrum of the Hamiltonian into discrete spectrum, and, for sufficiently large inverse temperature $\beta \gg 1$, the semigroup $e^{-\beta H}$ is trace class, and I call the corresponding state ρ_Λ.

For many questions concerning *static* thermodynamic properties (e.g., computation of the thermodynamic potentials or correlation functions), it usually suffices to work in finite boxes Λ, to prove estimates uniformly in $|\Lambda|$, and to pass finally to the thermodynamic limit, $\Lambda \nearrow \mathbb{R}^3$, by continuity. For example, the expectation value of a local observable A is then obtained in the thermodynamic limit as $\rho_\infty(A) := \lim_{\Lambda \nearrow \mathbb{R}^3} \rho_\Lambda(A)$.

For the study of *dynamical* questions, however, it may not be sufficient to work in finite boxes, but it might be necessary to formulate the dynamics in the thermodynamic limit right away. Indeed, the asymptotics of the time evolution, as $t \to \infty$, and the thermodynamic limit, $\Lambda \nearrow \mathbb{R}^3$, do not commute, in general. One example, for which this difference is crucial, is the property of *return to equilibrium*. If A_0 is an observable and $A_t := \alpha_t(A_0)$ its time evolution then the system under consideration is said to *return to equilibrium* iff, for all states ρ (with a certain trace-class property), I have

$$\text{(weak form)} \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T \rho(A_t)\, dt \;=\; \omega(A_0)\,, \tag{85}$$

$$\text{(strong form)} \qquad \lim_{t\to\infty} \rho(A_t) \;=\; \omega(A_0)\,. \tag{86}$$

Here, ω is a thermal equilibrium state, characterized by time-translation invariance and the KMS condition (see below). The existence and uniqueness of such a state is, in general, by no means trivial.

The framework for an infinite-volume theory at positive temperature was given in [24, 7, 6]. Two crucial properties that carry over from finite-volume Gibbs states, ρ_Λ, to the thermodynamic limit $\rho_\infty := \lim_{\Lambda \nearrow \mathbb{R}^3} \rho_\Lambda$ (provided it exists), are the time-translation invariance,

$$\rho_\infty\big(\alpha_t(A)\big) \;=\; \rho_\infty\big(A\big)\,, \tag{87}$$

for all t and A, and the KMS boundary condition,

$$\rho_\infty\big(A\,\alpha_t(B)\big) \;=\; \rho_\infty\big(\alpha_{-i\beta+t}(B)\,A\big)\,, \tag{88}$$

for A and B in a certain dense subalgebra $\mathring{\mathcal{A}}$ of the observable C^* algebra $\mathcal{A}$, invariant under α_t.

Using a GNS construction, the infinite-volume time evolution α_t of an observable $A \in \mathcal{A}$ can be unitarily implemented as

$$\rho_\infty\big(\alpha_t(A)\big) \;=\; \Big\langle \Omega_\beta \;\Big|\; e^{-itL_g}\,\ell[A]\,e^{itL_g}\,\Omega_\beta \Big\rangle\,, \tag{89}$$

where $\mathcal{H}_\beta := \mathcal{H} \otimes \mathcal{H}$, ℓ is a linear left-representation of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H}_\beta)$, the KMS state ρ_∞ is identified with the projection onto a cyclic (vacuum) vector in $\mathcal{H}_\beta$, e.g., $\Omega_\beta = \varphi_0 \otimes \varphi_0 \otimes \Omega \otimes \Omega \in \mathcal{H} \otimes \mathcal{H}$ (tensor factors swapped), and the dynamics is generated by the selfadjoint operator L_g on $\mathcal{H}_\beta$, the *Liouvillian.*

The difference between finite in infinite systems is manifest in the form of the Liouvillian. For finite systems (i.e., for the models discussed here, discretized momentum space $|\Lambda|^{-1/3}\mathbb{Z}^3$ replacing $\mathbb{R}^3$), $\ell_\Lambda[A] = A \otimes \mathbf{1}$ and $L_g^\Lambda = H_g^\Lambda \otimes \mathbf{1} - \mathbf{1} \otimes H_g^\Lambda$. For infinite systems, however, L_g is not of this form but rather

$$L_g \;=\; L_0 \,+\, \ell[W_g] - r[W_g] \;=\; H_0 \otimes \mathbf{1} - \mathbf{1} \otimes H_0 \,+\, \ell[W_g] - r[W_g]\,, \tag{90}$$

where $\ell[a(f)]$ is *not* simply $a(f) \otimes \mathbf{1}$ but, e.g.,

$$\ell[a(f)] \;=\; a\big(\sqrt{1+\rho_\beta}\,f\big) \otimes \mathbf{1} \,+\, \mathbf{1} \otimes a^*\big(\sqrt{\rho_\beta}\,\overline{f}\big)\,, \tag{91}$$

and $\rho_\beta(k) := (e^{\beta\omega(k)} - 1)^{-1}$.

The virtue of the GNS construction yielding the Liouvillian L_g is that it allows for tracing back the property of return to equilibrium to spectral properties of L_g. Namely, return to equilibrium follows if

- Zero is a simple eigenvalue of L_g, i.e., $\mathrm{Ker}\{L_g\} = \mathbb{C} \cdot \Omega_\beta(g)$, where $\Omega_\beta(g)$ is the unique KMS state of the system,
- Apart from zero, the spectrum is continuous, $\sigma_{\mathrm{cont}}(L_g) \setminus \{0\} = \sigma(L_g) \setminus \{0\}$. In this case, return to equilibrium holds at least in the weak form (85).
- If, apart from zero, the spectrum is even absolutely continuous, $\sigma_{\mathrm{ac}}(L_g) \setminus \{0\} = \sigma(L_g) \setminus \{0\}$, then return to equilibrium holds in the strong form (86).

This reformulation was proposed and applied to prove return to equilibrium for a system fulfilling hypotheses 2.1–5.1 in [34, 35, 36, 37]. The spectral analysis of the Liouvillian then goes through a complex deformation, similar to the analysis of resonances in section 3. The complex deformation used in [34, 35, 36, 37] is a special type of *complex translation.* This elegant method has the advantage that it yields fairly strong results already in second order perturbation theory, but the price to pay are the stringent analyticity assumptions on the coupling functions $w_{m,n}$ and the requirement of smallness of the coupling parameter g compared to the temperature $T > 0$.

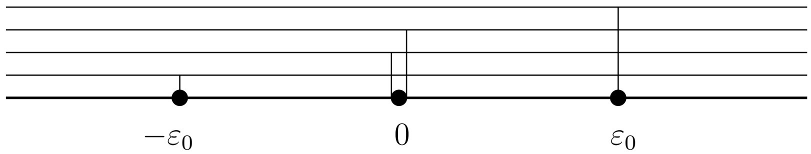

FIGURE 6. The spectrum of L_0 with $H_{el} = \mathrm{diag}[\varepsilon_0, -\varepsilon_0]$.

The approach in [34, 35, 36, 37] has been generalized in [13] to allow for coupling functions that merely fulfill hypotheses 2.1–5.1 and values of the coupling parameter uniform in the temperature $T \searrow 0$, by using complex dilatations. The trade-off here is that for the prove of return to equilibrium I need to use technically involved methods like the Feshbach renormalization map, described in section 7, below.

To understand the results from [34, 35, 36, 37] and those in [13], it is again useful to discuss the trivial decoupled case, $g = 0$. Recall that, according to hypothesis 1.2, I consider a simplified model of the particle system as a selfadjoint $N \times N$-matrix with non-degenerate eigenvalues, $H_{el} = \mathrm{diag}(E_0, E_1, \ldots, E_{N-1})$. Then the spectrum of $L_{el} := H_{el} \otimes \mathbf{1} - \mathbf{1} \otimes H_{el}$ is given by $\{E_{i,j} := E_i - E_j | 0 \leq i, j \leq N-1\}$. Note that zero is an eigenvalue of multiplicity N. Next, the spectrum of $L_f := H_f \otimes \mathbf{1} - \mathbf{1} \otimes H_f$ covers the entire real axis, and according to $L_0 = L_{el} \otimes \mathbf{1} + \mathbf{1} \otimes L_f$, I have that $\sigma(L_0) = \overline{\sigma(L_{el}) + \sigma(L_0)} = \mathbb{R}$, and all $E_{i,j}$ become eigenvalues embedded in the continuum, see fig. 6.

The complex translations used in [34, 35, 36, 37] now transform L_f into $L_f(\theta) = L_f - i\vartheta N_f$, where N_f is the number operator on $\mathcal{F} \otimes \mathcal{F}$ and $\theta = i\vartheta$, $\vartheta > 0$. Therefore,

$$\sigma[L_0(\theta)] \;=\; \{E_{i,j} := E_i - E_j | 0 \leq i, j \leq N-1\} \,\cup\, \bigcup_{N \in \mathbb{N}} \Big\{ \theta N + \mathbb{R} \Big\} . \tag{92}$$

I observe that the eigenvalues on the real axis are isolated. A simple application of second order perturbation theory now shows that, for $0 < g \ll |\vartheta|$, all non-zero eigenvalues are shifted into the lower half plane, $\mathrm{Im} E_{i,j}(g) < -\Gamma_{i,j} g^2$, $\Gamma_{i,j} > 0$. Furthermore, the N-fold degeneracy of the zero eigenvalue is lifted: all but one eigenvalues of $\mathrm{Ker} L_0(\theta)$ are also shifted into $\mathbb{C}_-$. The one vector remaining in $\mathrm{Ker} L_g$ in second order perturbation theory is, in fact, the approximate KMS state. Unfortunately, the domain of analyticity of the map $\theta \mapsto L_0(\theta)$ is the disk of radius T about 0, where $T > 0$ is the temperature. Thus, one has the restriction $|g| < T$.

Using a special form of complex dilatations, the unperturbed operator L_0 is mapped into $L_0(\theta) := L_{el} + \cos(\vartheta) L_f - i \sin(\vartheta) L_{\mathrm{aux}}$, where $L_{\mathrm{aux}} := H_f \otimes \mathbf{1} + \mathbf{1} \otimes H_f$ and $\theta = i\vartheta$, $\vartheta > 0$. Therefore, the spectrum of $L_0(\theta)$ is the union of sectors of

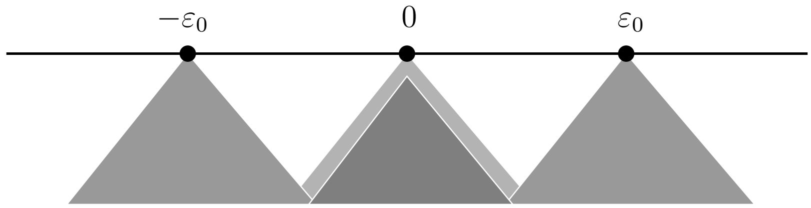

FIGURE 7. The spectrum of $L_0(\theta)$, for $\mathrm{Re}\,\theta = 0$, $\mathrm{Im}\,\theta = \vartheta > 0$.

opening angle $(\pi/2) - \vartheta$ in CC^-, with (real) eigenvalues $E_{i,j}$ as tips,

$$\sigma[L_0(\theta)] \;=\; \bigcup_{i,j=0}^{N-1} E_{i,j} \;+\; \big\{ a - ib \,\big|\, b > 0 \,, \;\; |a| \le \cot(\vartheta) b \big\} \,, \tag{93}$$

see fig. 7.

The domain of analyticity of the map $\theta \mapsto L_0(\theta)$ is now includes the open disk of radius $\pi/2$ about 0, uniformly in $T \to 0$. (I remark that this analytic continuation is more subtle than what is discussed in section 3 because $L_g(\theta)$ is not an analytic family of type A.) Note, however, that the eigenvalues $E_{i,j}$ of $L_0(\theta)$ on the real axis are *not* isolated anymore, and their behaviour under switching on the coupling parameter $g > 0$ cannot be studied by standard perturbation theory, in general. Nevertheless, an argument adapted from second order perturbation theory now shows that, for $0 < g \ll 1$, all sectors attached to non-zero eigenvalues are (possibly slightly defromed and) shifted into the lower half-plane, $\mathrm{Im} E_{i,j}(g) < -\Gamma_{i,j} g^2$, $\Gamma_{i,j} > 0$. Furthermore, the N-fold degeneracy of the zero eigenvalue is lifted: $N-1$ of the N overlapping sectors attached to the zero eigenvalues $E_{0,0} = \ldots = E_{N-1,N-1} = 0$ of $L_0(\theta)$ are also shifted into $\mathbb{C}_-$, and one sector at 0 remains there, for $g > 0$, in second order perturbation theory.

The most difficult part is now show that the form of the spectrum of $L_g(\theta)$ described above is stable *beyond second order perturbation theory*, that is, to prove that higher order terms in a perturbation series do not change it qualitatively (although the sectors may become slightly deformed). This is established by applying the Feshbach renormalization map described in section 7, below. As a result, the the following theorem is obtained in [13].

Theorem 6.1. *Assume hypotheses 1.2 and 2.1, 5.1, 5.2. Then*

(i) *There exist* $0 < \vartheta_0' < \vartheta_0$ *such that, for* $z \in \mathbb{C}^+$, *the resolvent* $(L_g(\theta) - z)^{-1}$ *has an analytic continuation from* $(z, 0)$ *to* $(z, i\vartheta)$, *for any* $\vartheta_0' < \vartheta < \vartheta_0$.
(ii) *Zero is a simple eigenvalue of* L_g *and* $L_g(i\vartheta)$ *corresponding to a KMS state of the system, which, therefore, exists and is unique.*

(iii) *For $\vartheta_0' < \vartheta < \vartheta_0$, there exists $0 < \varepsilon$ such that, for $0 < g \ll 1$, the spectrum of $L_g(i\vartheta)$ is contained in*

$$\sigma[L_0(i\vartheta)] \;=\; \big\{a - ib \,\big|\, b > 0\,,\;\; |a| \le \cot(\vartheta - g/\varepsilon) b\big\}$$
$$\cup \bigcup_{i,j=0, i\neq j}^{N-1} \Big\{E_{i,j}(g) \,+\, \big\{a - ib \,\big|\, b > 0\,,\;\; |a| \le \cot(\vartheta) b\big\} + D(g^{2+\varepsilon})\Big\}\,. \quad (94)$$

Therefore, the spectrum of L_g away from zero is absolutely continuous, and return to equlibrium *holds in the strong form (86).*

7. Renormalization Map

In this final section I describe how the *Feshbach Renormalization Map* is used for spectral analysis, e.g., of resonances (see section 3) or the zero eigenvalue of the Liouvillian (see section 6), to prove the existence of eigenvectors and to derive an explicit, convergent series expansions for eigenvalues and the corresponding eigenvectors, for perturbation problems which are not of the standard type with isolated unperturbed eigenvalues.

To be concrete, I study the ground state energy of H_g, assuming hypotheses 1.2, 2.1, 5.1 and (without loss of generality) that $E_1 - E_0 = 2$.

The key ingredient is the Feshbach map which is well-known in mathematics and physics, perhaps under a different name like "Grushin problem" or "Schur complement". I refer here to [11]. Given a closed operator H on a Hilbert space $\mathcal{H}$ and a bounded projection $P = P^2$, $\overline{P} := \mathbf{1} - P$ denoting the complement.

Lemma 7.1. *Assume that $\overline{H} := \overline{P} H \overline{P}$ is bounded invertible on* $\mathrm{Ran}\overline{P}$ *and that PHP, $PH\overline{P}(\overline{H})^{-1}\overline{P}$, and $\overline{P}(\overline{H})^{-1}\overline{P}HP$ are bounded. Then*
(i) The Feshbach map $H \mapsto \mathcal{F}_P(H)$,

$$\mathcal{F}_P(H) \;:=\; P\,H\,P \,-\, P\,H\,\overline{P}\,\big(\overline{H}\big)^{-1}\overline{P}\,H\,P\,, \quad (95)$$

defines a bounded operator on $\mathrm{Ran}P$.
(ii) H is invertible on $\mathcal{H}$ iff $\mathcal{F}_P(H)$ is invertible on $\mathrm{Ran}P$. *In this case*

$$H^{-1} \;=\; \Big(P - \overline{P}(\overline{H})^{-1}\overline{P}HP\Big)\,\mathcal{F}_P(H)\,\Big(P - PH\overline{P}(\overline{H})^{-1}\overline{P}\Big) \,+\, \overline{P}(\overline{H})^{-1}\overline{P}\,. \quad (96)$$

(iii) $\dim \mathrm{Ker}(H) = \dim \mathrm{Ker}(\mathcal{F}_P(H))$.

I refer to (ii) and (iii) as *isospectrality* of the Feshbach map.

As a first application, I set $P := |\varphi_0\rangle\langle\varphi_0| \otimes \chi[H_f < 1]$ and apply the Feshbach map to $H := H_g - E_0 - z$, for $|z| < 1/2$. It is easy to see that the assumptions of lemma 7.1 are fulfilled, and thus I obtain a family of bounded operators, $H_g^{(0)}[z]$

$$|\varphi_0\rangle\langle\varphi_0| \otimes H_g^{(0)}[z] \;:=\; \mathcal{F}_P(H_g - z)\,, \;\;\text{defined on } \mathcal{H}_{\mathrm{red}} := \mathrm{Ran}\chi[H_f < 1]\,. \quad (97)$$

To define the renormalization group map $\mathcal{R}_\rho$, I introduce a norm, $\|\cdot\|'$, on $\mathcal{B}(\mathcal{H}_{\mathrm{red}})$ that is stronger than the usual operator norm, $\|A\| \le \|A\|'$ (Details can be found in [11]). In a small $\|\cdot\|'$-ball, $B \subseteq \mathcal{H}_{\rm red}$, about H_f, and for $0 < \rho < 1/32$, the renormalization map $\mathcal{R}_\rho$ is defined by

$$\mathcal{R}_\rho \, : \, B \to B \, , \quad H \; \mapsto \; \frac{1}{\rho} U_\rho \Big(\mathcal{F}_{\chi[H_f<\rho]}(H) \; - \; \big\langle \mathcal{F}_{\chi[H_f<\rho]}(H) \big\rangle_\Omega \Big) U_\rho^* \, , \tag{98}$$

where $\langle \cdot \rangle_\Omega$ denotes vacuum expectation value, U_ρ is the unitary dilatation that maps $k \mapsto \rho k$, thus $\rho^{-1} U_\rho H_f U_\rho^* = H_f$, $U_\rho \chi[H_f < \rho] U_\rho^* = \chi[H_f < 1]$, and hence $U_\rho {\rm Ran} \chi[H_f < \rho] = \mathcal{H}_{\rm red}$. The most important property of $\mathcal{R}_\rho$ is that it is a contraction on B, with fixed point H_f. This leads to the following theorem

Theorem 7.2. *For $0 < g \ll 1$, there is a unique number $E_0(g) \in B_{1/2}(E_0)$ such that*

$$H_g^{(n)} \; := \; \mathcal{R}_\rho^n \big(H_g^{(0)}[E_0(g)] \big) \; \to \; H_f \, , \quad \textit{in } \|\cdot\|', \textit{ as } n \to \infty. \tag{99}$$

The number $E_0(g)$ can be interatively computed as $E_0(g) = \lim_{N\to\infty} E_0^{(N)}(g)$, where $E_0^{(N)}(g)$ is the unique solution of

$$E_0^{(N)}(g) = E_0 \, + \sum_{n=0}^{N} \rho^{-n} \big\langle \mathcal{F}_{\chi[H_f<\rho]} \big(H_g^{(n)} \big) \big\rangle_\Omega \, . \tag{100}$$

The isospectrality of the Feshbach map, according to lemma 7.1, and the fact that Ω is an eigenvector (corresponding to a zero eigenvalue) of the operator $H_f = \lim_{n\to\infty} H_g^{(n)}$ now yields the eigenvalue and eigenvector of H_g, sought for.

Corollary 7.3. *The number $E_0(g)$ defined in theorem 7.2 is an eigenvalue of H_g. The corresponding eigenvector can be written as a limit of a sequence of approximate eigenvectors determined by an iterative equation similar to (100).*

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