A Conjecture of De Giorgi on Symmetry for Elliptic Equations in $\mathbb{R}^n$

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Abstract. In 1978 De Giorgi formulated the following conjecture. *Let u be a solution of $\Delta u = u^3 - u$ in all of $\mathbb{R}^n$ such that $|u| \leq 1$ and $\partial_{x_n} u > 0$ in $\mathbb{R}^n$. Is it true that all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if* $n \leq 8$? Equivalently, does u depend only on one variable? When $n = 2$, this conjecture was proved in 1997 by N. Ghoussoub and C. Gui. More recently, L. Ambrosio and the author have proved it for $n = 3$. The question, however, remains open for $n \geq 4$. A connection with the Bernstein problem for minimal hypersurfaces suggests that the conjecture may be true at least if $n \leq 8$. The results for $n = 2$ and 3 apply also to the equation $\Delta u = F'(u)$ for every nonlinearity $F \in C^2$.

1. Introduction

In 1978 De Giorgi [7] stated the following conjecture:

Conjecture. ([7]) *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\Delta u = u^3 - u \quad \textit{in } \mathbb{R}^n$$

such that

$$|u| \leq 1 \qquad \textit{and} \quad \partial_{x_n} u > 0$$

in the whole $\mathbb{R}^n$. Is it true that all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \leq 8$?

When $n = 2$, this conjecture was proved by Ghoussoub and Gui [9] in 1997. More recently, Ambrosio and the author [2] have proved it in dimension $n = 3$. The conjecture, however, remains open for all $n \geq 4$.

Note that the level sets of u are hyperplanes if and only if u depends only on one variable. Thus, the question of De Giorgi is concerned with the one-dimensional character or symmetry of bounded solutions u of semilinear elliptic equations in the whole space $\mathbb{R}^n$, under the assumption that u is monotone in one direction,

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say, $\partial_{x_n} u > 0$ in $\mathbb{R}^n$. The proofs for $n = 2$ and 3 use some techniques developed by Berestycki, Caffarelli and Nirenberg in [4] for the study of symmetry properties of solutions of semilinear equations in half spaces.

We will see in section 3 that the conjecture of De Giorgi has some relations with the theories of minimal hypersurfaces and phase transitions. In particular, a connection with the Bernstein problem for minimal graphs is probably the reason why De Giorgi [7] includes "at least if $n \le 8$" in his statement of the question.

The positive answers to the conjecture in dimensions two and three apply not only to the scalar Ginzburg-Landau equation $\Delta u + u - u^3 = 0$, but to general nonlinearities. Indeed, we have the following:

Theorem 1.1. ([9, 2, 1]) *Assume that $F \in C^2(\mathbb{R})$. Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \text{in } \mathbb{R}^n \tag{1}$$

satisfying

$$\partial_{x_n} u > 0 \text{ in } \mathbb{R}^n . \tag{2}$$

If $n = 2$ or $n = 3$, then all level sets of u are hyperplanes, i.e., there exist $a \in \mathbb{R}^n$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \quad \text{for all } x \in \mathbb{R}^n .$$

For the model case $F'(u) = u^3 - u$, the function $\tanh(s/\sqrt{2})$ is the unique (up to a translation of the independent variable s) one-dimensional solution of the equation. Hence, in this case the conclusion of theorem 1.1 is that

$$u(x) = \tanh\left(\frac{a \cdot x - c}{\sqrt{2}}\right) \quad \text{in } \mathbb{R}^n ,$$

for some $c \in \mathbb{R}$ and $a \in \mathbb{R}^n$, with $|a| = 1$ and $a_n > 0$. It is also easy to verify that if $F \in C^2(\mathbb{R})$ satisfies $F > F(-1) = F(1)$ in $(-1, 1)$ and $F'(-1) = F'(1) = 0$, then $h'' - F'(h) = 0$ has an increasing solution $h(s)$ satisfying $\lim_{s\to\pm\infty} h(s) = \pm 1$, which is unique up to a translation in s.

Several articles have also considered the question of De Giorgi in a slightly simpler version. It consists of assuming the hypothesis of the conjecture and, in addition, that

$$\lim_{x_n\to\pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1} . \tag{3}$$

Here, the limits are *not* assumed to be uniform in $x' \in \mathbb{R}^{n-1}$. Even in this simpler form, the conjecture was first proved in [9] for $n = 2$, in [2] for $n = 3$, and it remains open for $n \ge 4$.

In theorem 1.1 the direction a of the variable on which u depends is not known apriori. Indeed, if u is a one-dimensional solution satisfying (2), we can "slightly" rotate coordinates to obtain a new one-dimensional solution still satisfying (2). The same remark applies to assumption (3). Instead, if one further assumes that the limits in (3) are uniform in $x' \in \mathbb{R}^{n-1}$, then an apriori choice of the direction a is imposed, namely $a \cdot x = x_n$. Furthermore, with this additional assumption one knows apriori that every level set of u is contained between two parallel hyperplanes. In this respect, it has been established in [3, 5, 8] (independently and using different techniques) that, for every dimension n, if the limits in (3) are assumed to be uniform in $x' \in \mathbb{R}^{n-1}$ then u only depends on the variable x_n, that is, $u = u(x_n)$. This result applies to equation (1) for various classes of nonlinearities F which always include the Ginzburg-Landau model $\Delta u + u - u^3 = 0$.

The first partial result towards the question of De Giorgi was proved by Modica and Mortola [14] in 1980. They gave a positive answer to the conjecture when $n = 2$ under the additional assumption that the level sets of u are the graphs of an equi-Lipschitz family of functions. Note that, since $\partial_{x_n} u > 0$, each level set of u is the graph of a function of x'. In 1985 Modica [11] proved that if $F \geq 0$ in $\mathbb{R}$ then every bounded solution u of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfies the gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq F(u) \quad \text{in } \mathbb{R}^n . \tag{4}$$

In 1994 Caffarelli, Garofalo and Segala [6] generalized this bound to more general equations. They also showed that, if equality occurs in (4) at some point of $\mathbb{R}^n$, then the conclusion of the conjecture is true.

2. Sketch of the Proofs in Dimensions Two and Three

The proof of theorem 1.1 relies on the following method, already used by Berestycki, Caffarelli and Nirenberg in [4] to establish (also for low dimensions) a very general result on the symmetry of positive solutions in half spaces. The idea is to consider the functions

$$\varphi := \partial_{x_n} u > 0 \quad \text{and} \quad \sigma_i := \frac{\partial_{x_i} u}{\partial_{x_n} u}$$

for each $i \in \{1, \dots, n-1\}$. Since

$$\varphi^2 \nabla \sigma_i = \partial_{x_n} u \, \nabla \partial_{x_i} u - \partial_{x_i} u \, \nabla \partial_{x_n} u$$

and $\Delta \partial_{x_j} u = F''(u) \partial_{x_j} u$, the function σ_i satisfies the equation

$$\operatorname{div}(\varphi^2 \nabla \sigma_i) = 0 \quad \text{in } \mathbb{R}^n . \tag{5}$$

The goal is to prove that σ_i is constant, since then the theorem follows. Indeed, if for every $i \in \{1, \dots, n-1\}$ we have

$$\partial_{x_i} u = c_i \, \partial_{x_n} u$$

for some constant c_i, then u is constant along the $n-1$ directions $\partial_{x_i} - c_i \partial_{x_n}$. We then conclude that u is a function of the variable $a \cdot x$ alone, where $a = (c_1, \dots, c_{n-1}, 1)$.

The proof that σ_i is necessarily constant in dimensions two and three uses the following Liouville theorem for equation (5). Its proof is already contained in the paper [4] by Berestycki, Caffarelli and Nirenberg; see also [2].

Proposition 2.1. *Let* $\varphi \in L^\infty_{\rm loc}(\mathbb{R}^n)$ *be positive in* $\mathbb{R}^n$. *Suppose that* $\sigma \in H^1_{\rm loc}(\mathbb{R}^n)$ *satisfies*

$$\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \geq 0 \quad \text{in } \mathbb{R}^n \tag{6}$$

in the distributional sense. For every $R > 1$, *let* $B_R = \{|x| < R\}$ *and assume that*

$$\int_{B_R} (\varphi\sigma)^2 \leq CR^2\,, \tag{7}$$

for some constant C *independent of* R. *Then* σ *is constant.*

The proof of this proposition is based in a simple Cacciopoli type estimate for the function σ.

Returning to the conjecture of De Giorgi, we note that

$$\varphi\sigma_i = \partial_{x_i} u\,.$$

Hence, assumption (7) in the Liouville theorem will hold if one shows that, for each $R > 1$,

$$\int_{B_R} |\nabla u|^2 \leq CR^2 \tag{8}$$

for some constant C independent of R. Using standard local elliptic estimates and that $u \in L^\infty(\mathbb{R}^n)$ is a solution of $\Delta u - F'(u) = 0$, we deduce that ∇u also belongs to $L^\infty(\mathbb{R}^n)$. Therefore, estimate (8) is obviously true if $n = 2$. This finishes the proof of theorem 1.1, and hence of the conjecture of De Giorgi, when $n = 2$.

In order to prove the theorem in dimension three, it suffices to establish (8). This bound is a consequence (when $n = 3$) of new energy estimates established in [2] and [1]. A first estimate which holds in all dimensions is given by the following:

Theorem 2.2. ([2]) *Let* $F \in C^2(\mathbb{R})$ *and* u *be a bounded solution of* $\Delta u - F'(u) = 0$ *in* $\mathbb{R}^n$. *Assume that*

$$\partial_{x_n} u > 0 \text{ in } \mathbb{R}^n \text{ and } \quad \lim_{x_n \to +\infty} u(x', x_n) = 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}\,.$$

Then, for every $R > 1$, *we have*

$$\int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right\} dx \leq CR^{n-1} \tag{9}$$

for some constant C *independent of* R.

Note that the previous estimate is clearly true (and optimal) for one-dimensional solutions. The energy functional in B_R,

$$E_R(u) = \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right\} dx\,,$$

has $\Delta u - F'(u) = 0$ as Euler-Lagrange equation. In 1989 Modica [12] proved that if $F - F(1) \geq 0$ in $\mathbb{R}$ and u is a bounded solution of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$, then the quantity

$$\frac{E_R(u)}{R^{n-1}}$$

is a nondecreasing function of R. Theorem 2.2 establishes that this quotient is, in addition, bounded from above. Moreover, the upper bound of theorem 2.2 is optimal. Indeed, if $E_R(u)/R^{n-1} \to 0$ as $R \to \infty$ then the monotonicity formula of Modica would give that $E_R(u) = 0$ for every $R > 0$, and hence that u is constant in $\mathbb{R}^n$.

The proof of the energy estimate of theorem 2.2 relies on the consideration of the path of functions $u^t(x) = u(x', x_n + t)$ connecting u (for $t = 0$) and 1 (for $t = +\infty$). Note that the functions u^t have different boundary values on $\partial B_R(0)$, but they are all solutions of the same Euler-Lagrange equation. The desired energy estimate follows from a bound for the variation of energy with respect to t, in a fixed ball $B_R(0)$, which is obtained using the key hypothesis $\partial_{x_n} u > 0$; see [2].

For the model case $F'(u) = u^3 - u$, we have $F(u) = (1 - u^2)^2/4$ and hence $F(u) - F(1) \geq 0$. It follows then that (8) is a consequence of (9) when $n = 3$. Moreover, the assumption $\lim_{x_n \to +\infty} u(x', x_n) = 1$ in theorem 2.2 may be removed when $n = 3$ (see [2]). In this way, the proof of the conjecture of De Giorgi in dimension three is completed. The argument applies not only to $F(u) = (1-u^2)^2/4$ but also to a large class of nonlinearities F.

The proof of the conjecture in dimension three for *every* nonlinearity $F \in C^2$ (see theorem 1.1) has been obtained by Alberti, Ambrosio and the author in [1]. It relies on the following local minimality property of u.

Theorem 2.3. ([1]) *Let $F \in C^2(\mathbb{R})$ and let u be a bounded solution of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfying $\partial_{x_n} u > 0$ in $\mathbb{R}^n$. Consider the functions*

$$\underline{u}(x') = \lim_{x_n \to -\infty} u(x', x_n) \quad \text{and} \quad \overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n)\,.$$

Then

$$\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + F(u) \right\} dx \leq \int_{B_R} \left\{ \frac{1}{2}|\nabla v|^2 + F(v) \right\} dx$$

for every function $v \in C^1(\mathbb{R}^n)$ such that $\{v \neq u\}$ has compact closure contained in B_R and

$$\underline{u}(x') < v(x', x_n) < \overline{u}(x') \quad \text{for all } x = (x', x_n) \in B_R\,. \tag{10}$$

The theorem states that every solution u which is monotone in one direction is a minimizer of the energy in every ball with respect to the class of functions v with same boundary values as u and satisfying condition (10). Before we discovered theorem 2.3, a hint for its validity had been given in [2], where we pointed out that the condition $\partial_{x_n} u > 0$ implies (by the maximum principle) that the second variation of energy at u is nonnegative definite with respect to compact perturbations —a necessary condition for minimality.

The proof of theorem 2.3 is based on the construction of an appropriate calibration or null-lagrangian (a classical tool in the Calculus of Variations) associated to the extremal field $u^t(x) = u(x', x_n + t)$ for $t \in \mathbb{R}$. Note that the graphs of the functions u^t are disjoint due to the monotonicity condition $\partial_{x_n} u > 0$.

In [1] we use theorem 2.3 to obtain an improved version of the energy estimate of theorem 2.2, in which $F(1)$ is replaced by the infimum of F on the range of u. The improved estimate leads to the gradient bound (8), and hence to the one-dimensional symmetry result (theorem 1.1 in dimension three) for all nonlinearities $F \in C^2$.

3. Relation with the Bernstein Problem for Minimal Graphs

In this section we present the heuristic argument that relates the conjecture of De Giorgi with the Bernstein problem for minimal graphs. We suppose that $F(u) = (1-u^2)^2/4$, for simplicity. With u as in the conjecture, one considers the blown-down sequence

$$u_R(y) = u(Ry) \quad \text{for } y \in B_1 \subset \mathbb{R}^n \,,$$

and the penalized energy of u_R in B_1

$$H_R(u_R) = \int_{B_1} \left\{ \frac{1}{2R} |\nabla u_R|^2 + RF(u_R) \right\} dy \,.$$

Note that $H_R(u_R)$ is a bounded sequence, by theorem 2.2. By a result of Modica and Mortola [13], as $R \to \infty$ the functionals H_R Γ-converge to a functional which is finite only for characteristic functions with values in $\{-1, 1\}$ and equal (up to the multiplicative constant $2\sqrt{2}/3$) to the area of the hypersurface of discontinuity.

The sequence u_R is therefore expected to converge to a characteristic function whose hypersurface of discontinuity S has minimal area, by the local minimality property of theorem 2.3. Moreover, S is expected to be the graph of a function defined on $\mathbb{R}^{n-1}$ (since the level sets of u are graphs due to hypothesis $\partial_{x_n} u > 0$), possibly with vertical parts. The set S describes the behavior at infinity of the level sets of u. The conjecture of De Giorgi states that the level sets of u are hyperplanes. The connection with the Bernstein problem (see Chapter 17 of [10] for a complete survey on this topic) is due to the fact that every minimal graph of a function defined on $\mathbb{R}^m = \mathbb{R}^{n-1}$ is known to be a hyperplane whenever $m \leq 7$, i.e., $n \leq 8$. Hence, S is expected to be a hyperplane when $n \leq 8$. Using the local minimality property of u (theorem 2.3), we have given in [1] a rigorous proof of this convergence result, up to subsequences of radii $R_k \to \infty$.

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