New Invariants of Legendrian Knots

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Abstract. We discuss two different ways to construct new invariants of Legendrian knots in the standard contact $\mathbf{R}^3$. These invariants are defined combinatorially, in terms of certain planar projections, and (sometimes) allow to distinguish Legendrian knots which are not Legendrian isotopic but have the same classical invariants.

1. Introduction

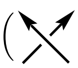
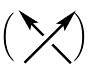
A smooth knot L in the standard contact space $(\mathbf{R}^3, \alpha) = (\{(p, q, u)\}, du - pdq)$ is called Legendrian if the restriction of α to L vanishes (or, in other words, if L is everywhere tangent to the 2-plane distribution given by $\alpha = 0$). The general, open, problem of the theory of Legendrian knots is to give a classification of Legendrian knots up to Legendrian isotopy. The latter is defined as follows: two Legendrian knots are said to be Legendrian isotopic if they can be connected by a path in the space of Legendrian embeddings (or, equivalently, if one can be sent to another by a diffeomorphism g of $\mathbf{R}^3$ such that $g^*\alpha = \varphi\alpha$, where $\varphi > 0$). It is easy to show that every smooth knot is isotopic to a Legendrian one. However, two different Legendrian knots belonging to the same smooth isotopy class may be not Legendrian isotopic.

The so-called classical invariants of an oriented Legendrian knot L are defined as follows. First of them is just the smooth isotopy type of L. The Thurston-Bennequin number $\beta(L)$ of L is the linking number (with respect to the orientation defined by $\alpha \wedge d\alpha$) between L and $s(L)$, where s is a small shift along the u direction. The Maslov number $m(L)$ (which actually is an invariant of Legendrian immersion) is twice the rotation number of the projection of L to the (p, q) plane (or, equivalently, the value of the Maslov 1-cohomology class on the fundamental class of L). That m and β are indeed invariant under Legendrian isotopy follows since vectors tangent to a Legendrian curve are never parallel to the u axis. The change of orientation on L changes the sign of $m(L)$ and preserves $\beta(L)$.

One can ask whether there exists a pair of Legendrian knots which have the same classical invariants but are not Legendrian isotopic. Eliashberg and Fraser showed that this cannot happen when the knots are trivial as smooth knots [4, 6]. However, it turns out that there exist Legendrian knots with the same classical invariants but not Legendrian isotopic to each other. In the present talk, we discuss two entirely different new constructions of invariants of Legendrian knots [1, 2] that sometimes allow to distinguish Legendrian knots with the same classical invariants. These new invariants are combinatorially defined in terms of projections of L to certain planes. They does not change when the orientation of the knot changes, so essentially they are invariants of non-oriented Legendrian knots. There is no visible relation between the two constructions. Neither of them is known to provide a priori stronger invariants; there are examples where the first construction works better. Both constructions extend, with minor modifications, to the case of Legendrian links, which we do not discuss here.

In order to visualize a knot in $\mathbf{R}^3$, it is convenient to use its projection to a plane. In the Legendrian case, the character of the resulting picture will depend on the choice of the projection. We shall use two of them: $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$, $(p, q, u) \mapsto (p, q)$ and $\sigma\colon \mathbf{R}^3 \to \mathbf{R}^2$, $(p, q, u) \mapsto (q, u)$.

We say that a Legendrian knot $L \subset \mathbf{R}^3$ is generic with respect to π, or π-generic, if all self-intersections of the immersed curve $\pi(L)$ are transverse double points. We can represent a π-generic Legendrian knot L by its (π-)diagram: the curve $\pi(L) \subset \mathbf{R}^2$, at every crossing of which the overpassing branch (the one with the greater value of u) is marked. Of course, not every abstract knot diagram in $\mathbf{R}^2$ is a diagram of a Legendrian knot, or is oriented diffeomorphic to such. Thus it requires a bit of extra work (which we are going to skip here) to check that the diagrams we draw indeed correspond to Legendrian knots. The Thurston-Bennequin number of a π-generic Legendrian knot can be computed by counting the crossings of its diagram with signs:

$\beta(L) = \#\Big($  $\Big) - \#\Big($  $\Big)$,

(where the p axis is horizontal and the q axis vertical). The Maslov number $m(L)$ is twice the rotation number of $\pi(L)$. Properties of the projection σ are discussed in section 3.

Theorem 1.1. ([1, 2]) *Legendrian knots L, L' whose π-diagrams are given in figure 1 have the same classical invariants (smooth knot type 5_2, $m = 0$, $\beta = 1$) but are not Legendrian isotopic.*

It is easy to check that the classical invariants are the same. Two different constructions of invariants distinguishing L and L' are given in section 2 and section 3.

2. First Construction: Differential Algebra

2.1. Introduction

In order to construct new invariants of Legendrian knots, we associate with every π-generic Legendrian knot L a differential graded algebra (A, ∂) over $\mathbf{Z}_2$. Our constructions can be viewed as an algebraically refined combinatorial version of certain particular case of a more general Morse-Witten-type theory, an outline of

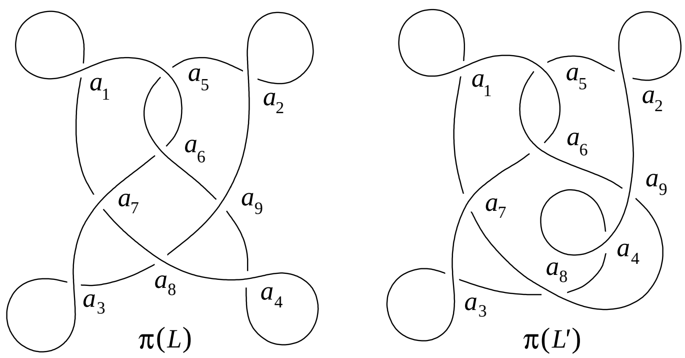


FIGURE 1

which was given by Eliashberg, Givental, and Hofer [5]. Similar results were also announced by Eliashberg.

2.2. Definitions and results

Let $\{a_1, \dots, a_n\}$ be the set of crossings of $Y = \pi(L)$. Define A to be the tensor algebra (free associative unital algebra) $T(a_1, \dots, a_n)$ generated by $a_1, \dots, a_n$. The grading on A, which takes values in the group $\mathbf{Z}/m(L)\mathbf{Z}$, is defined as follows. Given a crossing a, consider the points $z_+, z_- \in L$ such that $\pi(z_+) = \pi(z_-) = a$ and $u(z_+) > u(z_-)$ (where $u(z)$ denotes the u coordinate of a point $z \in \mathbf{R}^3$). These points divide L into two pieces, γ_1 and γ_2, which we orient from z_- to z_+. We can assume, without loss of generality, that the intersecting branches are orthogonal at a. Then, for $\varepsilon \in \{1, 2\}$, the rotation number of the curve $\pi(\gamma_\varepsilon)$ has the form $N_\varepsilon/2+1/4$, where $N_\varepsilon \in \mathbf{Z}$. Clearly, $N_1 - N_2$ is equal to $\pm m(L)$. Hence N_1 and N_2 represent the same element of the group $\Gamma = \mathbf{Z}/m(L)\mathbf{Z}$, which we define to be the degree of a.

We are going to define the differential ∂. For every natural k, fix a (curved) convex k-gon $\Pi_k \subset \mathbf{R}^2$ whose vertices $x_0^k, \dots, x_{k-1}^k$ are numbered anticlockwise. The form $dp \wedge dq$ defines an orientation on $\mathbf{R}^2$. Denote by $W_k(Y)$ the collection of smooth orientation-preserving immersions $f\colon \Pi_k \to \mathbf{R}^2$ such that $f(\partial \Pi_k) \subset Y$. Note that $f \in W_k(Y)$ implies $f(x_i^k) \in \{a_1, \dots, a_n\}$. Consider the set of non-parametrized immersions $\widetilde{W}_k(Y)$, which is the quotient of $W_k(Y)$ by the action of the group $\{g \in \mathrm{Diff}_+(\Pi_k) \mid g(x_i^k) = x_i^k\}$.

The diagram Y divides a neighbourhood of each of its crossings into four sectors. We call positive two of them which are swept out by the underpassing curve rotating anticlockwise, and negative the other two (see figure 2). For each vertex x_i^k of the polygon Π_k, a smooth immersion $f \in \widetilde{W}_k(Y)$ maps its neighbourhood in Π_k to either a positive or a negative sector; we shall say that x_i^k is, respectively, a positive or negative vertex for f. Define the set $W_k^+(Y)$ to consist of immersions $f \in \widetilde{W}_k(Y)$ such that the vertex x_0^k is positive for f, and all other vertices are negative. Let $W_k^+(Y, a_j) = \{f \in W_k^+(Y) \mid f(x_0^k) = a_j\}$.

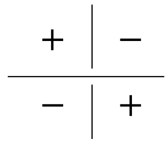


FIGURE 2

Denote $A_1 = \{a_1, \dots, a_n\} \otimes \mathbf{Z}_2 \subset A$, $A_k = (A_1)^{\otimes k}$. Then $A = \oplus_{l=0}^{\infty} A_l$. There is a decomposition $\partial = \sum_{k \geq 0} \partial_k$, where $\partial_k(A_i) \in A_{i+k-1}$. Define

$$\partial_k(a_j) = \sum_{f \in W_{k+1}^{+}(Y, a_j)} f(x_1) \cdots f(x_k)\,,$$

which, in particular, means $\partial_0(a_j) = \#(W_1^{+}(Y, a_j))$, and extend ∂ to A by linearity and the Leibniz rule.

Theorem 2.1. *The differential* ∂ *is well defined. We have* $\deg(\partial) = -1$ *and* $\partial^2 = 0$.

The above theorem allows us to consider the homology ring $\ker(\partial)/\operatorname{im}(\partial)$, which turns out to be a Legendrian isotopy invariant:

Theorem 2.2. *Let* (A, ∂), (A', ∂') *be the differential graded algebras associated with Legendrian isotopic (π-generic) Legendrian knots L, L'. Then the homology rings of* (A, ∂) *and* (A', ∂') *are isomorphic as graded rings.*

The hard part in the proof of theorem 2.1 is to show that $\partial^2 = 0$. The proof of this fact mimics, in a combinatorial way, the classical gluing-compactness argument of the Floer theory (cf. [7]). The proof of theorem 2.2 involves a careful study of the behaviour of the differential graded algebra associated with a Legendrian knot when the diagram goes through elementary bifurcations (Legendrian Reidemeister moves).

2.3. Examples

2.3.1. Consider the Legendrian knots L_1, L_2, L_3 whose diagrams are given in figure 3. The diagram $Y_1 = \pi(L_1)$ is the simplest possible diagram of a Legendrian knot. The classical invariants of L_1 are as follows: $m(L_1) = 0$, $\beta(L_1) = -1$, the knot is an unknot in the smooth category. Since $m(L_1) = 0$, the grading on the algebra $A = T(a)$ takes values in $\mathbf{Z}$. We have $\deg(a) = 1$. The set of immersions $\widetilde{W}_k(Y_1)$ is empty for $k > 1$ and consists of two elements $f_1, f_2 \in W_1^{+}(Y_1, a)$, whose images are the closures of the two bounded components of $\mathbf{R}^2 \setminus Y_1$. Hence $\partial(a) = 1 + 1 = 0$.

The Legendrian knot L_2 is a right-handed (with respect to the orientation defined by $\alpha \wedge d\alpha$) trefoil as a smooth knot. We have $m(L_2) = 0$, $\beta(L_2) = 1$, $A = T(a_1, \dots, a_5)$, where $\deg(a_1) = \deg(a_2) = 1$, $\deg(a_3) = \deg(a_4) = \deg(a_5) = 0$, $\partial(a_1) = 1 + a_3 + a_5 + a_3 a_4 a_5$, $\partial(a_2) = 1 + a_3 + a_5 + a_5 a_4 a_3$, $\partial(a_3) = \partial(a_4) = \partial(a_5) = 0$.

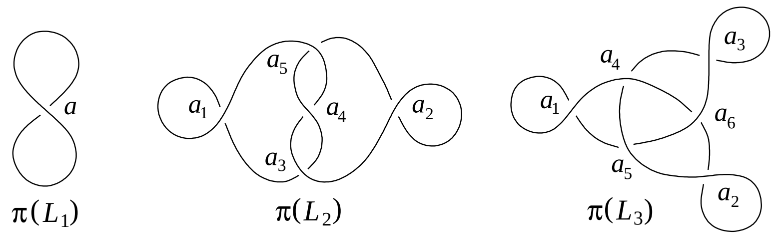


FIGURE 3

The Legendrian knot L_3 is a left-handed trefoil as a smooth knot. We have $m(L_3) = \pm 2$, $\beta(L_3) = -6$. $A = T(a_1, \dots, a_6)$, where all generators have degree $1 \in \mathbf{Z}_2$, $\partial(a_1) = 1 + a_4 a_6$, $\partial(a_2) = 1 + a_5 a_4$, $\partial(a_3) = 1 + a_6 a_5$, $\partial(a_4) = \partial(a_5) = \partial(a_6) = 0$.

2.3.2. Let $(A, \partial) = (T(a_1, \dots, a_9), \partial)$ be the differential graded algebra associated with the Legendrian knot L given in figure 1. We have $m(L) = 0$, $\beta(L) = 1$, $\deg(a_i) = 1$ for $i \leq 4$, $\deg(a_5) = 2$, $\deg(a_6) = -2$, $\deg(a_i) = 0$ for $i \geq 7$, $\partial(a_1) = 1 + a_7 + a_7 a_6 a_5$, $\partial(a_2) = 1 + a_9 + a_5 a_6 a_9$, $\partial(a_3) = 1 + a_8 a_7$, $\partial(a_4) = 1 + a_8 a_9$, $\partial(a_i) = 0$ for $i \geq 5$.

Let $(A', \partial) = (T(a_1, \dots, a_9), \partial)$ be the differential graded algebra associated with the Legendrian knot L' given in figure 1. We have $m(L') = 0$, $\beta(L') = 1$, $\deg(a_i) = 1$ for $i \leq 4$, $\deg(a_i) = 0$ for $i \geq 5$, $\partial(a_1) = 1 + a_7 + a_5 + a_7 a_6 a_5 + a_9 a_8 a_5$, $\partial(a_2) = 1 + a_9 + a_5 a_6 a_9$, $\partial(a_3) = 1 + a_8 a_7$, $\partial(a_4) = 1 + a_8 a_9$, $\partial(a_i) = 0$ for $i \geq 5$.

2.4. Poincaré polynomials

Homology rings of differential graded algebras are rather hard to handle. That is why we introduce another, a bit more subtle, invariant of Legendrian knots. This invariant is a finite subset of the group monoid $\mathbf{N}_0[\Gamma]$, where $\mathbf{N}_0 = \{0, 1, \dots\}$, $\Gamma = \mathbf{Z}/m(L)\mathbf{Z}$. Assume that $\partial_0 = 0$. Then $\partial_1^2 = 0$. Since $\partial(A_1) \subset A_1$, we can consider the homology $H(A_1, \partial_1) = \ker(\partial_1|_{A_1}) / \operatorname{im}(\partial_1|_{A_1})$, which is a vector space graded by the cyclic group Γ. Define the Poincaré polynomial $P_{(A,\partial)} \in \mathbf{N}_0[\Gamma]$ by

$$P_{(A,\partial)}(t) = \sum_{\lambda \in \Gamma} \dim\bigl(H_\lambda(A_1, \partial_1)\bigr) t^\lambda \,,$$

where $H_\lambda(A_1, \partial_1)$ is the degree λ homogeneous component of $H(A_1, \partial_1)$.

Define the group $\mathrm{Aut}_0(A)$ to consist of graded automorphisms of A such that for each $i \in \{1, \dots, n\}$ we have $g(a_i) = a_i + c_i$, where $c_i \in A_0 = \mathbf{Z}_2$. (of course, $c_i = 0$ when $\deg(a_i) \neq 0$). Consider the set $U_0(A, \partial)$ consisting of automorphisms $g \in \mathrm{Aut}_0(A)$ such that $(\partial^g)_0 = 0$ (where $\partial^g = g^{-1} \circ \partial \circ g$). Define

$$I(A, \partial) = \{P_{(A,\partial^g)} \mid g \in U_0(A, \partial)\} \,.$$

Since $\mathrm{Aut}_0(A)$ has at most 2^n elements, this invariant is not hard to compute. We can associate with every (π-generic) Legendrian knot L the set $I(L) = I(A_L, \partial_L)$. The set $I(L)$ can be empty (see section 4) but no examples are known where $I(L)$ contains more than one element.

Theorem 2.3. *If L is Legendrian isotopic to L' then $I(L) = I(L')$.*

Theorem 2.3 is not a direct consequence of theorem 2.2. It is proved by showing that the differential graded algebras associated with Legendrian isotopic Legendrian knots not only have isomorphic homology rings but satisfy a stricter equivalence relation: they become tame isomorphic after (iterated) stabilizations. We are skipping the details here.

By theorem 2.3, the following assertion, which is straightforward to verify, implies theorem 1.1:

Proposition 2.4. *For the knots L, L' given in figure 1, we have* $I(L) = \{t^{-2} + t^1 + t^2\}$, $I(L') = \{2t^0 + t^1\}$.

As an obvious corollary of the definitions we get

Proposition 2.5. *If $P \in I(L)$ then $P(-1) = \beta(L)$.*

Thus the relation between the Thurston-Bennequin number and the coefficients of the polynomial P is the same as between the Casson invariant of a homology 3-sphere and its Floer homology. This analogy was the starting point for developing the theory described above.

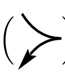
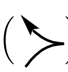
3. Second Construction: Decompositions of Fronts

In this section, we present the invariants of Legendrian knots constructed in [2]. These invariants are defined in terms of the σ-projection.

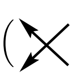
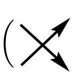
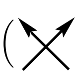
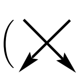
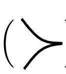
3.1. Fronts of Legendrian knots

Given a Legendrian knot $L \subset \mathbf{R}^3$, its σ-projection, or front, $\sigma(L) \subset \mathbf{R}^2$ is a singular curve with nowhere vertical tangent vectors (the q axis is horizontal and the u axis vertical). Its singularities, generically, are semi-cubic cusps. We say that L is σ-generic if, moreover, all self-intersections of $\sigma(L)$ are transverse double points with different q coordinates. Every closed planar curve with these types of singularities and nowhere vertical tangent vectors is a front of a Legendrian knot. Note that there is no need to explicitly indicate the type of a crossing of $\sigma(L)$ (which was necessary for $\pi(L)$): the overpassing branch (the one with the greater value of p) is always the one with the greater slope.

The Maslov number a σ-generic oriented Legendrian knot L can be computed by counting the right cusps of the front $\sigma(L)$ with signs depending on the orientations:

$m(L) = \#($  $) - \#($  $)$.

The Thurston-Bennequin number of L equals the sum of crossings of $\sigma(L)$ with signs minus half the total number of cusps:

$\beta(L) = \#($  $) + \#($  $) - \#($  $) - \#($  $) - \#($  $)$.

Given a σ-generic oriented Legendrian knot L, denote by $C(L)$ the set of points that correspond to cusps of $\sigma(L)$. Define the Maslov index $r\colon L \setminus C(L) \to \Gamma = \mathbf{Z}/m(L)\mathbf{Z}$ to be a locally constant function, uniquely defined up to adding a constant, whose value changes near points of $C(L)$ as shown in figure 4. We say that a crossing of $\Sigma = \sigma(L)$ is Maslov if r takes the same value on both its branches.

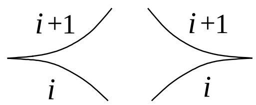


Figure 4

3.2. Admissible decompositions

Assume that $\Sigma = \sigma(L)$ is a union of closed curves $X_1, \dots, X_n$ that have finitely many self-intersections and meet each other at finitely many points. Then we call the set $\{X_1, \dots, X_n\}$ a decomposition of Σ.

We are going to formulate the main definition. A decomposition $\{X_1, \dots, X_n\}$ is called admissible if it satisfies the four conditions stated below. The first two of them are as follows:

(1) Each curve X_i bounds a topological disk: $X_i = \partial B_i$.
(2) For each $i \in \{1, \dots, n\}$, $q \in \mathbf{R}$, the set $B_i(q) = \{u \in \mathbf{R} \mid (q, u) \in B_i\}$ is either a segment, or consists of a single point u such that (q, u) is a cusp of Σ, or is empty.

Conditions (1) and (2) imply that each curve X_i has exactly two cusps (and hence the total number of cusps is $2n$). These cusps divide X_i into two pieces, on which the coordinate q is a monotone function. Near a crossing point $x \in X_i \cap X_j$, the curves X_i and X_j may look in one of three ways represented in figure 5, (one drawn by solid lines and the other by dashed ones). Conditions (1) and (2), in particular, forbid the case shown in figure 5a. We call the crossing point x switching if X_i and X_j are not smooth near x (figure 5b), and non-switching otherwise (figure 5c). We

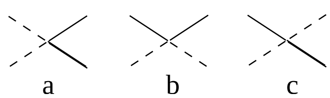


Figure 5

can now formulate the remaining two conditions:

(3) If $(q_0, u) \in X_i \cap X_j$ $(i \neq j)$ is switching then for each $q \neq q_0$ sufficiently close to q_0 the set $B_i(q) \cap B_j(q)$ either coincides with $B_i(q)$ or $B_j(q)$, or is empty.
(4) Every switching crossing is Maslov.

Denote by $\mathrm{Adm}(\Sigma)$ the set of admissible decompositions of Σ. Given $D \in \mathrm{Adm}(\Sigma)$, denote by $\mathrm{Sw}(D)$ the set of its switching points. Define $\theta(D) = \#(D) - \#(\mathrm{Sw}(D))$.

Theorem 3.1. *If σ-generic Legendrian knots $L, L' \subset \mathbf{R}^3$ are Legendrian isotopic then there exists a one-to-one mapping $g\colon \operatorname{Adm}(\sigma(L)) \to \operatorname{Adm}(\sigma(L'))$ such that $\theta(g(D)) = \theta(D)$ for all $D \in \operatorname{Adm}(\sigma(L))$. In particular, the number $\#(\operatorname{Adm}(\sigma(L)))$ is an invariant of Legendrian isotopy.*

3.3. Remarks

3.3.1. Decompositions of fronts were first considered by Eliashberg in [3]. He stated a theorem, a particular case of which asserts that if a σ-genericLegendrian knot L is Legendrian isotopic to the one whose front is shown in figure 6a then $\sigma(L)$ admits a decomposition satisfying conditions (1) and (2) (the proofs has not yet appeared). The results of [3] in their general formulation cannot be proved by the methods of [2].

3.3.2. In the definition of admissible decomposition one can skip condition (4), or weaken it by replacing the group Γ with its quotient $\mathbf{Z}/\widetilde{m}\mathbf{Z}$ (the condition becomes void when $\widetilde{m} = 1$). A complete analogue of theorem 3.1 will hold for thus defined larger sets $\operatorname{Adm}_{\widetilde{m}}(\sigma(L))$. However, it seems that the corresponding invariants are weaker.

3.3.3. The proof of theorem 3.1 goes as follows: we connect L with L' by a generic path in the space of Legendrian knots and define a canonical way to extend admissible decompositions through the points where the front is not σ-generic. The mapping g depends on the choice of the path: going along a loop in the space of Legendrian knots we can produce a non-trivial automorphism of $\operatorname{Adm}(\sigma(L))$ even when the loop is contractible. It is not clear whether this phenomenon has some geometrical meaning, or is due to imperfections of the construction.

3.4. Examples

3.4.1. Before considering examples illustrating the definition of an admissible decomposition, notice that every admissible decomposition D of a front Σ is uniquely defined by its set of switching points. Indeed, denote by $X(\Sigma)$ the set of crossings of Σ, then each subset $E \subset X(\Sigma)$ defines a decomposition $D(E)$ of Σ which near $x \in X(\Sigma)$ has the form shown in figure 5b if $x \in E$, and the form shown in figure 5c otherwise. Clearly, if $E = \operatorname{Sw}(D)$ then $D = D(E)$.

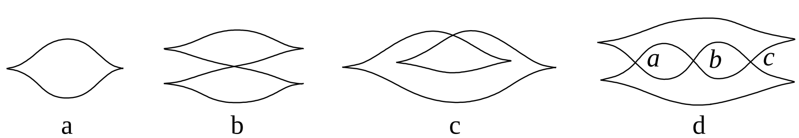


FIGURE 6

3.4.2. Each of the fronts shown in figure 6abc (the corresponding Legendrian knots are Legendrian isotopic to each other) admits a single admissible decomposition, all crossings being switching. Consider the front Σ_0 given in figure 6d. All of its three crossings a, b, c are Maslov. It is easy to that if $E \subset X(\Sigma_0)$ is empty or consists of 2 elements then the decomposition $D(E)$ consists of a single curve, and condition (1) is not satisfied. It is straightforward to check that each of the subsets $\{a\}$, $\{c\}$, $\{a, b, c\}$ defines an admissible decomposition. The decomposition $D(\{b\})$ is not admissible since condition (3) is not satisfied at the point b.

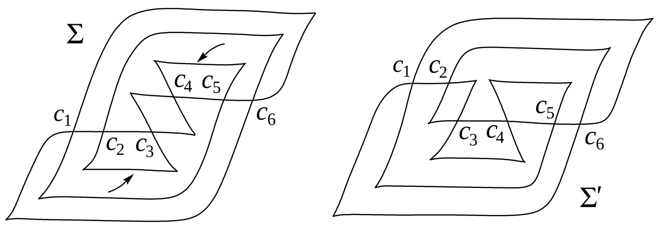


FIGURE 7

3.4.3. The Legendrian knots corresponding to the fronts Σ, Σ' represented in figure 7 are respectively Legendrian isotopic to the Legendrian knots L, L' defined in figure 1. We are going to show that $\#(\mathrm{Adm}(\Sigma)) \neq \#(\mathrm{Adm}(\Sigma'))$ and hence theorem 1.1 is a consequence of theorem 3.1.

Assume that $D \in \mathrm{Adm}(\Sigma)$. Consider the curve $X_1 \in D$ containing the piece of Σ indicated by the lower arrow. Being applied to X_1, conditions (1) and (2) imply that $c_2, c_3 \in \mathrm{Sw}(D)$. Similarly, looking at the curve $X_2 \in D$ containing the piece of Σ indicated by the upper arrow, one concludes that $c_4, c_5 \in \mathrm{Sw}(D)$. Since c_1 and c_6 are not Maslov, we have $\mathrm{Sw}(D) = \{c_2, c_3, c_4, c_5\}$. It is not hard to check that this decomposition is indeed admissible, and hence $\#(\mathrm{Adm}(\Sigma)) = 1$. Arguing similarly, one can find that $\#(\mathrm{Adm}(\Sigma')) = 2$, the admissible decompositions D_1, D_2 being defined by $\mathrm{Sw}(D_1) = \{c_2, c_3, c_4, c_5\}$, $\mathrm{Sw}(D_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$.

4. Instability of Invariants

There are two stabilizing operations, S_- and S_+, on Legendrian isotopy classes of oriented Legendrian knots, defined as follows. Given an oriented Legendrian knot L, we perform one of the operations described in figure 8 in a small neighbourhood of a point on L. One can check that, up to Legendrian isotopy, the resulting Legendrian knot $S_\pm(L)$ does not depend on the choices involved, and the operations S_-, S_+ commute. An important observation is that two Legendrian knots L, L' have the same classical invariants if and only if they are stable Legendrian isotopic in the sense that there exist $n_-, n_+ \in \mathbf{N}_0$ such that $S_-^{n_-}(S_+^{n_+}(L))$ is Legendrian isotopic to $S_-^{n_-}(S_+^{n_+}(L'))$ (see e. g. [8]).

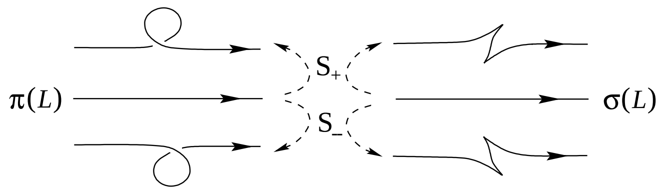


FIGURE 8

Thus the invariants constructed in sections 1 and 2 cannot be stable. In fact, they fail already after the first stabilization. The set $I(S_\pm(L))$ is always empty. The reason is that the differential graded algebra (A, ∂) for $S_\pm(L)$ can be obtained from that for L by adding a new generator a such that $\partial(a) = 1$ (while $I(S_\pm(L))$ being nonempty would imply $1 \notin \partial(A)$). The set $\mathrm{Adm}(\sigma(S_\pm(L)))$ is empty as well. This follows since conditions (1) and (2) cannot hold for the piece containing the newly created cusps.

It is not known whether there exists a pair of stable Legendrian isotopic but not Legendrian isotopic Legendrian knots such that at least one of them has the form $S_\pm(L)$.

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