Trace Formulas and Spectral Statistics of Diffractive Systems

Eugene Bogomolny

Abstract. Diffractive systems are quantum-mechanical models with point-like singularities where usual semiclassical approximation breaks down. An overview of recent investigations of such systems is presented. The following examples are considered in details: (i) billiards (both integrable and chaotic) with small-size scatterers, (ii) pseudo-integrable polygonal plane billiards, and (iii) billiards with the Bohr-Aharonov flux lines. First, the diffractive trace formulas are discussed with particular emphasis on models where the diffractive coefficient diverges in certain directions. Second, it is demonstrated that the spectral statistics of diffractive models are different from the statistics of both integrable and chaotic systems. The main part of the lecture is devoted to analytical calculations of spectral statistics for certain diffractive models.

1. Introduction

Quantum chaos (or chaology) is a part of theoretical physics whose aim is the investigation of (multi-dimensional) quantum problems in the semiclassical limit $\hbar \to 0$. The modern development of this field is based on the semiclassical representation of the Green function, $G(\vec{x}, \vec{x}\,')$, through the sum over classical trajectories connecting points $\vec{x}$ and $\vec{x}\,'$ (see e.g. [16])

$$G(\vec{x}, \vec{x}\,') = \sum_{cl.\ tr.} A_{tr}(\vec{x}, \vec{x}\,') \exp(\frac{i}{\hbar} S_{cl}(\vec{x}, \vec{x}\,')) \tag{1}$$

where the action, $S_{cl}(\vec{x}, \vec{x}\,')$, and the pre-factor, $A_{tr}(\vec{x}, \vec{x}\,')$ are computed from pure classical mechanics. In such an approach one implicitly assumes that the classical limit of quantum mechanical problem does exist which is not always the case. For a long time (see e.g. [17]) it was known that there are quantum systems without 'good' classical limit.

This lecture is devoted to the investigation of a particular type of such models, namely, diffractive systems whose characteristic property is the presence of point-like singularities (real or effective) in the quantum Hamiltonian. The existence of singularities in classical mechanics leads to the impossibility of continuing classical trajectories which hit these singularities. In quantum mechanics the situation is less dramatic [17]–[23]. Each point-like singularity can be described by a diffraction coefficient, $D(\vec{n},\vec{n}')$, which determines the scattering amplitude on the singularity. We normalize it in such a way that the semiclassical expansion of the Green function in the whole space in the presence of a singularity at point $\vec{x}_0$ has the following form

$$G(\vec{x},\vec{x}\,') = G_0(\vec{x},\vec{x}\,') + \frac{\hbar^2}{2m}\sum_{\vec{n},\vec{n}'} G_0(\vec{x},(\vec{x}_0,\vec{n}))D(\vec{n},\vec{n}')G_0((\vec{x}_0,\vec{n}'),\vec{x}\,') \qquad (2)$$

where $G_0(\vec{x},\vec{x}\,')$ is the Green function (1) in the absence of singularity and $G_0(\vec{x},(\vec{x}_0,\vec{n}))$ is the term in (1) corresponding to a classical trajectory starting from point $\vec{x}$ and ending at point $\vec{x}_0$ with momentum in the direction $\vec{n}$.

The knowledge of the Green function permits to write down the trace formula for diffractive systems [27, 19]. In the simplest case, when only one singularity is present, the density of states can be written as a sum of three terms

$$d(E) = \bar{d}(E) + d_p(E) + d_d(E) \qquad (3)$$

where $\bar{d}(E)$ is the smooth part of the level density given by the Thomas-Fermi term (plus corrections if necessary) [16], $d_p(E)$ is the contribution from classical periodic orbits, and $d_d(E)$ is the diffractive contribution.

For integrable billiards [4]

$$d_p(E) = \frac{A}{2\pi}\sum_{ppo}\sum_{n=1}^{\infty}\frac{1}{\sqrt{knl_p}}\cos(knl_p - \frac{\pi}{4} - \frac{\pi}{2}n\nu_p)\,. \qquad (4)$$

For chaotic billiard systems [16]

$$d_p(E) = \sum_{ppo}\frac{l_p}{\pi k}\sum_{n=1}^{\infty}\frac{1}{|\det(M_p^n - 1)|^{1/2}}\cos(knl_p - \frac{\pi}{2}n\nu_p)\,. \qquad (5)$$

In these formulas the summation is performed over all primitive periodic orbits (ppo) and their repetitions. l_p is the length of the ppo, ν_p is its Maslov index, and M_p is the monodromy matrix of the ppo.

In eq. (3) $d_d(E)$ is the contribution from classical orbits (called later diffractive orbits) which start from the singularity and end at it.

$$\begin{aligned} d_d(E) \;\; = \;\; & \sum_{m=1}^{\infty}\frac{1}{\pi m}\frac{\partial}{\partial E}\,\mathrm{Im}\sum_{\vec{n},\vec{n}'} G(\vec{n}_1,\vec{n}_1')D(\vec{n}_1',\vec{n}_2)G(\vec{n}_2,\vec{n}_2')D(\vec{n}_2',\vec{n}_3) \\ & \cdots G(\vec{n}_m,\vec{n}_m')D(\vec{n}_m',\vec{n}_1)\,. \end{aligned} \qquad (6)$$

Here $G(\vec{n},\vec{n}')$ is the contribution to the Green function (1) from a diffractive orbit which starts from the singular point with momentum in direction $\vec{n}$ and ends at this point with momentum in direction $\vec{n}'$. The sums in (6) can be transformed to

$$d_p(E) = -\frac{1}{\pi}\,\mathrm{Im}\,\frac{\partial}{\partial E}\ln\det(1-\hat{K}) \qquad (7)$$

where the operator $\hat{K}$ is defined as follows

$$(\hat{K}\phi)(\vec{n}) = \sum_{\vec{n}',\vec{n}''} G(\vec{n},\vec{n}')D(\vec{n}',\vec{n}'')\phi(\vec{n}'') \,. \quad (8)$$

Eqs. (7)–(8) mean that the formal 'quantization' condition of diffractive systems (from which energy levels can be computed) is

$$\det(1-\hat{K}) = 0 \,. \quad (9)$$

The main purpose of this paper is to discuss statistical properties of energy levels (i.e. the spectral statistics) of diffractive systems. The plan of the paper is the following. In section 2 integrable systems with a diffractive center are considered and it is demonstrated that the spectral statistics of these systems can be computed analytically. In section 3 the pseudo-integrable polygonal billiards and in section 4 the billiards with an Aharonov-Bohm flux line are investigated. These models are examples of diffractive systems where the diffractive coefficient diverges in certain directions which considerably complicates all calculations. First, the trace formulas for these systems are derived and it is demonstrated that they differ from eq. (6). Second, by combining numerical and analytical arguments we investigate their spectral statistics. Finally, in section 5 it is proved that the addition of a diffractive center to chaotic models does not change its spectral statistics.

2. Integrable Models with Diffractive Center

The simplest example of diffractive systems with a constant diffractive coefficient has been proposed in [21] and it consists of an integrable model (below we shall consider a 2-dimensional rectangular billiard) with a δ-function potential

$$V(\vec{x}) = \lambda\delta(\vec{x}-\vec{x}_0) \,. \quad (10)$$

The quantization condition (9) in this case takes a particular simple form

$$\lambda G(\vec{x}_0,\vec{x}_0) = 1 \quad (11)$$

or (ignoring renormalization problems [2])

$$\lambda \sum_{n=1}^{N} \frac{|\psi_n(\vec{x}_0)|^2}{E-e_n} = 1 \quad (12)$$

where e_n and $\psi_n(\vec{x})$ are eigenvalues and eigenfunctions of the unperturbed system (i.e. without the δ-function potential). The natural question arises: what is the distribution of the new eigenvalues, E_n, provided the distributions of e_n and $\psi_n(\vec{x}_0)$ are known.

Let us for simplicity assume that

(i) e_n are independent random variables (as for integrable systems) with a step-like common distribution

$$d\mu(e) = \begin{cases} \frac{1}{2W}de \,, & \text{if } |e| \le W \\ 0 \,, & \text{otherwise} \end{cases} \quad (13)$$

and (ii) $|\psi(\vec{x}_0)|^2 = 1$ (as for a rectangular billiard with periodic boundary conditions).

Under these assumptions eq. (12) takes the form

$$\sum_{j=1}^{N} \frac{1}{E - e_j} = \frac{1}{\lambda} \tag{14}$$

where each term in the sum is an independent random variable.

The exact density of solutions of this equation can be written in the following form

$$\rho(E) = \delta(\sum_{j=1}^{N} \frac{1}{E - e_j} - \frac{1}{\lambda}) \sum_{k=1}^{N} \frac{1}{(E - e_k)^2}\,. \tag{15}$$

Representing the δ-function as a Fourier integral one can express the density of states through the characteristic function of the left-hand side of eq. (14)

$$\rho(E) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp(i\alpha(\sum_{j=1}^{N} \frac{1}{E - e_j} - \frac{1}{\lambda})) \sum_{k=1}^{N} \frac{1}{(E - e_k)^2}\,. \tag{16}$$

Because all e_j are assumed to be independent random variables, the correlation functions of E can, in principle, be computed straightforwardly.

In particular, the 2-point correlation function

$$R_2(E_1, E_2) = \langle \rho(E_1)\rho(E_2) \rangle \tag{17}$$

can be expressed in the following form

$$\begin{aligned} R_2(E_1, E_2) &= \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} [N(f(\alpha_1, \alpha_2))^{N-1} g(\alpha_1, \alpha_2) \qquad (18) \\ &+ N(N-1)(f(\alpha_1, \alpha_2))^{N-2} \Psi_1(\alpha_1, \alpha_2) \Psi_2(\alpha_1, \alpha_2)] e^{-i(\alpha_1 + \alpha_2)/\lambda}\,, \end{aligned}$$

where

$$\begin{aligned} f(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i\frac{\alpha_1}{E_1 - e} + i\frac{\alpha_2}{E_2 - e})\,, \\ g(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i\frac{\alpha_1}{E_1 - e} + i\frac{\alpha_2}{E_2 - e}) \frac{1}{(E_1 - e)^2 (E_2 - e)^2}\,, \\ \Psi_i(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i\frac{\alpha_1}{E_1 - e} + i\frac{\alpha_2}{E_2 - e}) \frac{1}{(E_i - e)^2}\,. \end{aligned}$$

When $N \to \infty$ the direct (but tedious) calculation of these integrals gives [6]

$$R_2(E_1, E_2) = \bar{\rho}^2 r_2(\epsilon)\,, \tag{19}$$

where

$$\begin{aligned} & r_2(\epsilon) = 1 + \qquad (20) \\ & + \int_0^{\infty} d\alpha_1 \int_0^{\infty} d\alpha_2 (J_0^2(2\sqrt{\alpha_1\alpha_2}\,) + J_1^2(2\sqrt{\alpha_1\alpha_2}\,)) e^{-2\pi\epsilon J(\alpha_1, \alpha_2) + 2i(\alpha_1 + \alpha_2)}, \end{aligned}$$

and

$$J(\alpha_1,\alpha_2)=\frac{1}{2}(\alpha_1+\alpha_2)+\frac{i}{2\pi\lambda_r}(\alpha_1-\alpha_2)$$
$$+ie^{i(\alpha_1+\alpha_2)}[(\alpha_1-\alpha_2)G(\alpha_1,\alpha_2)-i\alpha_1 J_0(2\sqrt{\alpha_1\alpha_2}\,)-\sqrt{\alpha_1\alpha_2}J_1(2\sqrt{\alpha_1\alpha_2}\,)],$$
$$G(\alpha_1,\alpha_2)=e^{i\alpha_2}\int_{\alpha_1}^{\infty}J_0(2\sqrt{\alpha_2 t}\,)e^{it}dt\,,$$
$$\frac{1}{\lambda_r}=\frac{1}{\bar{\rho}\lambda}+\ln\frac{W-E}{W+E},\ \epsilon=\bar{\rho}(E_1-E_2),\ \bar{\rho}=\frac{N}{2W}.$$

This exact formula permits, in particular, to find the limiting behavior of the 2-point correlation function.

When $\epsilon\to 0$

$$r_2(\epsilon)\to A\epsilon \tag{21}$$

where

$$A=3\pi^2\int_0^{\infty}xJ_0((3+\delta)x)J_0^3(x)dx=\frac{\pi\sqrt{3}}{2}\approx 2.72 \tag{22}$$

is independent of the coupling constant λ, and differs from the GOE prediction $(r_2(\epsilon)\to\pi^2/6\epsilon\approx 1.64\epsilon)$.

When $\epsilon\to\infty$

$$r_2(\epsilon)\to 1+\frac{2}{\epsilon^2(\pi^2+1/\lambda_r^2)}\,. \tag{23}$$

For GOE $r_2(\epsilon)\to 1-1/(\pi\epsilon)^2$.

These calculations give rise to different generalizations.

(i) The spectral statistics can also be computed for models where the residues, $|\psi(\vec{x}_0)|^2$ are either fixed quantity (different from 1) or independent random variables. In particular, for the rectangular billiard with the Dirichlet boundary conditions the introduction of the potential (10) (when coordinates of the scatterer are non-commensurable with the sides) leads to the following behavior of the 2-point correlation function at small ϵ

$$r_2(\epsilon)\to\frac{\epsilon}{8\pi^3}\ln^4(\epsilon)\,. \tag{24}$$

(ii) The spectral statistics of the Bohr-Mottelson model [14] which describes the interaction of one level with all others can be calculated by a similar method.

(iii) The asymptotic behavior of the 2-point correlation function (21)–(23) can be derived without the knowledge of the exact solution (20) even for more general cases.

(iv) In [9] it was demonstrated that in the summation over periodic orbits of integrable systems there exists hidden saddle points whose contributions give correct off-diagonal terms. This result permits to construct a specific perturbation theory by which one can compute successive terms of the expansion at large ϵ of the 2-point correlation function for integrable systems with finite diffraction coefficient by semiclassical methods.

(v) Using the results of [9] it is possible to obtain an analog of trace formulas for composite operators (build from a product of the Green functions (1)), e.g. for conductance fluctuations [10].

3. Pseudo-Integrable Billiards

An interesting and important class of diffractive systems is 2-dimensional polygonal billiards with angles, α_i, equal rational multiples of π

$$\alpha_i = \frac{m_i}{n_i}\pi \tag{25}$$

where m_i and n_i are coprime integers. These models (called pseudo-integrable billiards [20, 22]) have a characteristic property that their classical trajectories belong to a 2-dimensional surface with the genus

$$g = 1 + \frac{N}{2}\sum_i \frac{m_i - 1}{n_i} \tag{26}$$

where N is the least common multiplier of n_i.

Classical trajectories in these billiards are not defined after they hit a corner (25) with $m_i \neq 1$. These singular angles play the role of diffraction centers and the quantum diffraction coefficient can be read off from the exact Sommerfeld solution near a wedge with angle α [26]

$$D(\theta_f, \theta_i) = \frac{2}{\gamma}\sin\frac{\pi}{\gamma}\left[\frac{1}{\cos\frac{\pi}{\gamma} - \cos\frac{\theta_f+\theta_i}{\gamma}} - \frac{1}{\cos\frac{\pi}{\gamma} - \cos\frac{\theta_f-\theta_i}{\gamma}}\right] \tag{27}$$

where $\gamma = \alpha/\pi$.

An important difference between these models and the ones considered in the previous section is the fact that the diffraction coefficient (27) diverges in certain directions (called optical boundaries) to compensate the discontinuous behavior of classical trajectories. Consequently, diffractive orbits lying on optical boundaries cannot be described by eq. (6) and, first of all, a modification of the trace formulas is required [11] which can conveniently be done by using the Kirchoff approximation [26] valid in a vicinity of optical boundaries [11]. The basis of this approximation is the representation of the free (2-dimensional) Green function, $G(\vec{x}, \vec{x}\,')$ as an integral over a line separating points $\vec{x}$ and $\vec{x}\,'$

$$G(\vec{x}, \vec{x}\,') = \int [G(\vec{q}, \vec{x}\,')\frac{\partial}{\partial \vec{n}_q}G(\vec{x}, \vec{q}\,) - G(\vec{x}, \vec{q}\,)\frac{\partial}{\partial \vec{n}_q}G(\vec{q}, \vec{x}\,')]d\vec{q} \tag{28}$$

where $\vec{n}_q$ is the normal to the line (parameterized by $\vec{q}\,$).

This formula is exact provided the integration is performed over the whole (infinite) line. The existence of the wedge reduces the integration to a semi-infinite region thus producing the Kirchoff approximation to the diffraction problem. The detailed calculations of many particular types of diffractive orbits in pseudo-integrable billiards have been performed in ref. [11]. Here we present only the contribution to the density of states from diffractive orbits lying on the boundary of multiple repetitions of a primitive periodic orbit of length l_0

$$d_{l_0}(E) = -\frac{l_0}{8\pi k}\left(\sum_{q=1}^{\infty}\frac{1}{q^{1/2}}e^{iqkl_0}\right)^2 + c.c. \tag{29}$$

This contribution with fixed l_0 and $k \to \infty$ is smaller that periodic orbit contribution (4) but bigger that the contribution of diffractive orbits with finite diffraction coefficient (6).

The diffraction coefficient (27) can be written near an optical boundary as a pole term plus a finite part, D^{reg}. Eq. (29) corresponds to the contribution from the pole term. It is also possible to find analytically contributions from the interference of D^{reg} with the pole term

$$d'_{l_0}(E) = -\frac{1}{2\pi i}\frac{\partial}{\partial E}\ln\left(1 - \frac{D^{reg}}{\sqrt{8\pi k l_0}}\sum_{q=1}^{\infty}\frac{1}{q^{3/2}}e^{iqkl_0 - 3\pi i/4 - i\pi\nu_d/2}\right) + c.c. \tag{30}$$

To derive these results it was necessary to compute analytically certain multi-dimensional integrals. In particular, it has been shown in [11] that

$$\int_0^{\infty}\cdots\int_0^{\infty} dx_1\cdots dx_n e^{-\Phi(\vec{x})} = \\ = \frac{1}{n+1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_n e^{-\Phi(\vec{x})} = \frac{\pi^{n/2}}{(n+1)^{3/2}} \tag{31}$$

where $\Phi(\vec{x}) = x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2$, and

$$\int_0^{\infty} dx[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dy_1\cdots dy_n e^{-\Psi(x,\vec{y})} - \\ -\int_0^{\infty}\cdots\int_0^{\infty} dy_1\cdots dy_n e^{-\Psi(x,\vec{y})}] = \frac{\pi^{(n-1)/2}}{4}\sum_{q=1}^{n-1}\frac{1}{\sqrt{q(n-q)}} \tag{32}$$

where $\Psi(x,\vec{y}) = (x - y_1)^2 + (y_1 - y_2)^2 + \cdots + (y_{n-1} - y_n)^2 + (y_n - x)^2$. The calculation of these integrals is based on the invariance of the quadratic form $\Phi(\vec{x})$ under the action of a finite group generated by reflections.

Spectral statistics of pseudo-integrable billiards in the shape of the right triangle with one angle equals π/n for different n has been investigated numerically in ref. [8] and it was demonstrated that the energy level distribution for all n (except the integrable cases $n = 3, 4, 6$) differs from both, the Poisson distribution typical for integrable models and the random matrix distribution [18] typical for chaotic systems. It was observed numerically that the spectral statistics of these billiards has the following characteristic properties:

1. The 2-point correlation function $R_2(\epsilon) \to A\epsilon$ when $\epsilon \to 0$, i.e. there exist linear level repulsion as for the Gaussian orthogonal ensemble of random matrices.

2. The nearest-neighbor distribution $p(s) \to e^{-\Lambda s}$ when $s \to \infty$ as for the Poisson distribution.
3. The number variance $\Sigma^2(L) \to \chi L$, when $L \to \infty$.
4. The distribution of energy levels do not change with increasing of energy (up to 30000 levels).
5. With increasing n distributions stabilize quite far from random matrix predictions.
6. For $n = 5, 8, 10, 12$ the spectral statistics is close (better than 10^{-2}) to the so-called semi-Poisson distribution [7] which has the following correlation functions

$$R_2(\epsilon) = 1 - e^{-4\epsilon}\ , \ \ p(s) = 4se^{-2s}, \ \ \chi = \frac{1}{2}\,. \tag{33}$$

Even the next-to-nearest distribution for these billiards agrees quite well with the prediction of the semi-Poisson model

$$p(2, s) = \frac{8}{3} s^3 e^{-2s}\,. \tag{34}$$

The first step of analytical investigation of spectral statistics for these models is the calculation of the 2-point correlation form-factor in the diagonal approximation [5]. It is possible to prove that at small τ it is necessary to take into account only the contribution from periodic orbits which in these cases is the same as for integrable billiards [5]

$$K(\tau) = \frac{1}{2\pi^2 \bar{d}} \sum_p \frac{A_p}{l_p} \delta(l - 4\pi k \bar{d} \tau)\,. \tag{35}$$

The summation here is performed over all periodic orbits. l_p is the periodic orbit length, A_p is the area occupied by primitive periodic orbit family.

The billiard in the shape of the right triangle with one angle equals π/n belongs to the so-called Veech polygons [28] for which there exists a hidden group structure which permits the explicit calculation of the number of periodic orbits and surfaces occupied by them. By generalizing arguments of [28] it is possible to prove [12] that for these triangular billiards

$$K(0) = \frac{n + \epsilon(n)}{3(n-2)} \tag{36}$$

where $\epsilon(n) = 0$ for n odd, $\epsilon(n) = 2$ for n even but not divisible by 3, and $\epsilon(n) = 6$ for n even and divisible by 3. (Note that $K(0) > 1/3$.) These values of $K(0)$ are different from any known distributions. The existence of non-zero $K(0)$ leads in particular to the linear grows of the number variance, $\Sigma^2(L) \to K(0)L$ when $L \to \infty$.

The contributions from diffractive and non-diagonal terms are more difficult to find. The main problem is the divergent character of the diffraction coefficient which leads to the existence of terms as in eq. (29). Such terms grow too quickly with l and cannot be treated in the diagonal approximation. These terms should be cancelled by other terms and this cancellation is a delicate procedure.

4. Billiards with Aharonov-Bohm Flux Line

Another interesting diffractive model close to the above polygonal billiards is a rectangular billiard with the Aharonov-Bohm flux line [1]. This model is defined by the Schrödinger equation

$$[E + (\partial_\mu - iA_\mu)^2]\Psi = 0 \tag{37}$$

with the vector potential of a flux line $A_\phi = \alpha/r$ and, say, the Dirichlet boundary conditions on the boundary of the billiard.

The introduction of the flux line does not change classical trajectories. The only difference is that any time the trajectory encircles the flux line it is necessary to add the phase $2\pi\alpha$ to the semiclassical expansion (1).

The exact quantum-mechanical solution of the flux line in the whole space gives the value of the diffraction coefficient for the scattering on the flux line [1, 24]

$$D(\theta_f, \theta_i) = \frac{2\sin\pi\alpha}{\cos(\frac{\theta_f-\theta_i}{2})} e^{i(\theta_f-\theta_i)/2} \, . \tag{38}$$

This diffraction coefficient diverges in the forward direction and, as for polygonal billiards, the trace formula requires the careful study of multiple forward diffraction. This can be done, as in section 3, by using the (generalized) Kirchoff approximation [11]. For example, the contribution to the trace formula from the simplest diffractive orbit parallel to the base of the rectangle (with 2-points of forward diffraction) takes the form

$$d(E) = -\frac{2\sqrt{l_0(a-l_0)}}{\pi^2 k} \sin^2 \pi\alpha \cos(2ka) \tag{39}$$

where a is the base of the rectangle and l_0 is the distance of the flux line from the rectangle side. Note that this expression has the same dependence on momentum and periodic orbit length as eq. (29).

For rectangular billiard with the flux line it is also possible to compute the value of the 2-point correlation function at small τ [12] When the ratios x_0/a and y_0/b of coordinates of the flux line to the corresponding rectangle sides are non-commensurable irrational numbers

$$K(0) = 1 - 3\bar{\alpha} + 4\bar{\alpha}^2 \tag{40}$$

where $\bar{\alpha}$ is the fractional part of α, $0 \le \bar{\alpha} \le 1/2$ and it is symmetric with respect to $\bar{\alpha} = 1/2$ when $1/2 \le \bar{\alpha} \le 1$.

The numerically computed spectral statistics for this model shows considerable deviations from any known distributions.

5. Chaotic Systems with Diffractive Center

Consider now a chaotic system perturbed by a point-like scatterer. New energy levels, as before, should be computed from eq. (12). The only difference with the case considered in section 2 is that here we shall assume that old (non-perturbed) energy levels, e_n, are distributed as eigenvalues of one of the standard random matrix ensembles [18]

$$P(\{e_k\}) \propto \prod_{i<j} |e_i - e_j|^{\beta} \tag{41}$$

where $\beta = 1, 2, 4$ for, respectively, GOE, GUE, and GSE cases. (We ignore here the one-body potential needed for the confinement of eigenvalues. One can e.g. assume that the eigenvalues are lying on a large radius circle.)

Within the random matrix theory the distribution of $v_k = |\psi(\vec{x_0})|$ is independent of the eigenvalues and is given by [18]

$$P(\{v_k\}) = \prod_i (\frac{\beta N}{2\pi})^{1-\beta/2} \exp(-\frac{\beta}{2} N v_i^2)\,. \tag{42}$$

The knowledge of the statistical distributions of the poles and the residues of eq. (12) permits the calculation of the distribution of new energy levels. The first step has been done in ref. [3] where the joint distribution of the new, E_j, and the old, e_i, levels has been computed

$$P(\{E_j\}, \{e_k\}) \propto \frac{\prod_{i<j}(e_i - e_j)(E_i - E_j)}{\prod_{i,j} |e_i - E_j|^{1-\beta/2}} \exp(-\rho \sum_i (E_i - e_i))\,, \tag{43}$$

where $\rho = \beta/(2\lambda N)$ and, due to the positivity of $|v_k|^2$, $e_i \leq E_i \leq e_{i+1}$. Here we assume that $\rho > 0$ and energy levels are ordered, $e_1 \leq e_2 \leq \cdots \leq e_N$.

The resulting distribution of the new eigenvalues is defined by the expression

$$P(\{E_j\}) = \int_{-\infty}^{E_1} de_1 \int_{E_1}^{E_2} de_2 \cdots \int_{E_{N-1}}^{E_N} de_N P(\{E_j\}, \{e_k\})\,. \tag{44}$$

In ref. [13] it was demonstrated that these integrals can be computed and the distribution $P(\{E_j\})$ has exactly the same form as the distribution of non-perturbed eigenvalues given by eq. (34)

$$P(\{E_j\}) \propto \prod_{i<j} |E_i - E_j|^{\beta}\,. \tag{45}$$

This result is not surprising. The random matrix theory distribution is often considered as the result of action of large number of small-size scatterers. Consequently, the addition of one more center of scattering should not change the spectral statistics.

In ref. [25] the contribution to the 2-point correlation form-factor from diffractive orbits in the diagonal approximation has been computed for f-dimensional chaotic systems

$$K_d(\tau) = \frac{\tau^2}{8\beta\pi^2} (\frac{k}{2\pi})^{2f-4} \int |D(\vec{n}, \vec{n}')|^2 dO_{\vec{n}} dO_{\vec{n}'}\,. \tag{46}$$

According to the above arguments this additional contribution should be removed by other (non-diagonal) terms. In ref. [13] it has been demonstrated that it is the interference between diffractive orbits in the forward direction and periodic orbits close to the diffraction center that exactly cancel this term. In the derivation of this result two main ingredients were important. First, the uniformity principle for periodic orbits of chaotic systems, in particular

$$\sum_p \frac{\chi(\vec{q}_p,\vec{p}_p)}{|\det(M_p-1)|}\delta(T-T_p)=\frac{1}{\Sigma}\int \chi(\vec{q},\vec{p}\,)d^{f-1}qd^{f-1}p \tag{47}$$

where $\chi(\vec{q},\vec{p}\,)$ is a test function defined on a Poincaré surface of section $(\vec{q},\vec{p}\,)$. $(\vec{q}_p,\vec{p}_p)$ are coordinates of points of intersections of a periodic orbit with the surface of section. T_p is the periodic orbit period and $\Sigma=\int d^f q d^f p\delta(E-H(q,p))$ is the phase-space volume of the constant energy surface.

The second important point is the optical theorem for the diffractive coefficient which is a consequence of the unitarity of the scattering S-matrix [13]

$$\text{Im } D(\vec{n},\vec{n})=-\frac{1}{8\pi}(\frac{k}{2\pi})^{f-2}\int |D(\vec{n},\vec{n}')|^2 dO_{\vec{n}'}\,. \tag{48}$$

Using these relations one recovers [13] the invariance of random matrix results under the addition of short-range scatterers from the semiclassical methods. One can easily check that ignoring the optical theorem can change the spectral statistics.

6. Conclusion

In this paper a few typical examples of diffractive models has been considered. An integrable model with a small-size scatterer with a constant diffraction coefficient is the simplest and the most investigated case. Under the assumption that the unperturbed system obeys the Poisson statistics it is possible rigorously compute the spectral statistics of this model. The main result is that adding a diffractive center changes completely the spectral statistics. The resulting statistics is characterized by level repulsion and depends on the value of the diffractive coefficient. Models with finite diffraction coefficient permit the construction of a specific perturbation theory which allows the term-by term computation of expansion of 2-point correlation function $R_2(\epsilon)$ (and other correlation functions as well) in powers of $1/\epsilon$. The models such as pseudo-integrable polygonal billiards and rectangular billiards with a flux line are another type of diffractive models characterized by divergence of the diffraction coefficient. In this case semiclassical contribution of classical periodic orbits to the trace formula differs from unperturbed case and the spectral statistics of these models characterized, first of all, by non-standard value of the 2-point correlation form factor at small τ, $K(0)<1$, and consequently the linear growths of the number variance. For higher order terms of the expansion of correlation functions, the cancellation of rapidly growing terms (as in (29)) is not yet fully understood, and calculations are in progress. A simple example of a such cancellation is provided by chaotic systems with a point-like scatterer. Though it is evident (and can rigorously be proved) that, if the spectral statistics of unperturbed chaotic system is described by one of standard random matrix ensembles, the addition of a diffraction center cannot change the statistics, in semiclassical approach this invariance requires a compensation between diagonal and off-diagonal terms. The important point is that this strong cancellation is a general phenomenon connected mostly with the unitarity of quantum mechanical scattering.

Diffractive systems are an interesting and promising class of quantum-mechanical models. Their properties, in general, differ from standard expectations but often are accessible to analytical calculations and they open new perspectives in the investigation of semiclassical limit in quantum mechanics.

Acknowledgments

The author is very grateful to C. Schmit, N. Pavloff, P. Leboeuf, U. Gerland, and O. Giraud, whose collaboration permits to obtain most of the results of this paper.

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Laboratoire de Physique Théorique et
Modelès Statistiques,
Université de Paris XI, Bât. 100,
F–91405 Orsay Cedex, France
E-mail address: `bogomol@ipno.in2p3.fr`