Some Aspects of Mean Curvature Flow in Presence of Nonsmooth Anisotropies

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Abstract. We discuss some aspects of motion by mean curvature of hypersurfaces in presence of nonsmooth anisotropies. We include the crystalline case in three dimensions.

1. Introduction

In this short note we discuss some aspects of anisotropic motion by mean curvature. One of the most interesting examples is when the anisotropy is crystalline in three dimensions (i.e. the Wulff shape is a polytope). In this respect, pioneeristic papers have been written by J. Taylor, see for instance [20, 21, 22]. Our main idea is to work in relative geometry, looking at the anisotropy as a norm inducing a new geometry in the ambient space. A number of difficulties arise, due to the nonstrict convexity and nonsmoothness of the Wulff shape. The starting paper leading to this approach is [12]. The reference list is largely incomplete. We refer to [22, 15, 24, 16, 17] for a more detailed bibliography.

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2. Notation

Anisotropies on $\mathbb{R}^n$. We indicate by $\mathcal{M}(\mathbb{R}^n)$ the class of all anisotropies of $\mathbb{R}^n$ (Finsler metrics), i.e. $\phi \in \mathcal{M}(\mathbb{R}^n)$ if $\phi\colon \mathbb{R}^n \to [0, +\infty[$ is convex and satisfies the properties

$$\phi(\xi) \geq \lambda|\xi| \,, \qquad \phi(a\xi) = a\phi(\xi) \,, \qquad \xi \in \mathbb{R}^n \,,\ a \geq 0 \,,$$

for a suitable constant $\lambda \in]0, +\infty[$. The dual function $\phi^o\colon \mathbb{R}^n \to [0, +\infty[$ of ϕ is defined as $\phi^o(\xi^*) := \sup\{\xi^* \cdot \xi \ :\ \phi(\xi) \leq 1\}$ for $\xi^* \in \mathbb{R}^n$, and belongs to $\mathcal{M}(\mathbb{R}^n)$; ϕ^o plays the rôle of the surface tension, see (1). We set

$$\mathcal{B}_\phi := \{\xi \in \mathbb{R}^n : \phi(\xi) \leq 1\} \,, \qquad \mathcal{B}^o_\phi := \{\xi^* \in \mathbb{R}^n : \phi^o(\xi^*) \leq 1\} \,.$$

$\mathcal{B}_\phi$ and $\mathcal{B}^o_\phi$ are sometimes called the Wulff shape and the Frank diagram, respectively. We say that ϕ is smooth if $\mathcal{B}_\phi$ and $\mathcal{B}^o_\phi$ are of class C^2 and strictly convex.

We say that ϕ is crystalline if $\mathcal{B}_\phi$ (and therefore $\mathcal{B}_\phi^o$) is a polytope. In general, we say that ϕ is nonsmooth if $\mathcal{B}_\phi$ is not at the same time smooth and strictly convex. By a facet of $\partial\mathcal{B}_\phi$ we mean a facet of $\partial\mathcal{B}_\phi$ of dimension $(n-1)$.

$\mathbb{R}^n$ endowed with the norm ϕ becomes a finite dimensional Banach space (sometimes called Minkowski space); our approach consists to properly introduce a notion of ϕ-surface measure, of ϕ-regular boundary (this is necessary only in the nonsmooth case), of ϕ-normal vector and of ϕ-mean curvature, in order to define (intrinsically) and to study the associated geometric evolution problems.

Duality mappings. Let $\phi \in \mathcal{M}(\mathbb{R}^n)$. By T and T^o we denote the possibly multivalued duality mappings defined by

$$T := \frac{1}{2}D^-(\phi^2), \qquad T^o := \frac{1}{2}D^-((\phi^o)^2),$$

where we drop the explicit dependence on the anisotropy in the notation, and D^- denotes the subdifferential. T, T^o are maximal monotone operators, and T (resp. T^o) takes $\partial\mathcal{B}_\phi$ (resp. $\partial\mathcal{B}_\phi^o$) onto $\partial\mathcal{B}_\phi^o$ (resp. onto $\partial\mathcal{B}_\phi$). The geometric properties of the maps $T_{|\partial\mathcal{B}_\phi}$, $T^o_{|\partial\mathcal{B}_\phi^o}$ are of basic importance for describing the geometry of ϕ-regular boundaries in the nonsmooth case: if $\xi \in \partial\mathcal{B}_\phi$, $T(\xi)$ is the intersection of the closed outward normal cone to $\partial\mathcal{B}_\phi$ with $\partial\mathcal{B}_\phi^o$.

ϕ-distance function. Given a nonempty set $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$\mathrm{dist}_\phi(x,E) := \inf_{y\in E} \phi(x-y), \qquad \mathrm{dist}_\phi(E,x) := \inf_{y\in E} \phi(y-x),$$

$$d_\phi^E(x) := \mathrm{dist}_\phi(x,E) - \mathrm{dist}_\phi(\mathbb{R}^n \setminus E, x).$$

The function d_ϕ^E is therefore the oriented ϕ-distance function negative inside E; since in general $\mathcal{B}_\phi$ is not symmetric with respect to the origin, $-d_\phi^E$ does not necessarily coincide with $d_\phi^{\mathbb{R}^n\setminus E}$. At each point x where d_ϕ^E is differentiable, there holds $\nabla d_\phi^E(x) \in \partial\mathcal{B}_\phi^o$ (eikonal equation at x).

The surface energy. The surface energy functional P_ϕ is defined as follows: if $E \subset \mathbb{R}^n$ is a finite perimeter set, then

$$P_\phi(E) := \int_{\partial E} \phi^o(\nu^E)\, d\mathcal{H}^{n-1}, \tag{1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hasudorff measure and ν^E is the outward unit normal to the (reduced) boundary of E.

P_ϕ coincides with the Minkowski content of ∂E (at least if ∂E is sufficiently smooth) defined by means of the distance dist_ϕ and does not necessarily coincide with the $(n-1)$-dimensional Hausdorff measure with respect to dist_ϕ [12]. The functional P_ϕ can be written as $\sup\{\int_E \mathrm{div}\sigma\, dx \colon \sigma \in C_0^1(\mathbb{R}^n; \mathcal{B}_\phi)\}$ [3], and coincides with the $\Gamma - L^2(\mathbb{R}^n)$ limit, as $\epsilon \to 0^+$, of the sequence of functionals

$$M_\epsilon(u) := \frac{1}{2c}\int_{\mathbb{R}^n} \left[\epsilon(\phi^o(\nabla u))^2 + \frac{1}{\epsilon}W(u)\right]\, dx, \qquad u \in W^{1,2}(\mathbb{R}^n), \tag{2}$$

where $W(s) := (1-s^2)^2$, $c := \int_{-1}^{1} \sqrt{W(s)}\, ds$, and $M_\epsilon := +\infty$ in $L^2(\mathbb{R}^n)\setminus W^{1,2}(\mathbb{R}^n)$. [12].

In the sequel, the function $g\colon \mathbb{R}^n \times [0,+\infty[\to \mathbb{R}$ denotes a fixed bounded function, which plays (depending of the context) either the rôle of prescribed mean curvature or of the forcing term (external field).

3. Definitions and Results in the Smooth Case

In this section $\phi \in \mathcal{M}(\mathbb{R}^n)$ is smooth.

ϕ-normal vectors, ϕ-mean curvature: smooth case. Let E be a bounded open set of class C^∞. We set $\nu_\phi^E := \nabla d_\phi^E$ on ∂E (which has unit ϕ^o-norm) and we define [12] the vector field $n_\phi^E\colon \partial E \to \partial \mathcal{B}_\phi$ as

$$n_\phi^E := T^o(\nu_\phi^E) \qquad \text{on } \partial E\,. \tag{3}$$

n_ϕ^E is the natural ϕ-normal vector field to ∂E, and $n_\phi^E \cdot \nu_\phi^E = 1$. It is sometimes called the Cahn-Hoffman vector field; it can be defined in a suitable neighbourhood of ∂E, keeping the fundamental constraint $\phi(n_\phi^E) = 1$, as $n_\phi^E := T^o(\nabla d_\phi^E)$. The ϕ-mean curvature κ_ϕ^E of ∂E is then defined [12, 11] as

$$\kappa_\phi^E := \operatorname{div}\, n_\phi^E \qquad \text{on } \partial E\,. \tag{4}$$

It turns out that κ_ϕ^E coincides with the tangential divergence of n_ϕ^E and that $\partial \mathcal{B}_\phi$ has ϕ-mean curvature equal to $n-1$.

First variation of P_ϕ: smooth case. Let $E \subset \mathbb{R}^n$ be a bounded open set of class C^∞. Let $\Psi \in C^\infty(\mathbb{R}^{n+1};\mathbb{R}^n)$ and define $\Psi_\lambda(x) := \Psi(x,\lambda)$ for any $(x,\lambda) \in \mathbb{R}^{n+1}$. Assume that $\Psi_0 = \mathrm{Id}$ and that $\Psi_\lambda - \mathrm{Id}$ has compact support, and let $X := \frac{\partial \Psi_\lambda}{\partial \lambda}_{|\lambda=0}$. Then one can prove [11] that

$$\frac{d}{d\lambda} P_\phi\left(\Psi_\lambda(E)\right)_{|\lambda=0} = \int_{\partial E} \kappa_\phi^E\ \nu_\phi^E \cdot X\ d\mathcal{P}_\phi\,, \tag{5}$$

where, here and in the following, $d\mathcal{P}_\phi := \phi^o(\nu^E) d\mathcal{H}^{n-1}$. Therefore, if $\mathcal{F}$ is the functional defined by

$$\mathcal{F}(E) := P_\phi(E) + \int_E g\ dx\,, \tag{6}$$

we have

$$\frac{d}{d\lambda} \mathcal{F}\left(\Psi_\lambda(E)\right)_{|\lambda=0} = \int_{\partial E} \left(\kappa_\phi^E - g\right) \nu_\phi^E \cdot X\ d\mathcal{P}_\phi\,. \tag{7}$$

Denoting by dP_ϕ the variation of P_ϕ as an element of the normed space $L^2_\phi(\partial E;\mathbb{R}^n)$, and denoting by $\langle \cdot,\cdot \rangle$ the duality, one can also prove that a scalar multiple of the vector field $\kappa_\phi^E n_\phi^E$ is a solution of the problem

$$\min \left\{ \langle dP_\phi, X\rangle : X \in L^2_\phi(\partial E;\mathbb{R}^n),\ \|X\|^2_{L^2_\phi(\partial E;\mathbb{R}^n)} := \int_{\partial E} \phi(X)^2 d\mathcal{P}_\phi \le 1 \right\}.$$

For the computation of the second variation of P_ϕ see [5].

Geometric evolution law: smooth case. Assume that g is smooth. Let $E(t)$ be a family of smooth bounded open sets, varying smoothly with $t \in [0,T]$. Set $d_\phi(x,t) := d_\phi^{E(t)}(x)$. We say that $t \in [0,T] \to E(t)$ is a ϕ-smooth flow on $[0,T]$ with forcing term g (and initial set $E(0)$) if

$$\frac{\partial d_\phi}{\partial t}(x,t) = \kappa_\phi^{E(t)}(x) + g(x,t), \qquad x \in \partial E(t), \quad t \in [0,T]\,. \tag{8}$$

Remark 3.1.

(i) *Under the evolution law (8), $\mathcal{B}_\phi$ shrinks self-similarly.*

(ii) *No mobility factor is present in (8).*

(iii) *If $n = 2$, setting $\phi^o(\xi) = |\xi|\phi^o(\frac{\xi}{|\xi|}) =: \rho\psi(\theta)$ (polar coordinates), it turns out that the velocity of the front along $n_\phi^{E(t)}$ is equal to $\kappa^{E(t)}(\psi + \psi'')$ (where $\kappa^{E(t)}$ is the euclidean curvature of $\partial E(t)$), while along the euclidean direction is equal to $\kappa^{E(t)}\psi(\psi + \psi'')$, see* [11].

(iv) *(8) admits local existence and uniqueness of a smooth solution, and the comparison principle holds. Different type of weak solutions (giving sense to the evolution after the onset of singularities) can be defined, see for instance* [2, 14].

(v) *Smooth evolutions under (8) can be approximated, as $\epsilon \to 0^+$, with the solutions of the reaction-diffusion type equation*

$$u_t = \mathrm{div}(T^o(\nabla u)) - \frac{1}{2\epsilon^2}W'(u) - \frac{1}{2c\epsilon}g\,, \tag{9}$$

obtained as the gradient flow of the functionals M_ϵ in (2), after a time rescaling, see [11, 4] *for more details.*

We conclude this section with some comments. The definitions of P_ϕ, n_ϕ^E and κ_ϕ^E, as well as the convergence result of solutions of (9), can be extended for smooth inhomogeneous anisotropies $\phi(x,\xi)$. The results do not cover the case when $\phi(x,\xi)$ vanishes (or becomes infinite) at some point x, or when $\phi(x,\xi)$ depends in a discontinuous way on x. Finally, the extension of our approach to the evolution of manifolds with arbitrary codimension is still an open problem.

4. Extension to the Nonsmooth Case

In this section $\phi \in \mathcal{M}(\mathbb{R}^n)$ is nonsmooth. Unless otherwise stated, we assume also $g = 0$ for simplicity. Our aim is to extend the approach of Section 3. The first step is to define the vector field n_ϕ^E. It is clear that the definition in (3) must be modified, since $T^o(\nu_\phi^E)$ is now a (convex) set, whose dimension may vary from point to point on ∂E. Moreover (as it clearly happens in the crystalline case) we must be able to let evolve nonsmooth (for instance polyhedral) sets, which do not admit a normal vector field everywhere defined. These (and others) considerations lead to redefine the concept of smooth boundary [6, 7].

Definition 4.1. *Let $\phi \in \mathcal{M}(\mathbb{R}^n)$ be nonsmooth. Let $E \subset \mathbb{R}^n$ be a bounded open set. We say that E is Lipschitz ϕ-regular, and we write $E \in \mathcal{R}_\phi(\mathbb{R}^n)$, if ∂E is Lipschitz continuous and there exists a vector field $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$, where*

$$\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n) := \{v \in \mathrm{Lip}(\partial E;\mathbb{R}^n) : v(x) \in T^o(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1} - \text{a.e. } x \in \partial E\}\,.$$

Remark 4.2.

(i) *Lipschitz ϕ-regular sets are the analog, in the euclidean case, of the sets of class $C^{1,1}$.*

(ii) *It has been shown in* [8] *that, if $\phi \in \mathcal{M}(\mathbb{R}^3)$ is crystalline, there is a Lipschitz ϕ-regular set E, which is polyhedral, convex and very close to the Wulff shape $\mathcal{B}_\phi$, whose evolution, under crystalline mean curvature flow, does not remain a polyhedral set (i.e. a facet of ∂E bends instantly). See also* [19] *for numerical simulations. This is why, even for a crystalline ϕ, we impose ∂E to be Lipschitz: we cannot restrict definition 4.1 to polyhedral sets ∂E. It must be said that, if $n = 2$, polygonal curves remain polygonal during the crystalline flow.*

(iii) *The lipschitzianity of η is a regularity requirement making more difficult the proof of a short time existence result of a ϕ-mean curvature flow in the class $\mathcal{R}_\phi(\mathbb{R}^n)$ (which is an open problem). In this respect, one could relax the regularity of η, for instance by requiring η to be only bounded with divergence in $L^2(\partial E)$ or, alternatively, with divergence in $L^\infty(\partial E)$, see* [8]. *A fourth definition can be given by imposing that η admits an extension in a suitable neighbourhood of ∂E which is bounded with bounded divergence* [7, 9]. *The three last definitions are expected to coincide for a rather large class of ϕ and E, and hopely to coincide with definition 4.1 for some choice of ϕ and E.*

(iv) *$\mathcal{B}_\phi$ belongs to $\mathcal{R}_\phi(\mathbb{R}^n)$: take $\eta(x) := x$.*

(v) *Let $n = 2$, $\phi(\xi) := \max\{|\xi_1|, |\xi_2|\}$ and $B := \{\xi \in \mathbb{R}^2 : |\xi| < 1\}$ be the euclidean ball. Then there exists no $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial B;\mathbb{R}^2)$ (and no η in one of the other three classes introduced in (iii)). Therefore B is not Lipschitz ϕ-regular. The regularity of B, in our relative approach, is analogous to the regularity of the square in the euclidean geometry.*

(vi) *The structure and classification of Lipschitz ϕ-regular sets in $n = 2$ dimensions is essentially known: for instance, in the crystalline case, roughly speaking the boundary of E is a closed Lipschitz curve which is a sequence (with a precise order) of segments which are parallel to some edge of $\partial\mathcal{B}_\phi$ and of segments or arcs which correspond to vertices of $\partial\mathcal{B}_\phi$* [23, 18]. *In $n = 2$ dimensions, there is a natural choice of a special vector field, which is the one which makes the edges and the arcs of ∂E of constant ϕ-curvature.*

(vii) *The structure and classification of Lipschitz ϕ-regular sets in $n = 3$ dimensions, even for special ϕ's, is an open problem. Some of their basic properties are studied in* [9]. *If ϕ is crystalline and E is a polyhedron with the property that at any of its vertices v, the intersection of*

$T^o(\nu_\phi^E(x))$ ($x \in \mathrm{int}(Q)$) over all facets Q containing v is non empty, then $E \in \mathcal{R}_\phi(\mathbb{R}^3)$.

First variation of P_ϕ: nonsmooth case. Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. In [9] it is rigorously computed the first variation of the functional P_ϕ at E. Roughly speaking, it turns out that the expression of (minus) the norm of the gradient of P_ϕ at E is given by

$$-\inf_{N \in Y} \left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2 \, d\mathcal{P}_\phi \right)^{1/2} . \tag{10}$$

The operator $\mathrm{div}_{\phi,\tau}$ denotes the tangential divergence with respect to ϕ, here it suffices to say that it is the natural definition of tangential divergence in relative geometry, and coincides, on flat regions of ∂E, with the usual tangential divergence. The space Y is the class of all vector fields in $L^2(\partial E; \mathbb{R}^n)$ satisfying the constraint to belong to $T^o(\nu_\phi^E)$ and having ϕ-tangential divergence in $L^2(\partial E)$. It can be proved that the minimum problem (10) has a solution, which we denote by $N_{\min}^E$, which is *unique in the divergence*. It can then be proved that the direction of minimal slope for the functional P_ϕ at E is given by $N_{\min}^E$: this is one of the motivations for studying problem (10) in connection with the geometric evolution problem. It is clear that $\mathrm{div}_{\phi,\tau} N_{\min}^E$ is expected to identify the initial velocity of the front.

Definition 4.3. *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. We define* [9] *the ϕ-mean curvature κ_ϕ^E of E as $\kappa_\phi^E := \mathrm{div}_{\phi,\tau} N_{\min}^E \in L^2(\partial E)$.*

It turns out that the ϕ-mean curvature of $\mathcal{B}_\phi$ is constantly equal to $n-1$.

Let $E \in \mathcal{R}_\phi(\mathbb{R}^2)$. If C is an edge of ∂E and W_C is the corresponding edge in the Wulff shape, then κ_ϕ^E is constant on C and is equal to $\delta_C \frac{|W_C|}{|C|}$, where $\delta_C \in \{0, \pm 1\}$ is a convexity factor. $N_{\min}^E$ is, on C, the linear combination of the vector η at the vertices of C (all $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^2)$ coincide on the vertices of ∂E).

One of the main results is the following global regularity result on the minimizers of (10), see [9, 24].

Theorem 4.4. *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. Then $\kappa_\phi^E \in L^\infty(\partial E)$. Moreover, κ_ϕ^E has bounded variation on all facets of ∂E corresponding to facets of $\partial \mathcal{B}_\phi$.*

Open problem. Is there a solution $\overline{N}$ of problem (10) with $\overline{N} \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$?

Theorem 4.4 makes possible to speak of the jump set of κ_ϕ^E on the facets of ∂E corresponding to facets of $\partial \mathcal{B}_\phi$. If $F \subset \partial E$ is such a facet, it is of particular interest to find necessary and sufficient conditions on E and F ensuring that the jump set of κ_ϕ^E on F is empty: that is, to prove that $\mathrm{div}_{\phi,\tau} N_{\min}^E$ is continuous on F. For small times in the evolution problem, F is expected to translate parallely to itself, possibly changing its shape, if κ_ϕ^E is constant on F (in this case we say that F is ϕ-calibrable), or to bend if κ_ϕ^E is continuous but not constant on F. In the first example of [8] it is shown a crystalline mean curvature flow of a

Lipschitz ϕ-regular set having a facet F which instantly subdivides (κ_ϕ^E is piecewise constant on F). In the second example of [8] the bending phenomenon of a facet of a Lipschitz ϕ-regular polyhedral set is described. Both these two evolutions are ϕ-regular evolutions, i.e. they are not examples of singularities of the flow.

In general, it is interesting to find the structure and the properties of the jump set of κ_ϕ^E on F, see [9] for some results in this direction. A characterization of ϕ-calibrability for convex facets F of convex sets E will appear in [10].

Geometric evolution problem: nonsmooth case. We now define the notion of ϕ-regular flow: to avoid technical difficulties, we skip some details. Let $E(t)$, for $t \in [0,T]$, be a family of bounded open sets. We say that $t \in [0,T] \to E(t)$ is a ϕ-regular flow on $[0,T]$ if $E(t)$ is Lipschitz ϕ-regular and there exists a family $t \in [0,T] \to N(\cdot,t)$ of vector fields $N(\cdot,t)\colon \partial E(t) \to \mathbb{R}^n$, such that $N(x,t) \in T^o(\nu_\phi^{E(t)}(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E(t)$, $\mathrm{div}_{\phi,\tau} N(\cdot,t) \in L^2(\partial E(t))$ and such that, setting $d_\phi(x,t) := d_\phi^{E(t)}(x)$, there holds

$$\frac{\partial d_\phi}{\partial t}(x,t) = \mathrm{div}_{\phi,\tau} N(x,t)\,, \qquad \mathcal{H}^{n-1} - \text{a.e. } x \in \partial E(t)\,, \quad \text{a.e. } t \in [0,T]\,. \tag{11}$$

By analogy with semigroup theory (see [13, Theorem 3.1]), an open problem is to prove that

$$\frac{\partial d_\phi}{\partial t}(x,t) = \mathrm{div}_{\phi,\tau} N_{\min}^{E(t)}(x)\,, \qquad \mathcal{H}^{n-1} - \text{a.e. } x \in \partial E(t)\,, \quad t \in [0,T] \tag{12}$$

(possibly $\frac{\partial d_\phi}{\partial t}$ being the right derivative), where $N_{\min}^{E(t)}$ solves (10) with $E(t)$ in place of E.

Remark 4.5.

(i) *In* [6] *it is proved that, in two dimensions, the solutions of the reaction-diffusion type inclusion*

$$u_t - \mathrm{div}(T^o(\nabla u)) + \frac{1}{2\epsilon^2} W'(u) \ni 0 \tag{13}$$

approximate, as $\epsilon \to 0^+$, the ϕ-curvature flow with a quasi-optimal error estimate of order $\epsilon^2|\log\epsilon|^2$. A similar result holds in any space dimensions, with a sub-optimal error estimate of order $\epsilon|\log\epsilon|^2$ [7].

(ii) *In* [7] *it is proved a comparison principle between ϕ-regular flows (see* [18] *for a comparison result in two dimensions), where the notion of ϕ-regularity used is the fourth one in (iii) of remark 4.2: this result implies that, if a ϕ-regular flow exists, then it is* unique *(in that class). This result is also sufficient to ensure that the two examples constructed in* [8] *are the unique crystalline evolutions starting from their initial data.*

(iv) *Another notion of flow can be given by imposing that each $E(t)$ is Lipschitz ϕ-regular, and $\partial E(t)$ admits a vector field $\eta(\cdot,t) \in \mathrm{Lip}_{\nu,\phi}(\partial E(t);\mathbb{R}^n)$ such that (11) holds with η in place of N (Lipschitz ϕ-regular flow). Finding conditions ensuring that a ϕ-regular flow is also a Lipschitz ϕ-regular flow is an open problem, strictly related to the problem addressed after* *theorem 4.4. We do not know whether the first evolutionary example constructed in* [8] *(subdivision of a facet at the initial time) is a Lipschitz ϕ-regular flow.*

(v) *The extension of the above results in presence of a forcing term g depending on x is a largely open problem.*

References

[1] F. J. Almgren and J. E. Taylor, *Flat flow is motion by crystalline curvature for curves with crystalline energies,* J. Diff. Geom. **42** (1995), 1–22.

[2] F. J. Almgren, J. E. Taylor and L. Wang, *Curvature-driven flows: a variational approach,* SIAM J. Control Optim., **31** (1993), 387–437.

[3] M. Amar and G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **11** (1993), 91–133.

[4] G. Bellettini, P. Colli Franzone and M. Paolini, *Convergence of front propagation for anisotropic bistable reaction-diffusion equations,* Asymp. Anal., **15** (1997), 325–358.

[5] G. Bellettini and I. Fragalà, *Elliptic approximations to prescribed mean curvature surfaces in Finsler geometry,* Asymp. Anal., **22** (2000), 87–111.

[6] G. Bellettini, R. Goglione and M. Novaga, *Approximation to driven motion by crystalline curvature in two dimensions,* Adv. Math. Sci. and Appl., to appear.

[7] G. Bellettini and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in R^N,* Math. Mod. Meth. Appl. Sc., to appear.

[8] G. Bellettini, M. Novaga and M. Paolini *Facet-breaking for three-dimensional crystals evolving by mean curvature,* Interfaces and Free Boundaries **1** (1999), 39–55.

[9] G. Bellettini, M. Novaga and M. Paolini, *On a crystalline variational problem,* Preprint Univ. Pisa, December (1999), submitted.

[10] G. Bellettini, M. Novaga and M. Paolini, *Characterization of facets breaking for nonsmooth mean curvature flow in 3D in the convex case,* in preparation.

[11] G. Bellettini and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry,* Hokkaido Mathematical J., **89** (1996), 537–566.

[12] G. Bellettini, M. Paolini and S. Venturini, *Some results on surface measures in the calculus of variations,* Ann. Mat. Pura Appl., **CLXX** (1996), 329–359.

[13] H. Brezis, Operateurs Maximaux Monotones et Semi-groupes de Contractions Dans les Espaces de Hilbert, North-Holland Mathematics Studies, 1973.

[14] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Diff. Geom., **3** (1991), 749–786.

[15] M. E. Gage, *Evolving plane curves by curvature in relative geometries. II.,* Duke Math. J., **75** (1994), 79–98.

[16] M.-H. Giga and Y. Giga, *Evolving graphs by singular weighted curvature,* Arch. Rational Mech. Anal., **141** (1998), 117–198.

[17] M.-H. Giga and Y. Giga, *Crystalline and level set flow-convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane,* Preprint Univ. of Hokkaido, February (2000).

[18] Y. Giga and M. E. Gurtin, *A comparison theorem for crystalline evolutions in the plane,* Quarterly of Applied Mathematics, **IV** (1996) 727–737.

[19] M. Paolini and F. Pasquarelli, *Numerical simulations of crystalline curvature flow in 3D by interface diffusion,* in: Free Boundary Problems: theory and applications II, GAKUTO Intern. Ser. Math. Sci. Appl. **14** (N. Kenmochi ed.), Gakkötosho (2000), 376–389, to appear.

[20] J. E. Taylor, *Crystalline variational problems,* Bull. Amer. Math. Soc. (N.S.), **84** (1978), 568–588.

[21] J. E. Taylor, *Complete catalog of minimizing embedded crystalline cones,* Proc. Symposia Math., **44** (1986), 379–403.

[22] J. E. Taylor, *II-Mean curvature and weighted mean curvature,* Acta Metall. Mater., **40** (1992), 1475–1485.

[23] J. E. Taylor, *Motion of curves by crystalline curvature, including triple junctions and boundary points,* Proc. of Symposia in Pure Math., **54** (1993) 417–438.

[24] J. Yunger, *Facet stepping and motion by crystalline curvature,* Ph.D. Thesis, Rutgers University (1998).

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