From Symplectic Packing to Algebraic Geometry and Back

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Abstract. In this paper we survey various aspects of the symplectic packing problem and its relations to algebraic geometry, going through results of Gromov, McDuff, Polterovich and the author.

1. Introduction

1.1. Symplectic packing

Let (M^{2n}, Ω) be a $2n$-dimensional symplectic manifold with finite volume. Fix an integer $N \geq 1$ and consider *symplectic packing of* (M, Ω) *by* N *equal balls* of radius λ, that is symplectic embeddings

$$\varphi\colon \underbrace{B(\lambda) \coprod \cdots \coprod B(\lambda)}_{N \text{ times}} \to (M, \Omega) \tag{1}$$

of N disjoint copies of $B(\lambda)$ —the closed $2n$-dimensional ($2n = \dim M$) Euclidean ball of radius λ, endowed with the standard symplectic structure of $\mathbb{R}^{2n}$, $\omega_{\mathrm{std}} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$.

While Darboux theorem assures that such packings always exist for small enough λ's, when trying to increase the radii one runs into an obvious *volume obstruction*: φ being a symplectic embedding must also be volume preserving. We thus have:

$$N \operatorname{Vol} B(\lambda) = \operatorname{Vol}(\operatorname{Image}(\varphi)) \leq \operatorname{Vol}(M, \Omega), \quad \text{or equivalently} \quad \lambda^{2n} \leq \frac{\int_M \Omega^n}{\pi^n N}. \tag{2}$$

The symplectic packing problem is the following question: *Does the symplectic structure impose other obstructions on symplectic packings, beyond inequality (2)?*

1.2. Singularities of plane algebraic curves

Let $p_1, \dots, p_N \in \mathbb{C}P^2$ be N general points in the complex projective plane. Consider *irreducible* algebraic curves $C \subset \mathbb{C}P^2$ which pass through $p_1, \dots, p_N$ with given multiplicities $m_1, \dots, m_N$. The following is a classical question in singularity theory: *What is the minimal possible degree of a curve C with the above* *prescribed singularities?* An old (and still open) conjecture of Nagata asserts that when $N \geq 9$ the degree of C must satisfy

$$\deg(C) \geq \frac{m_1 + \cdots + m_N}{\sqrt{N}} .$$

Surprisingly, this purely algebro-geometric problem is intimately related to the symplectic packing problem.

In this paper we shall survey the symplectic packing problem and its mutual relations with algebraic geometry and the conjecture of Nagata.

2. Packing Obstructions and Algebraic Geometry

First results in this direction were established by Gromov in 1985. Using his theory of pseudo-holomorphic curves he proved the following:

Theorem 2.1. (Gromov [14]**)** *If $B(\lambda_1) \coprod B(\lambda_2)$ embeds symplectically into $B^{2n}(R)$ then $\lambda_1^2 + \lambda_2^2 < R^2$.*

Notice that for $\lambda_1 = \lambda_2 = \lambda$ this inequality becomes $\lambda^2 < \frac{1}{2}R^2$, or when written differently:

$$2 \operatorname{Vol} B(\lambda) < \frac{1}{2^{n-1}} \operatorname{Vol} B^{2n}(R) .$$

In other words, the maximal portion of the volume of $B^{2n}(R)$ that can be filled via 2 disjoint symplectic balls of equal radius is smaller than $\frac{1}{2^{n-1}}$. This is a purely symplectic phenomenon! Indeed it is not hard to prove that if one replaces the condition "symplectic" on the embedding φ by "volume preserving" then the problem becomes trivial in the sense that no obstructions beyond inequality (2) exist, namely an arbitrarily large portion of the volume of (M, Ω) can be filled by N disjoint copies of equal balls embedded via a volume preserving map. This holds for every manifold M and any $N \geq 1$.

Trying to formalize the study of the preceding phenomenon one introduces the following quantity:

$$v_N(M, \Omega) = \sup_{\lambda} \frac{N \operatorname{Vol} B(\lambda)}{\operatorname{Vol}(M, \Omega)} ,$$

where λ passes over all possible radii for which there exists a symplectic embedding φ as in (1) above. The number $v_N(M, \Omega)$ measures the "maximal" portion of the volume of M that can be symplectically packed by N equal balls. When $v_N(M, \Omega) = 1$ we say that (M, Ω) admits a *full symplectic packing* by N equal balls, while the case $v_N(M, \Omega)$ is referred to as a *packing obstruction*.

Generalizing Gromov's work, McDuff and Polterovich discovered the following interesting result:

Theorem 2.2. (McDuff-Polterovich [26]**)** *For each of the manifolds $B^4(1)$ and $\mathbb{C}P^2$ (both endowed with their standard symplectic structures) we have:*

N	1	2	3	4	5	6	7	8	9
v_N	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

Moreover, $v_N = 1$ *when* $N = k^2$.

2.1. Relations to algebraic geometry

McDuff and Polterovich discovered that the symplectic packing problem is intimately related to algebraic geometry. The first ingredient in building up this connection is the symplectic blowing-up construction. We shall only give a general overview of it here referring the reader to [26, 27] for more details.

The symplectic blowing-up construction is a symplectic version of the classical blowing-up procedure in algebraic geometry. It was discovered by Gromov [15] and further developed by Guillemin and Sternberg [16]. Topologically symplectic blowing-up is the same as in the algebraic framework, but it also defines a symplectic form on the blown-up manifold. The deep connection between this operation and symplectic packing was first established by McDuff in [23].

Topologically, blowing-up amounts to removing a point p from M and replacing it by the (complex) projectivization of the tangent space of M at p. The same construction also allows us to blow-up M at N distinct points $p_1, \ldots, p_N \in M$.

It turns out that in order to define a symplectic form on the blow-up of M at $p_1, \ldots, p_N$ one has to specify a symplectic embedding

$$\varphi\colon B(\lambda_1) \coprod \cdots \coprod B(\lambda_N) \to (M, \Omega)$$

which sends the centre of the i'th ball to p_i for every i. Roughly speaking, once such an embedding is given, one cuts out from M the images of the embedded balls and collapses their boundaries to copies of $\mathbb{C}P^{n-1}$ (called exceptional divisors) using the Hopf map. If we denote by $\Theta\colon \widetilde{M} \to M$ the blow-up of M at $p_1, \ldots, p_N$, then the symplectic blowing-up construction defines a symplectic form $\widetilde{\Omega}$ on $\widetilde{M}$ such that $\widetilde{\Omega}|_{T(\Sigma_i)} = \lambda_i^2 \sigma_{\mathrm{std}}$ and $\widetilde{\Omega} = \Theta^*\Omega$ outside the images of the embedded balls. Here we have denoted by Σ_i the exceptional divisor corresponding to p_i, $\Sigma_i = \Theta^{-1}(p_i) \approx \mathbb{C}P^{n-1}$, and by σ_{std} the standard symplectic structures of $\mathbb{C}P^{n-1}$ normalized so that the area of a projective line $\mathbb{C}P^1 \subset \mathbb{C}P^{n-1}$ is π. Writing $E_1, \ldots, E_N$ for the homology classes of the exceptional divisors in $H_{2n-2}(\widetilde{M})$ and $e_1, \ldots, e_N \in H^2(\widetilde{M})$ for their Poincaré duals we thus have:

$$[\widetilde{\Omega}] = \Theta^*[\Omega] - \pi\lambda_1^2 e_1 \ldots - \pi\lambda_1^2 e_N\,. \tag{3}$$

Conversely, given a symplectic form $\widetilde{\Omega}$ on the blow-up $\widetilde{M}$ of a manifold one can, by *blowing down*, obtain a symplectic form Ω on M, and a symplectic packing of (M, Ω) by balls. The radii of these balls are determined by $\widetilde{\Omega}|_{\Sigma_i}$ as can be easily seen from (3) above.

In the language of symplectic blow-ups, the problem of existence of symplectic packings with balls of given radii, is equivalent to determining how much it is possible to blow-up symplectically the manifold. In view of the blowing down construction this amounts to determination of those cohomology classes on the blow-up of the manifold which can be represented by symplectic forms. For example, in order to show that $\mathbb{C}P^2$ admits a symplectic packing by N equal balls of radius λ one has to show that the cohomology class

$$a = \pi l - \pi\lambda^2(e_1 + \cdots + e_N) \tag{4}$$

on the blow-up of $\mathbb{C}P^2$ has a symplectic representative. Here $l \in H^2(\mathbb{C}P^2, \mathbb{Z})$ stands for the (positive) generator.

Although there are no general tools to attack this type of problem on a general symplectic manifold, some information can be still obtained using tools from classical algebraic geometry. The idea is to produce a *Kähler* form (rather than just symplectic) in the given cohomology class. The main tool for this purpose is the following criterion which is due independently to Nakai and to Moishezon (see [17]). Henceforth we shall denote by $H^{(1,1)}(S, \mathbb{R}) = H^{(1,1)}(S) \cap H^2(S, \mathbb{R})$ the space of cohomology classes which can be represented by *real* $(1,1)$-forms on a given smooth algebraic variety S.

Theorem 2.3. (Nakai-Moishezon criterion) *Let S be a smooth algebraic surface and $a \in H^{(1,1)}(S, \mathbb{R})$. The class a can be represented by a Kähler form if and only if the following two conditions are satisfied:*

1. $\int_S a \cup a > 0$.
2. $\int_C a > 0$ *for every algebraic curve* $C \subset S$.

Note that in order to apply this criterion one has to have some information on *all* the possible homology classes $[C] \in H_2(S)$ which can be represented by algebraic curves $C \subset S$. In general this might be hard, however when S has a positive anti-canonical class a great deal of information is available on the relevant classes $[C] \in H_2(S)$. Such surfaces S are called *Fano* (or *del-Pezzo*) and it is not hard to check that blow-ups of $\mathbb{C}P^2$ at no more than 8 points belong to this category. It turns out that for Fano surfaces one can replace the second condition in the Nakai-Moishezon criterion by the following much weaker one: "$\int_C a > 0$ *for every exceptional rational curve* $C \subset S$". By an *exceptional rational curve* we mean a rational algebraic curve with self intersection -1. Furthermore, it is a classical fact that Fano surfaces have only finitely many such curves and a complete list of their homology classes is available (see [8]). Therefore in this case the Nakai-Moishezon criterion boils down to finitely many explicit inequalities. All this leads to a precise computation of the maximal λ for which the class a in (4) is Kähler. In particular this gives a lower bound on the supremum of all λ for which the class in (4) is symplectic, hence a lower bound on $v_N(\mathbb{C}P^2)$ when $N \leq 8$.

To obtain an upper bound on λ (that is, a packing obstruction) it is not possible to use algebraic geometry any longer since a priori it is not clear that the symplectic forms obtained by the symplectic blowing-up operation are always Kähler. Nevertheless, using the theory of pseudo-holomorphic curves McDuff and Polterovich managed to obtain limitations on the possible radii λ. Roughly speaking, the idea is that the symplectic forms obtained from blowing-up can be deformed to Kähler forms via symplectic deformations. The point is that certain type of algebraic curves (such as exceptional rational curves) persist under deformations of the symplectic structure and so one can still find symplectic surfaces in the same homology classes as the rational exceptional curves for every symplectic blow-up of $\mathbb{C}P^2$. The blown-up form $\widetilde{\Omega}$ must have positive area on such surfaces and so one gets from each such symplectic surface an inequality which bounds λ from above. In other words, the same set of inequalities on λ as in the algebraic case applies here also to the (more general) symplectic case. As a result, the upper and lower bounds mentioned above coincide. This leads to the *precise* values of v_N which appear in theorem 2.2.

The case $N > 9$ could not be attacked by algebraic geometry since the blow-up of $\mathbb{C}P^2$ at $N > 9$ is not Fano anymore and much less is known on the possible homology classes of irreducible algebraic curves on such surfaces.

Notice that in order to prove that $\mathbb{C}P^2$ admits a full symplectic packing by N equal balls one has to show that the cohomology class

$$a_\epsilon = l - (\frac{1}{\sqrt{N}} - \epsilon)(e_1 + \cdots + e_N) \tag{5}$$

has a symplectic representative for every $\epsilon > 0$. McDuff and Polterovich discovered that surprisingly this would follow from the old conjecture of Nagata made in the early 1950's which (in an equivalent formulation) asserts that for every $N \geq 9$ the cohomology class a_ϵ has a Kähler representative for every $\epsilon > 0$.

We shall discuss Nagata's conjecture in some more detail in section 5 below. For the moment let us only mention that Nagata's conjecture holds true in the case $N = k^2$ due to a very simple argument, hence $v_N(\mathbb{C}P^2) = 1$ when $N = k^2$.

3. Existence of Full Packings: First Iteration

Although Nagata's conjecture is still not settled the following was proved in [2]:

Theorem 3.1. *For every $N \geq 9$ we have $v_N(B^4(1)) = v_N(\mathbb{C}P^2) = 1$. That is, both $B^4(1)$ and $\mathbb{C}P^2$ admit a full symplectic packing by N equal balls for every $N \geq 9$.*

Let us outline the main ideas leading to this theorem. As already mentioned, the existence of symplectic packing is equivalent to existence of symplectic forms representing certain cohomology classes on the blow-up of the manifold. We therefore need a version of the Nakai-Moishezon criterion which will be valid in the symplectic category. The main tool for establishing such a criterion (in dimension 4) is the theory of *Seiberg-Witten invariants* and their interpretation, due to Taubes, as *Gromov* invariants in the language of pseudo-holomorphic curves. The reader is referred to [31] and [24] for excellent presentations of this theory and its applications to symplectic geometry.

In order to state our criterion we need to introduce a class $\mathcal{C}$ of symplectic 4-manifolds for which it is valid. The precise definition of the class $\mathcal{C}$ involves knowledge of the the theory of Seiberg-Witten invariants which we shall not attempt to give here, but for the purpose of symplectic packing it is enough to mention that the class $\mathcal{C}$ contains the following types of symplectic 4-manifolds:

- Manifolds with $b_2^+ = 1$ and $b_1 = 0$.
- Ruled surfaces.
- If $(M, \Omega) \in \mathcal{C}$ then any (symplectic) blow-up of (M, Ω) also belongs to $\mathcal{C}$.

Next we need the notion of *exceptional classes*. In analogy to algebraic geometry we call a homology class $A \in H_2(M, \mathbb{Z})$ exceptional if it can be represented by a symplectic 2-sphere $\Sigma \subset (M, \Omega)$ with $\Sigma \cdot \Sigma = -1$. It is a standard fact (that can be proved using Gromov compactness theorem) that if Ω_0 and Ω_1 are symplectic forms that can be joint by a path of symplectic forms $\{\Omega_t\}$ then the set of exceptional classes for Ω_0 and and for Ω_1 coincide. Thus the set of Ω-exceptional classes $\mathcal{E}_\Omega \subset H_2(M, \mathbb{Z})$ depends only on the deformation class of Ω.

Our symplectic version of the Nakai-Moishezon criterion is (see [1]):

Theorem 3.2. *Let M be a closed 4-manifold and $\alpha \in H^2(M, \mathbb{R})$. Suppose that M admits a symplectic form Ω such that the following conditions are satisfied:*

1. *(M, Ω) is in the class $\mathcal{C}$.*
2. *$\int_M [\Omega] \cup \alpha > 0$ and $\int_M \alpha \cup \alpha > 0$.*
3. *$\alpha(E) \geq 0$ for every exceptional class $E \in \mathcal{E}_\Omega$.*

Then arbitrarily close to α in $H^2(M, \mathbb{R})$ there exist cohomology classes which can be represented by symplectic forms. Moreover, these symplectic forms may be assumed to be in the deformation class of Ω.

The proof of this theorem consists of two main ingredients. The first one is the *inflation* procedure which was introduced into symplectic geometry by Lalonde and McDuff (see [20, 21]) in the context of the problem of symplectic isotopies (see section 4.1 below). This procedure can be formulated as follows: *If $C \subset (M^4, \Omega)$ is a 2-dimensional symplectic submanifold with $C \cdot C \geq 0$ then there exists a closed 2-form ρ, supported arbitrarily close to C, whose cohomology class is Poincaré dual to $[C]$ and such that for every $t > 0$, $s \geq 0$ the form $\Omega_{t,s} = t\Omega + s\rho$ is symplectic.*

In our context this is an important tool for obtaining symplectic cohomology classes (i.e. classes which can be represented by symplectic forms). Indeed by taking $t > 0$ smaller and smaller the cohomology class $[\Omega_{t,s}]$ remains symplectic and becomes closer and closer to the class Poincaré dual to $s[C]$. Therefore, in order to prove that a given cohomology class, say a, can be approximated by symplectic cohomology classes it is enough to produce a symplectic submanifold $C \subset (M, \Omega)$ with non-negative self intersection whose homology class is Poincaré dual to a positive multiple of a. We would like to point out that a similar principle applies in the algebraic category as well (see section 5.1 below).

At present the problem of existence of symplectic hypersurfaces in given homology classes is in general out of reach, however in dimension 4 for manifolds which belong to the class $\mathcal{C}$ one can apply the machinery of Taubes-Seiberg-Witten theory of Gromov invariants to obtain the wanted submanifold C as a pseudo-holomorphic curve. Condition 2 of the theorem is exactly the technical assumption needed in order to force these invariants to be non-trivial on manifolds in the class $\mathcal{C}$. In general non-triviality of the Gromov invariants gives a reducible pseudo-holomorphic curve C which may have some components of negative self intersection. If such components appear the inflation cannot be performed. It is this point at which condition 3 of theorem 3.2 comes into the play. This assumption, it turns out, assures that C has only one irreducible component. We refer the reader to [24, 25] for more details on the structure of the components of curves arising from non-zero Seiberg-Witten Gromov invariants (see also [2]).

Returning to symplectic packings, theorem 3.2 implies that the "main" obstruction (beyond having positive volume) for a cohomology class α to carry a symplectic representative comes from the exceptional classes $E \in \mathcal{E}_\Omega$ (on which α must be positive). Applying this to the blow-up of $\mathbb{C}P^2$ at $N \geq 9$ points, an easy computation shows that when α is the class a_ϵ from (5) the restrictions coming from exceptional classes (i.e. $a_\epsilon(E) > 0$ for every $E \in \mathcal{E}_\Omega$) are *weaker* than the volume inequality $\int_M a_\epsilon \cup a_\epsilon > 0$. Consequently $v_N(\mathbb{C}P^2) = 1$ for every $N \geq 9$.

4. Other Directions

Apart from the problem of existence of packings there are other important related aspects on which extensive research has been carried out. We shall give here a brief overview of some of them without any attempt to be complete.

4.1. Isotopies of balls

Two symplectic packings

$$\varphi_0, \varphi_1 \colon B(\lambda_1) \coprod \cdots \coprod B(\lambda_N) \to (M, \Omega) \tag{6}$$

are called symplectically isotopic if there exists a smooth family $\{\varphi_t\}$ of symplectic packings

$$\varphi_t \colon B(\lambda_1) \coprod \cdots \coprod B(\lambda_N) \to (M, \Omega), \quad 0 \leq t \leq 1$$

which interpolates between φ_0 and φ_1.

The main question in this context is: *Given N and radii $\lambda_1, \ldots, \lambda_N$, are every two symplectic packings of (M, Ω) by balls of these radii symplectically isotopic?* Or put in a different way, *Is the space of symplectic packings of (M, Ω) by balls of these radii connected?*

This problem can be easily translated to an equivalent question on uniqueness of the symplectic blowing-up construction: *Does symplectic blowing-up really depend on the symplectic packing φ used to define it, or in fact solely on the "weights" $\lambda_1, \ldots, \lambda_N$?* In other words, *If $\widetilde{\Omega}_0$ and $\widetilde{\Omega_1}$ are two symplectic forms on the blow-up $\Theta \colon \widetilde{M} \to M$ of M, constructed using symplectic packings φ_0 and φ_1 as in (6), are $\widetilde{\Omega}_0$ and $\widetilde{\Omega_1}$ symplectomorphic? Are they isotopic?*

First results in this direction were obtained by McDuff [23] for the case of $\mathbb{C}P^2$. These results have been later generalized to other cases by Lalonde [20] and later on by the author [5]. At present, the most general result concerning the uniqueness problem is the following theorem due to McDuff [25]:

Theorem 4.1. (McDuff) *Let (M^4, Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Then any two cohomologous symplectic forms $\widetilde{\Omega}_0$ and $\widetilde{\Omega}_1$ obtained from symplectic blowing up are isotopic. In particular, any two symplectic packings of such a manifold by balls of given radii are symplectically isotopic.*

The arguments used to prove this theorem are somewhat parallel to those used for the proof of theorem 3.2, the main techniques being the inflation procedure and Taubes-Seiberg-Witten theory.

At present the problem of isotopies is still open for symplectic manifolds which *do not* belong to the class $\mathcal{C}$ (e.g. $\mathbb{T}^4$). In higher dimension than 4 the problem is completely open for *all* manifolds (see section 7.1 below).

4.2. Explicit packing constructions

Another interesting direction deals with explicit packing constructions that realize the maximal values of v_N. The methods proving existence of symplectic packings are very indirect, thus explicit constructions are important in order to gain intuition about how these symplectic embeddings really look like.

First such constructions were obtained by Karshon [18] for $\mathbb{C}P^n$ with $1 \leq N \leq n+1$ balls. Traynor [32] found explicit constructions realizing full packings of $\mathbb{C}P^n$ by k^n equal balls, the maximal packings of $\mathbb{C}P^2$ by $1 \leq N \leq 6$ equal balls, as well as some constructions for other manifolds (see also [19] for some generalizations and [28] for other types of packings). All these constructions are based on the fact that $\mathbb{C}P^n$ is a toric manifold and make use of its moment map.

To the best of the author's knowledge no explicit constructions for the *maximal* packings of $\mathbb{C}P^2$ by 7, 8 or any $N > 9$ ($N \neq k^2$) equal balls are known.

5. Nagata's Conjecture

In the early 1950's Nagata made the following conjecture regarding singularities of plane algebraic curves (see [29]):

Conjecture 5.1. (Nagata) *Let $p_1, \dots, p_N \in \mathbb{C}P^2$ be $N \geq 9$ very general[1] points. Then for every algebraic curve $C \subset \mathbb{C}P^2$ the following inequality holds:*

$$\deg(C) \geq \frac{\mathrm{mult}_{p_1} C + \cdots + \mathrm{mult}_{p_N} C}{\sqrt{N}} \, .$$

[1] By very general we mean that $(p_1, \dots, p_N)$ is allowed to vary in a subset of the configuration space $\mathcal{C}_N = \{(x_1, \dots, x_N) \mid x_i \neq x_j \ \forall\, i \neq j\}$ whose complement contains at most a countable union of proper subvarieties.

This conjecture can be easily translated via the Nakai-Moishezon criterion to a statement about the *Kähler cone* of blow-ups of $\mathbb{C}P^2$. By the Kähler cone of an algebraic manifold S we mean the subset $\mathcal{K}_S \subset H^{(1,1)}(S,\mathbb{R})$ of all cohomology classes a which admit a Kähler representative. It also relevant to consider the closure $\overline{\mathcal{K}}_S$ of $\mathcal{K}_S$, and we shall call classes in $\overline{\mathcal{K}}_S$ by the name *semi-Kähler*.

Returning to blow-ups $\Theta\colon V_N \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at N points we have:

$$H^{(1,1)}(V_N,\mathbb{R}) = H^2(V_N,\mathbb{R}) = \mathbb{R}l \oplus \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_N \quad \text{and}$$
$$H_2(V_N,\mathbb{R}) = \mathbb{R}L \oplus \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_N\,.$$

Here, the E_i's denote the homology classes of the exceptional divisors, L the homology class of the proper transform of a projective line which does not pass through the blown-up points, and $l, e_1, \dots, e_N$ the Poincaré duals to $L, E_1, \dots, E_N$. Note that if $C \subset \mathbb{C}P^2$ is a an algebraic curve then its proper transform $\overline{C} \subset V_N$ lies in the homology class

$$[\overline{C}] = \deg(C)L - \mathrm{mult}_{p_1}(C)E_1 - \cdots - \mathrm{mult}_{p_N}(C)E_N\,.$$

With these notations, Nagata's conjecture is equivalent to saying that the cohomology class

$$a = l - \frac{1}{\sqrt{N}}(e_1 + \cdots + e_N)$$

is non-negative when evaluated on each homology class $[\overline{C}] \in H_2(V_N)$ that can be represented by an algebraic curve. Adding this to the fact that $a \cdot a = 0$, it follows from the Nakai-Moishezon criterion that Nagata's conjecture is equivalent to the the class a being semi-Kähler.

5.1. The special case $N = k^2$

Interestingly, Nagata's conjecture can be easily confirmed in case the number of points N is a square. This is based on the following simple observation: *Let V be a smooth surface and $D \subset V$ an irreducible curve with non-negative self intersection. Then the cohomology class Poincaré dual to $[D]$ lies in $\overline{\mathcal{K}}_V$.* This follows easily from Nakai-Moishezon criterion and the irreducibility of C.

Now let $C \subset \mathbb{C}P^2$ be a smooth curve of degree $k > 0$ and choose $N = k^2$ points $p_1, \dots, p_N$ on C. Let $\Theta\colon V_N \to \mathbb{C}P^2$ be the blow-up of $\mathbb{C}P^2$ at $p_1, \dots, p_N$ and denote by D the proper transform of C in V_N. Since D is irreducible and $D \cdot D = 0$, the class Poincaré dual to D (namely $kl - (e_1 + \cdots + e_N)$) is semi-Kähler, which is the statement of Nagata's conjecture for $N = k^2$. The fact that we have chosen the points $p_1, \dots, p_N$ to lie in a specific position rather than a very general one is not restrictive. Indeed, it is not hard to show that if Nagata's conjecture holds for a specific choice of points $p_1, \dots, p_N$ then it continues to hold also for a very general choice of points.

5.2. Numerical invariants of the Kähler cone

Let X be an algebraic manifold and $a \in H^{(1,1)}(X, \mathbb{R})$. Given real numbers $\lambda_1, \ldots, \lambda_N$ we say that the vector $(X, a; \lambda_1, \ldots, \lambda_N)$ is Kähler (resp. semi-Kähler) if there exist N points $p_1, \ldots, p_N \in X$ such that the cohomology class $\Theta^* a - \lambda_1 e_1 - \cdots - \lambda_N e_N$ lies in $\mathcal{K}_{\widetilde{X}}$ (resp. $\overline{\mathcal{K}}_{\widetilde{X}}$), where $\Theta \colon \widetilde{X} \to X$ is the blow-up of X at $p_1, \ldots, p_N$. Now, given a class $a \in \mathcal{K}_X$ one can introduce the following quantity which we call the *N'th remainder* of a:

$$\mathcal{R}_N(a) = 1 - \frac{\sup_\lambda \{N\lambda^n \mid a_\lambda = (X, a; \lambda, \ldots, \lambda) \text{ is Kähler}\}}{\int_X a^n},$$

where $n = \dim_{\mathbb{C}} X$. The numbers $\mathcal{R}_N(a)$ are analogous to the quantities introduced in section 2 and play the role of $1 - v_N$ in the algebraic category. Similar quantities have been previously introduced in algebraic geometry and are commonly called *Seshadri constants* (see [7, 11] and [12]). Denoting by $l \in H^2(\mathbb{C}P^2, \mathbb{Z})$ the positive generator, Nagata's conjecture can now be reformulated as:

$$\mathcal{R}_N(l) = 0 \quad \textit{for every} \quad N \geq 9.$$

As already mentioned, Nagata's conjecture is still far from being settled (except for the case $N = k^2$). This makes it interesting to try to bound $\mathcal{R}_N(l)$ from above, or to find asymptotics on $R_N(l)$. For example:

Theorem 5.2. **(Xu** [33]**)** *$\mathcal{R}_N(l) \leq \frac{1}{N}$ for every N.*

At present, this seems to be the best general asymptotic on $\mathcal{R}_N(l)$, however for some special families of N's it can be improved quadratically. The following was proved in [3]:

Theorem 5.3. *Let $a, r \in \mathbb{N}$.*

1. *For $N = a^2r^2 + 2r$, $\mathcal{R}_N(l) \leq \frac{1}{(a^2r+1)^2}$.*
2. *For $N = a^2l^2 - 2l$, $\mathcal{R}_N(l) \leq \frac{1}{(a^2r-1)^2}$.*
3. *For $N = a^2r^2 + r$ with $ar \geq 3$, $\mathcal{R}_N(l) \leq \frac{1}{(2a^2r+1)^2}$.*

This result was obtained using a recursive algorithmic procedure for constructing Kähler classes on algebraic manifolds. Although this algorithm has purely algebro-geometric foundations it originated from very simple ideas coming from symplectic geometry which we shall now explain.

5.3. An algorithm for constructing Kähler classes on algebraic manifolds

The first ingredient in our algorithm is the following general theorem from [3]:

Theorem 5.4. *Let X^n be a smooth algebraic variety of (complex) dimension n and let $a \in H^{(1,1)}(X, \mathbb{R})$. Suppose that:*

1. *$(X, a; m)$ is Kähler (resp. semi-Kähler) for some $m \in \mathbb{R}$.*
2. *$(\mathbb{C}P^n, ml; \alpha_1, \ldots, \alpha_r)$ is semi-Kähler.*

Then $(X, a; \alpha_1, \ldots, \alpha_r)$ is Kähler (resp. semi-Kähler).

As a corollary we get:

Corollary 5.5. *Suppose that* $(\mathbb{C}P^n, dl; m_1, \ldots, m_k, m)$ *is Kähler (resp. semi-Kähler) and* $(\mathbb{C}P^n, ml; \alpha_1, \ldots, \alpha_r)$ *is semi-Kähler. Then* $(\mathbb{C}P^n, dl; m_1, \ldots, m_k, \alpha_1, \ldots, \alpha_r)$ *is Kähler (resp. semi-Kähler).*

This means that one can reduce questions about the Kähler cone of blow-ups of $\mathbb{C}P^n$ to the same questions on blow-ups of $\mathbb{C}P^n$ at *less* points.

Before we describe the algorithm itself let us explain the symplectic rationale behind theorem 5.4. We remark in advance that the following is not a rigorous mathematical argument but just a sequence of intuitive ideas which are supposed to explain how one can guess that such a statement should be true. The mathematical proof can be carried out using purely algebro-geometric techniques (see [3]).

To start with, note that the first condition of theorem 5.4 means that if Ω is a symplectic form with $[\Omega] = a$ then (X, Ω) admits a symplectic embedding of a ball of radius $\sqrt{\frac{m}{\pi}}$,

$$\varphi \colon B\left(\sqrt{\frac{m}{\pi}}\right) \to (X, \Omega)\,.$$

The second assumption means that $\mathbb{C}P^n$ endowed with the symplectic form $m\sigma_{\mathrm{std}}$ admits a symplectic packing by r balls of radii $\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_r}$. Here, σ_{std} stands for the standard symplectic form of $\mathbb{C}P^n$, normalized so that $[\sigma_{\mathrm{std}}] = \pi l \in H^2(\mathbb{C}P^n, \mathbb{R})$. By a standard procedure in symplectic geometry (see [26]) we may assume that all these r balls lie in the complement of some complex hyperplane $H \approx \mathbb{C}P^{n-1}$. Now, it is well known that $(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}, m\sigma_{\mathrm{std}})$ is symplectomorphic to $\operatorname{Int} B(\sqrt{m})$. Therefore we get a symplectic packing of $B(\sqrt{m})$ by r balls of radii $\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_r}$. Rescaling everything we obtain a symplectic packing

$$\psi \colon B\left(\sqrt{\frac{\alpha_1}{\pi}}\right) \coprod \cdots \coprod B\left(\sqrt{\frac{\alpha_r}{\pi}}\right) \to B\left(\sqrt{\frac{m}{\pi}}\right)\,.$$

The composition $\varphi \circ \psi$ will give us a symplectic packing of (X, Ω) by r balls of radii $\sqrt{\frac{\alpha_1}{\pi}}, \ldots, \sqrt{\frac{\alpha_1}{\pi}}$. This in turn implies that the cohomology class

$$\Theta^* a - \alpha_1 e_1 - \cdots - \alpha_r e_r \tag{7}$$

on the blow-up $\Theta \colon \widetilde{X} \to X$ at r points has a symplectic representative. Of course, this still does not mean that this class admits a Kähler representative, however since everything in this "argument" consisted of Kähler classes it seems reasonable to expect the class in (7) to be also Kähler. It turns out that this intuitive argument can be translated to a rigorous algebro-geometric proof using a so called *degeneration* argument (see [3]. See also [30] and [6] for more about degenerations and their applications).

In this context it is worth mentioning here Lazarsfeld's work [22] on Seshadri constants of Abelian varieties which is also based on a simple "symplectic packing" argument.

The second ingredient in our algorithm is the *Cremona* group action on the Kähler cone of blow-ups of $\mathbb{C}P^2$. Let $\Theta \colon V_N \to \mathbb{C}P^2$ be a blow-up of $\mathbb{C}P^2$

at $N \geq 3$ points. The Cremona group is a group of reflections which acts on $H^{(1,1)}(V_N, \mathbb{R})$. A very important feature of this action is that it preserves the Kähler cone $\mathcal{K}_{V_N} \subset H^{(1,1)}(V_N, \mathbb{R})$. The reader is referred to [10] and to [3] for the definition and basic properties of the action of this group.

In order to detect Kähler (or semi-Kähler) classes on V_N we use theorem 5.4 and the Cremona action successively until we arrive to a vector on which we can easily verify Kählerness. Here is a simple example which illustrates how the algorithm works. For brevity let us write

$$(\mathbb{C}P^2, dl; \alpha_1^{\times r_1}, \dots, \alpha_k^{\times r_k}) = (\mathbb{C}P^2, dl; \underbrace{\alpha_1, \dots, \alpha_1}_{r_1 \text{ times}}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{r_k \text{ times}}).$$

Consider the case $N = 15$. We shall show now that the vector $v = (\mathbb{C}P^2, 15l; 4^{\times 14})$ is semi-Kähler. Note that this would imply that

$$\mathcal{R}_{15}(l) \leq 1 - \frac{14 \cdot 4^2}{15^2} = \frac{1}{225}.$$

To start with, recall that for every $k > 0$ the vector $(\mathbb{C}P^2, kl; 1^{\times k^2})$ is semi-Kähler, hence for every $m > 0$ so is also $(\mathbb{C}P^2, mkl; m^{\times k^2})$. In particular, $(\mathbb{C}P^2, 8l; 4^{\times 4})$ is semi-Kähler. Therefore according to corollary 5.5, in order to show that our vector v is semi-Kähler it is enough to show that the vector $v_1' = (\mathbb{C}P^2, 15l; 4^{\times 10}, 8)$ is semi-Kähler. Applying suitable Cremona transformations to this vector we obtain the vector $v_1 = (\mathbb{C}P^2, 10l; 3^{\times 11})$. Since the Cremona group preserves semi-Kähler classes, v_1 is semi-Kähler if and only if v_1' is. Note that the vector v_1 is "simpler" then v_1' in the sense that its entries are smaller than those of v_1'. Now, in order to show that v_1 is semi-Kähler we decompose it into $v_2' = (\mathbb{C}P^2, 10l; 3^{\times 7}, 6)$ and $(\mathbb{C}P^2, 6l; 3^{\times 4})$. The latter being semi-Kähler, it is enough by corollary 5.5 to show that v_2' is semi-Kähler. Applying suitable Cremona transformations to v_2' we get the vector $v_2 = (\mathbb{C}P^2, l; 0^{\times 8})$. This vector is obviously semi-Kähler since it stands for the cohomology class that is Poincaré dual to a projective line on V_8 not passing through the exceptional divisors. All the above imply now that our original vector v is semi-Kähler.

Similar applications of the algorithm lead to the asymptotics of theorem 5.3. In fact this algorithm gives rise to a mysterious pattern which links upper bounds on $\mathcal{R}_N(l)$ with *continued fractions* expansions of the number $\sqrt{N}$ (see [3]).

Another result of a successive application of corollary 5.5 is the following, somewhat amusing, corollary:

Corollary 5.6. *If Nagata's conjecture holds for N_1 and N_2 then it holds also for $N_1 N_2$.*

It is of course a pity that the product of two squares is also a square.

5.4. Nagata's conjecture via ellipsoids

Recall that our algorithm for producing Kähler classes on blow-ups of $\mathbb{C}P^2$ (or more generally on any algebraic manifold) originated from the very simple symplectic picture that if we can embed symplectically a ball B into (X, Ω) then any packing of this ball would give rise also to a packing of M by balls of the same sizes. Note that the role played by the ball B is merely auxiliary and in the end we are only interested in the balls which pack (X, Ω).

In view of this there is no a priori reason not to replace B by a different manifold or shape. In fact it is reasonable to expect that doing so may lead to sharper estimates on $\mathcal{R}_N(l)$. Of course balls have the important advantage in that they have a nice characteristic foliation on the boundary thus giving rise to the symplectic blowing-up construction. Therefore, good candidates to replace the ball B seem to be ellipsoids. By a *symplectic ellipsoid with multi-radius* $\underline{r} = (r_1, \dots, r_n)$ we mean the following subset of $\mathbb{C}^n$

$$E(\underline{r}) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{|z_i|^2}{r_i^2} \leq 1 \right\},$$

endowed with the standard symplectic form of $\mathbb{C}^n$. Now let $\underline{w} = (w_1, \dots, w_n)$ be a vector of integral weights and $m \in \mathbb{R}$. As before, if we can embed symplectically the ellipsoid $E(m\underline{w})$ into (X, Ω) then any symplectic packing

$$\varphi \colon B(\alpha_1) \coprod \cdots \coprod B(\alpha_r) \to E(m\underline{w}) \tag{8}$$

would give rise to a symplectic packing of (X, Ω) by the same balls.

It seems reasonable that this scheme would translate well into pure algebro-geometric terms. The idea is that precisely as $\mathbb{C}P^n$ is the algebro-geometric analogue of a symplectic ball, *weighted projective spaces* should correspond to symplectic ellipsoids. Weighted projective spaces are defined as follows: given a vector of integral weights $\underline{w} = (w_1, \dots, w_n)$, define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus 0$ by $\lambda \cdot (z_0, \dots, z_n) = (z_0, \lambda^{w_1} z_1, \dots, \lambda^{w_n} z_n)$ for every $\lambda \in \mathbb{C}^*$. The quotient space is a singular (unless $w_i = 1$ for all i) toric algebraic variety which we denote by $\mathbb{C}P^n(\underline{w})$. Existence of a packing φ as in (8) should translate here to *ampleness* of some divisor class[2] on blow-ups of $\mathbb{C}P^n(\underline{w})$ at r points.

The next step is a bit more involved. We have to translate the fact that (X, Ω) admits a symplectic embedding of an ellipsoid $E(m\underline{w})$. For this end we would like to define a weighted blowing-up of X at a point $p \in X$ with weights $\underline{w}$. The problem is that this procedure is not well defined since there does not exist any canonical weighted-action of $\mathbb{C}^*$ on the tangent space $T_p(X)$. In other words, such an action depends on a choice of coordinates (unless $\underline{w} = (1, \dots, 1)$ of course). Nevertheless, when X is a toric variety this difficulty should be tractable and one expects to obtain a weighted blown-up variety $\mathrm{Bl}^{(\underline{w})}(X)$.

If all the above is indeed feasible then one would expect a theorem of the following type to hold:

[2]We cannot work here with Kähler classes for these varieties are not smooth manifolds. Nevertheless the notion of ample divisor exits and essentially the same theory (with some modifications) applies to them as in the smooth case.

If $(\mathrm{Bl}^{(\underline{w})}(X), a; m)$ *is ample (resp. semi-ample) and* $(\mathbb{C}P^n(\underline{w}), ml; \alpha_1, \ldots, \alpha_r)$ *is ample (resp. semi-ample) then* $(X, a; \alpha_1, \ldots, \alpha_r)$ *is also ample (resp. semi-ample).* Here a is a given divisor class on X and l is a suitable hyperplane in $\mathbb{C}P^n(\underline{w})$.

A successive application of this splitting procedure in combination with the Cremona group action (on blow-ups of weighted projective planes) would hopefully improve the existing asymptotics on the remainders $\mathcal{R}_N(l)$ of $\mathbb{C}P^2$ and may be shed some new light on Nagata's conjecture.

6. Back to Symplectic Packing: a Stability Phenomenon

Let S be a smooth algebraic surface and $C \subset S$ a smooth irreducible curve. Consider the ruled surface

$$P_{C/S} = \mathbb{P}(N_{C/S} \oplus \mathcal{O}_C) \xrightarrow{p_C} C\,,$$

where $N_{C/S} \to C$ is the normal bundle of C in S. Inside the surface $P_{C/S}$ we have the curve $Z_{C/S} = \mathbb{P}(N_{C/S} \oplus 0)$. Denote by $z_C, f \in H^{(1,1)}(P_{C/S}, \mathbb{R})$ the Poincaré duals to $[Z_{C/S}]$ and to the fibre of $p_C\colon P_{C/S} \to C$ respectively.

The following theorem allows us to reduce questions on the Kähler cone of blow-ups of an arbitrary algebraic surface to the same questions on blow-ups of ruled surfaces. It can be proved using a degeneration argument similarly to theorem 5.4 (see [3]).

Theorem 6.1. *Let S and C be as above and denote by $c \in H^{(1,1)}(S, \mathbb{R})$ the class Poincaré dual to $[C]$. Let $a \in H^{(1,1)}(S, \mathbb{R})$, and suppose that the following two conditions are satisfied:*

1. *The class $a - mc$ is Kähler (resp. semi-Kähler) for some $m \in \mathbb{R}$.*
2. *The vector $\left(P_{C/S}, (\int_C a) f + m z_C; \alpha_1, \ldots, \alpha_k\right)$ is Kähler (resp. semi-Kähler).*

Then the vector $(S, a; \alpha_1, \ldots, \alpha_k)$ is Kähler (resp. semi-Kähler).

Remark 6.2. *When the class a of the theorem is Poincaré dual itself to a smooth curve in S we may take C to be this curve and $m = 1$. Then the class $a - mc$ in condition 1 of the theorem is just* 0 *hence semi-Kähler. Thus in this case only condition 2 has to be checked.*

With this remark in mind, let us return to the problem of packing of a symplectic 4-manifold (M, Ω). It turns out that the degeneration argument (see [3]) used to prove theorem 6.1 can be neatly translated to symplectic geometry using a construction called *Gompf surgery*. The ability to find a smooth curve C which represents the Poincaré dual to the class $a = [\Omega]$ as in remark 6.2 follows from Donaldson's symplectic hypersurface theorem (see [9]). The upshot of all this is that in order to obtain symplectic packings of an arbitrary symplectic 4-manifold it is enough to pack a symplectic ruled surface constructed in an analogous way to $P_{C/S}$ above. The main advantage is that symplectic ruled surfaces belong to the class of manifolds $\mathcal{C}$ described in section 3 and so our symplectic version of Nakai-Moishezon criterion 3.2 applies to them. An analysis of the possible exceptional classes on blow-ups of ruled surfaces shows that similarly to $\mathbb{C}P^2$ ruled surfaces admit full symplectic packing by N equal balls for *every* large enough N. The conclusion is that the same thing holds also for (M, Ω). More precisely we have the following (see [4]):

Theorem 6.3. *Let (M, Ω) be a closed symplectic 4-manifold with $[\Omega] \in H^2(M, \mathbb{Q})$. Then, there exists N_0 such that for every $N \geq N_0$, (M, Ω) admits a full symplectic packing by N equal balls. In fact, if for some $k_0 \in \mathbb{Q}$ the Poincaré dual to $k_0[\Omega]$ can be represented by a symplectic submanifold of genus at least 1, then one can assume that $N_0 = 2k_0^2 \operatorname{Vol}(M, \Omega)$, where $\operatorname{Vol}(M, \Omega) = \frac{1}{2}\int_M \Omega \wedge \Omega$.*

This means that packing obstructions occur for at most a finite number of values of N. Thus on the phenomenological level, essentially on every symplectic manifold we have stability of the process of symplectic packing: starting from some N all obstructions disappear and the quantities $v_N(M, \Omega)$ stablize on the value 1.

Theorem 6.3 also allows us to bound from above the number N starting from which the process becomes stable. Here are some concrete examples:

Corollary 6.4. *In each of the following cases N_0 is a number (not necessarily minimal) for which the relevant symplectic manifold admits full symplectic packing by N equal balls for every $N \geq N_0$:*

1. *Let $S \subset \mathbb{C}P^n$ be an irrational smooth projective surface of degree d, and let Ω be the restriction of the standard Kähler form of $\mathbb{C}P^n$ to S. Then for (S, Ω) we have $N_0 = d$.*
2. *For $(\mathbb{T}^2 \times \mathbb{T}^2, \sigma \oplus \sigma)$ the 4-dimensional symplectic split-torus, where σ is an area form on $\mathbb{T}^2$, we have $N_0 = 2$.*
3. *Let (C_1, σ_1), (C_2, σ_2) be (real) symplectic surfaces with $\int_{C_1} \sigma_1 = \int_{C_2} \sigma_2$, and let $a, b \in \mathbb{N}$. Then, for $(C_1 \times C_2, a\sigma_1 \oplus b\sigma_2)$ we have $N_0 = 8ab$.*

7. Open Problems, New Directions and Some Speculations

7.1. Symplectic packing in higher dimensions

Most of the results mentioned up to now are special to dimension 4. This is not a coincidence. While (part) of the algebro-geometric results we mentioned can be generalized to any dimension, the results on the stability of symplectic packing rely strongly on Seiberg-Witten theory which is very special to dimension 4, and it is not clear how to generalize it (if possible at all) to higher dimensions.

The only known results about symplectic packings in higher dimensions are Gromov's theorem 2.1 and a result due to McDuff and Polterovich [26] which states that $v_N(B^{2n}(1)) = v_N(\mathbb{C}P^n) = 1$ whenever $N = k^n$ (the latter can be easily generalized to any algebraic manifold).

The following two problems seem to be of particular interest:

1. Does the process of symplectic packing stabilize starting from some number of balls N_0, as in dimension 4?
2. Given a symplectic manifold (M, Ω), does there exist a radius λ_0 such that for every $\lambda \leq \lambda_0$ any two symplectic embeddings of a ball of radius λ are symplectically isotopic?

7.2. What is the symplectic nature of the phenomenon of packing obstructions?

Although powerful tools such as pseudo-holomorphic curves allow us to obtain quite a lot of information on symplectic packings there is at least one aspects of the problem which remains mysterious: the *geometric nature* of this phenomenon.

It seems reasonable to categorize packing obstructions (for $N \geq 2$) as a *symplectic intersection phenomenon* (e.g. if one tries to embed symplectically two large enough balls into $\mathbb{C}P^n$ then their images must intersect). Intersections play a fundamental role in symplectic geometry and appear in many forms and disguises. Probably the most fundamental known such phenomenon is that of *Lagrangian intersections*. This phenomenon has to do with non-removable intersections between Lagrangian submanifolds that cannot be detected by means of classical topology or differential geometry. Lagrangian intersections have been studied intensively over the last decade by numerous people and nowadays there is both a systematic machinery to study them (see [13] for example) as well as a fair geometric intuition about their symplectic nature. In fact, this feel or intuition preceded the invention of the first mathematical tools used to study Lagrangian intersections by more than a decade. Let us also mention that several symplectic rigidity phenomena can be well understood and studied in the framework of Lagrangian intersections, most notably Arnold conjecture on fixed points of Hamiltonian diffeomorphisms.

It seems to be a question of conceptual relevance to figure out whether packing obstructions are in fact a "hidden" Lagrangian intersection phenomenon or something else, of a genuinely different nature.

A perhaps naive, but still worth exploring, idea would be to try to approach this question using Floer homology. For example, given a symplectic embedding of a ball $\phi\colon B(\lambda) \to \mathbb{C}P^n$ with $2\lambda^2 > 1$ one may consider a Lagrangian submanifold L which lies on the boundary of Image(ϕ) (a Lagrangian torus for example). If it can be shown that the Floer homology $HF_*(L, L)$ does not vanish, then it would follow that any two such symplectic balls $\varphi_1, \varphi_2\colon B(\lambda) \to \mathbb{C}P^n$ which are Hamiltonian isotopic *must have non-empty intersection*. Although this would not recover Gromov's packing theorem 2.1 in its full generality[3], it would strongly indicate that packing obstructions are in fact a Lagrangian intersection phenomenon.

At present the implementation of this plan using Floer homology unfortunately runs into (technical) difficulties due to analytic problems which cause Floer homology not to be well defined in our setting. Hopefully the techniques will be refined in the future so as to enable us to settle this "dilemma".

[3]Since, except in dimension 4 (see [23]), we do not know whether or not any two symplectic balls in $\mathbb{C}P^n$ are Hamiltonian isotopic.

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