

The Calibration Method for Free Discontinuity Problems

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Abstract. The calibration method is used to identify some minimizers of the Mumford-Shah functional. The method is then extended to more general free discontinuity problems.

1. Introduction

In [5] De Giorgi introduced the name *free discontinuity problems* to denote a wide class of minimum problems for functionals of the form

$$F(u) := \int_{\Omega \setminus S_u} f(x, u(x), \nabla u(x)) dx + \int_{S_u} \psi(x, u^+(x), u^-(x), \nu_u(x)) d\mathcal{H}^{n-1}, \quad (1)$$

where Ω is a given bounded domain in \mathbf{R}^n with Lipschitz boundary, $f: \Omega \times \mathbf{R} \times \mathbf{R}^n \rightarrow [0, +\infty]$ and $\psi: \Omega \times \mathbf{R} \times \mathbf{R} \times \mathbf{S}^{n-1} \rightarrow [0, +\infty]$ are given Borel functions, $\mathbf{S}^{n-1} = \{v \in \mathbf{R}^n : |v| = 1\}$, \mathcal{H}^{n-1} is the $(n-1)$ -dimensional Hausdorff measure, and the unknown function $u: \Omega \rightarrow \mathbf{R}$ is assumed to be regular out of a (partially regular) singular set S_u of dimension $n-1$, with unit normal ν_u , on which u admits unilateral traces u^+ and u^- . The main feature of these problems is that the shape and location of the discontinuity set S_u are not prescribed. Thus minimizing F means optimizing both the function u and the singular set S_u , which is indeed often regarded as an independent unknown.

These problems have an increasing importance in many branches of applied analysis, such as image processing (Mumford-Shah functional for image segmentation) and fracture mechanics (Griffith's criterion and Barenblatt cohesive zone model).

The *Mumford-Shah functional* was introduced in [10] in the context of a variational approach to image segmentation problems (for which we refer to [9]). It can be written, in dimension n , as

$$F_g^{\alpha, \beta}(u) := \int_{\Omega \setminus S_u} |\nabla u(x)|^2 dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_{\Omega \setminus S_u} |u(x) - g(x)|^2 dx, \quad (2)$$

where g is a given function in $L^\infty(\Omega)$ (interpreted as the grey level of the image to be analysed), and $\alpha > 0$ and $\beta \geq 0$ are constants. When $n = 2$ (the only case

considered in image processing), the singular set S_u of a minimizer u of $F_g^{\alpha,\beta}$ is interpreted as the set of the most relevant segmentation lines of the image.

Using different classes of infinitesimal variations, one can show that every minimizer must satisfy certain equilibrium conditions, which could be globally called *Euler-Lagrange equations* for $F_g^{\alpha,\beta}$. For instance, u must satisfy the equation $\Delta u = \beta(u - g)$ on $\Omega \setminus S_u$, with Neumann boundary conditions on $S_u \cup \partial\Omega$. Moreover, there is a link between the mean curvature of S_u (where defined) and the traces of u and ∇u on the two sides of S_u ; for instance, when $\beta = 0$, the mean curvature of S_u must be equal to the difference of the squares of the norms of the traces of ∇u . Additional conditions have been derived for the two-dimensional case. We refer the reader to [10] and [2] for a precise description of these equilibrium conditions.

However, since $F_g^{\alpha,\beta}$ is not convex, all conditions which can be derived by infinitesimal variations are necessary for minimality, but never sufficient. The purpose of this note is precisely to present a *sufficient condition* for minimality (theorem 3.1 for $F_g^{\alpha,\beta}$ and theorem 3.4 for F), and give a few applications (examples 4.1–4.8). Detailed proofs and further results will be given in the forthcoming paper [1].

2. Notation and Preliminaries

For a complete mathematical treatment of the minimum problems for the functional F considered in (1), we use the space $SBV(\Omega)$ of *special functions of bounded variation*, introduced by De Giorgi and Ambrosio in [6]. A self-contained presentation of this space can be found in the recent book [2], which contains also the complete proof of the existence of a minimizer u of $F_g^{\alpha,\beta}$, and of the partial regularity of the corresponding singular set S_u (the regularity of u on $\Omega \setminus S_u$ follows from the standard theory of elliptic equations).

We recall that for every $u \in SBV(\Omega)$ the *approximate upper and lower limits* $u^+(x)$ and $u^-(x)$ at a point $x \in \Omega$ are defined by

$$u^\pm(x) := \pm \inf \{ t \in \mathbf{R} : \lim_{\rho \rightarrow 0^+} \rho^{-n} \mathcal{L}^n(\{\pm u > t\} \cap B_\rho(x)) = 0 \},$$

where $B_\rho(x)$ is the open ball with centre x and radius ρ . The *singular set* (or *jump set*) of u is defined by $S_u := \{x \in \Omega : u^-(x) < u^+(x)\}$. It is known that S_u is countably $(\mathcal{H}^{n-1}, n-1)$ -rectifiable and that there exists a Borel measurable function $\nu_u : S_u \rightarrow \mathbf{S}^{n-1}$ such that for \mathcal{H}^{n-1} -a.e. $x \in S_u$ we have

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^n} \int_{B_\rho^\pm(x)} |u(y) - u^\pm(x)| dy = 0, \quad (3)$$

where $B_\rho^\pm(x) := \{y \in B_\rho(x) : \pm(y - x) \cdot \nu_u(x) > 0\}$ and \cdot denotes the scalar product in \mathbf{R}^n (see [7, Theorem 4.5.9]). Condition (3) says that $\nu_u(x)$ points from the side of S_u corresponding to $u^-(x)$ to the side corresponding to $u^+(x)$.

The gradient Du of u is a measure that can be decomposed as the sum of two measures $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to the Lebesgue measure \mathcal{L}^n . The density of $D^a u$ with respect to \mathcal{L}^n is denoted by ∇u . Since $u \in SBV(\Omega)$, for every Borel set B in Ω we have

$$(Du)(B) = \int_B \nabla u(x) dx + \int_{B \cap S_u} (u^+(x) - u^-(x)) \nu_u(x) d\mathcal{H}^{n-1}.$$

The *graph* of u is defined as

$$\Gamma_u := \{(x, t) \in \Omega \times \mathbf{R} : u^-(x) \leq t \leq u^+(x)\}.$$

The characteristic function of the subgraph $\{(x, t) \in \Omega \times \mathbf{R} : t \leq u(x)\}$ is denoted by 1_u . It is defined by $1_u(x, t) := 1$ if $t \leq u(x)$, and $1_u(x, t) := 0$ if $t > u(x)$. It belongs to $SBV(\Omega \times \mathbf{R})$ and its gradient $D1_u$ is a measure concentrated on Γ_u .

3. The Main Results

We fix an open subset U of $\Omega \times \mathbf{R}$ of the form

$$U := \{(x, t) \in \Omega \times \mathbf{R} : \tau_1(x) < t < \tau_2(x)\}, \quad (4)$$

where τ_1 and τ_2 are two continuous functions on $\bar{\Omega}$ such that $-\infty \leq \tau_1(x) \leq \tau_2(x) \leq +\infty$ for every $x \in \bar{\Omega}$.

Let F be the functional introduced in (1). We say that a function $u \in SBV(\Omega)$, with graph \mathcal{H}^n -contained in U (i.e., $\mathcal{H}^n(\Gamma_u \setminus U) = 0$), is a *Dirichlet U -minimizer* of F , if $F(u) \leq F(v)$ for every $v \in SBV(\Omega)$ with the same trace as u on $\partial\Omega$ and with graph \mathcal{H}^n -contained in U . If the inequality $F(u) \leq F(v)$ holds for every $v \in SBV(\Omega)$ with graph \mathcal{H}^n -contained in U , we say that u is a *U -minimizer* of F . We omit U when $U = \Omega \times \mathbf{R}$.

The symbol ϕ will always denote a bounded Borel measurable vectorfield defined on U with values in $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$, with components $\phi^x \in \mathbf{R}^n$ and $\phi^t \in \mathbf{R}$. The divergence of ϕ is then $\operatorname{div} \phi(x, t) = \operatorname{div}_x \phi^x(x, t) + \partial_t \phi^t(x, t)$.

We begin with a theorem concerning the functional $F_g^{\alpha, \beta}$ introduced in (2).

Theorem 3.1. *Let $u \in SBV(\Omega)$ with graph \mathcal{H}^n -contained in U . Assume that there exists a bounded vectorfield ϕ of class C^1 on U with the following properties:*

- (a1) $\frac{1}{4} |\phi^x(x, t)|^2 \leq \phi^t(x, t) + \beta |t - g(x)|^2$
for \mathcal{L}^n -a.e. $x \in \Omega$ and for every $\tau_1(x) < t < \tau_2(x)$;
- (a2) $\phi^x(x, u(x)) = 2 \nabla u(x)$ and $\phi^t(x, u(x)) = |\nabla u(x)|^2 - \beta |u(x) - g(x)|^2$
for \mathcal{L}^n -a.e. $x \in \Omega$;
- (b1) $\left| \int_{t_1}^{t_2} \phi^x(x, t) dt \right| \leq \alpha$
for \mathcal{H}^{n-1} -a.e. $x \in \Omega$ and for every $\tau_1(x) < t_1 < t_2 < \tau_2(x)$;

$$(b2) \quad \int_{u^-(x)}^{u^+(x)} \phi^x(x, t) dt = \alpha \nu_u(x)$$

for \mathcal{H}^{n-1} -a.e. $x \in S_u$;

$$(c1) \quad \operatorname{div} \phi(x, t) = 0$$

for every $(x, t) \in U$.

Then u is a Dirichlet U -minimizer of $F_g^{\alpha, \beta}$. If, in addition, $\phi^x(x, t)$ satisfies the boundary condition

$$(c2) \quad \lim_{(y, s) \rightarrow (x, t)} \phi^x(y, s) \cdot \nu(x) = 0$$

for \mathcal{H}^{n-1} -a.e. $x \in \partial\Omega$ and for \mathcal{L}^1 -a.e. $t \in [\tau_1(x), \tau_2(x)]$,

where $\nu(x)$ is the outer unit normal to $\partial\Omega$, then u is a U -minimizer of $F_g^{\alpha, \beta}$.

A vectorfield ϕ which satisfies conditions (a1)–(c1) of theorem 3.1 is called a *calibration* for the functional $F_g^{\alpha, \beta}$ on U . If ϕ satisfies also (c2), it is called a *Neumann calibration*. Theorem 3.1 is an immediate consequence of the following lemmas.

Lemma 3.2. *Let ϕ be a vectorfield which satisfies conditions (a1) and (b1) of theorem 3.1. Then for every $u \in SBV(\Omega)$ with graph \mathcal{H}^n -contained in U we have*

$$F_g^{\alpha, \beta}(u) \geq \int_U \phi \cdot d(D1_u). \quad (5)$$

Moreover, equality holds in (5) for a given u if and only if conditions (a2) and (b2) of theorem 3.1 are satisfied.

The next lemma is a consequence of the divergence theorem.

Lemma 3.3. *Suppose that ϕ is of class C^1 and that $\operatorname{div} \phi = 0$ on U . Then*

$$\int_U \phi \cdot d(D1_u) = \int_U \phi \cdot d(D1_v) \quad (6)$$

for every pair of functions u, v in $BV(\Omega)$ with the same trace on $\partial\Omega$ and with graphs \mathcal{H}^n -contained in U . If, in addition, ϕ satisfies condition (c2) of theorem 3.1, then (6) holds for every pair of functions u, v in $BV(\Omega)$ with graphs \mathcal{H}^n -contained in U .

As a matter of fact, the method of calibrations can be easily adapted to the functional F defined in (1).

Theorem 3.4. *Let $u \in SBV(\Omega)$ with graph \mathcal{H}^n -contained in U . Assume that there exists a bounded vectorfield ϕ of class C^1 on U with the following properties:*

$$(a1) \quad \phi^x(x, t) \cdot v \leq \phi^t(x, t) + f(x, t, v)$$

for \mathcal{L}^n -a.e. $x \in \Omega$, for every $\tau_1(x) < t < \tau_2(x)$, and for every $v \in \mathbf{R}^n$;

$$(a2) \quad \phi^x(x, u(x)) \cdot \nabla u(x) = \phi^t(x, u(x)) + f(x, u(x), \nabla u(x))$$

for \mathcal{L}^n -a.e. $x \in \Omega$;

$$(b1) \quad \nu \cdot \int_{t_1}^{t_2} \phi^x(x, t) dt \leq \psi(x, t_1, t_2, \nu)$$

for \mathcal{H}^{n-1} -a.e. $x \in \Omega$, for every $\tau_1(x) < t_1 < t_2 < \tau_2(x)$, and for every $\nu \in \mathbf{S}^{n-1}$;

$$(b2) \quad \nu_u(x) \cdot \int_{u^-(x)}^{u^+(x)} \phi^x(x, t) dt = \psi(x, u^-(x), u^+(x), \nu_u(x))$$

for \mathcal{H}^{n-1} -a.e. $x \in S_u$;

$$(c1) \quad \operatorname{div} \phi(x, t) = 0$$

for every $(x, t) \in U$.

Then u is a Dirichlet U -minimizer of F . If $\phi^x(x, t)$ satisfies also the boundary condition (c2) of theorem 3.1, then u is a U -minimizer of F .

Remark 3.5. We note that in theorem 3.4 there is no regularity or convexity hypothesis on f or ψ . If $f^*(x, t, v^*)$ is the the convex conjugate of $f(x, t, v)$ with respect to v , condition (a1) is equivalent to

$$(a1') \quad f^*(x, t, \phi^x(x, t)) \leq \phi^t(x, t)$$

for \mathcal{L}^n -a.e. $x \in \Omega$ and for every $\tau_1(x) < t < \tau_2(x)$.

If this condition is satisfied, and $f(x, t, v)$ is convex and differentiable with respect to v , then condition (a2) is equivalent to

$$(a2') \quad \begin{cases} \phi^x(x, u(x)) = \partial_v f(x, u(x), \nabla u(x)) \\ \phi^t(x, u(x)) = f^*(x, u(x), \phi^x(x, u(x))) \end{cases} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega.$$

Remark 3.6. In theorems 3.1 and 3.4 the hypothesis that ϕ is of class C^1 is too strong for many applications. It is used only in lemma 3.3 and it can be relaxed in several ways (see [1] for details). For instance, one may consider piecewise C^1 vectorfields, which may be discontinuous along sufficiently regular interfaces. In this case the divergence-free condition (c1) must be understood in the distributional sense, i.e., the pointwise divergence vanishes (where defined) and the normal component of ϕ is continuous across the discontinuity surfaces.

4. Some Examples

The following examples show that the calibration method is very flexible, and can be used to prove the minimality of a given function u in many different situations. In the first examples we will consider only the “homogeneous” functional $F^\alpha := F_g^{\alpha, 0}$, in which the lower order term $\beta \int_\Omega |u - g|^2 dx$ vanishes.

Example 4.1. (Affine function in one dimension) Let $n := 1$, $\Omega :=]0, a[$, and $u(x) := \lambda x$, with $\lambda > 0$. It is easy to see that u is a Dirichlet minimizer of F^α if and only if $a\lambda^2 \leq \alpha$. In this case a calibration is given by the piecewise constant

function

$$\phi(x, t) := \begin{cases} (2\lambda, \lambda^2), & \text{if } \frac{\lambda}{2}x \leq t \leq \frac{\lambda}{2}(x+a), \\ (0, 0), & \text{otherwise.} \end{cases} \quad (7)$$

Another calibration is given by

$$\phi(x, t) := \begin{cases} \left(2\frac{t}{x}, \left(\frac{t}{x}\right)^2\right), & \text{if } 0 \leq t \leq \lambda x, \\ \left(2\frac{\lambda a - t}{a-x}, \left(\frac{\lambda a - t}{a-x}\right)^2\right), & \text{if } \lambda x \leq t \leq \lambda a, \\ (0, 0), & \text{otherwise.} \end{cases} \quad (8)$$

If $a\lambda^2 > \alpha$, then the function $u(x) := \lambda x$ is not a Dirichlet minimizer of F^α , but it is still a Dirichlet U -minimizer with

$$U := \{(x, t) \in]0, a[\times \mathbf{R} : \lambda x - \frac{\alpha}{4\lambda} < t < \lambda x + \frac{\alpha}{4\lambda}\}.$$

A calibration on U is given by $\phi(x, t) := (2\lambda, \lambda^2)$.

Example 4.2. (Jump in one dimension) Let $n := 1$, $\Omega :=]0, a[$, $u(x) := 0$ for $0 < x < c$, and $u(x) := h$ for $c < x < a$, with $0 < c < a$ and $h > 0$. It is easy to see that u is a Dirichlet minimizer of F^α if and only if $\alpha \leq h^2$. In this case two different calibrations are given by (7) and (8) with $\lambda = \sqrt{\alpha}/\sqrt{a}$.

Suppose now that $\alpha > h^2$. Let $\varepsilon > 0$ be a constant such that $2\varepsilon + \sqrt{2\alpha\varepsilon} \leq h$, let

$$\tau_1(x) = \begin{cases} -\varepsilon, & \text{if } x \leq c, \\ -\varepsilon + \frac{h}{\varepsilon}(x - c), & \text{if } c \leq x \leq c + \varepsilon, \\ h - \varepsilon, & \text{if } c + \varepsilon \leq x, \end{cases}$$

let $\tau_2(x) = \tau_1(x + \varepsilon) + 2\varepsilon$, and let U be the open set defined by (4). Then u is a Dirichlet U -minimizer of F^α , and a calibration on U is given by the piecewise constant function

$$\phi(x, t) := \begin{cases} (2\lambda, \lambda^2), & \text{if } c - \varepsilon < x < c + \varepsilon \text{ and} \\ & \varepsilon + \frac{\lambda}{2}(x - c + \varepsilon) < t < \varepsilon + \frac{\lambda}{2}(x - c + \varepsilon) + \frac{\alpha}{2\lambda}, \\ (0, 0), & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is any constant such that $\varepsilon + \varepsilon\lambda + \frac{\alpha}{2\lambda} \leq h - \varepsilon$, for instance $\lambda = \sqrt{\alpha}/\sqrt{2\varepsilon}$.

Example 4.3. (Harmonic function) Let Ω be a bounded domain in \mathbf{R}^n , n arbitrary, and let u be a harmonic function on Ω . As pointed out by Chambolle [3], u is a Dirichlet minimizer of F^α if

$$\operatorname{osc}_\Omega \sup_\Omega |\nabla u| \leq \alpha, \quad (9)$$

where $\operatorname{osc}_\Omega u := \sup_\Omega u - \inf_\Omega u$. Note that for $n = 1$ this condition reduces to the constraint $a\lambda^2 \leq \alpha$ of example 4.1. Inspired by the one dimensional case (see (7)),

we construct the calibration

$$\phi(x, t) := \begin{cases} (2\nabla u(x), |\nabla u(x)|^2), & \text{if } \frac{1}{2}(u(x) + m) \leq t \leq \frac{1}{2}(u(x) + M), \\ (0, 0), & \text{otherwise,} \end{cases} \quad (10)$$

where $m := \inf_{\Omega} u$ and $M := \sup_{\Omega} u$. Another calibration (see (8)) is given by

$$\phi(x, t) := \begin{cases} (2\frac{t-m}{u(x)-m}\nabla u(x), (\frac{t-m}{u(x)-m})^2|\nabla u(x)|^2), & \text{if } m \leq t \leq u(x), \\ (2\frac{M-t}{M-u(x)}\nabla u(x), (\frac{M-t}{M-u(x)})^2|\nabla u(x)|^2), & \text{if } u(x) \leq t \leq M, \\ (0, 0), & \text{otherwise.} \end{cases}$$

If (9) is not satisfied, u is still is a Dirichlet U -minimizer of F^α , for

$$U := \{(x, t) \in \Omega \times \mathbf{R} : u(x) - \frac{\alpha}{4}|\nabla u(x)|^{-1} < t < u(x) + \frac{\alpha}{4}|\nabla u(x)|^{-1}\}, \quad (11)$$

and a calibration in U is given by $\phi(x, t) := (2\nabla u(x), |\nabla u(x)|^2)$.

Example 4.4. (Pure jump) Let $n \geq 2$ and let $\Omega :=]0, a[\times V$, where V is a bounded domain in \mathbf{R}^{n-1} with Lipschitz boundary. Denoting the first coordinate of x by x_1 , let $u(x) := 0$ for $0 < x_1 < c$, and $u(x) := h$ for $c < x_1 < a$, with $0 < c < a$ and $h > 0$. Using the results of example 4.2 it is easy to see that u is a Dirichlet minimizer of F^α if $\alpha \leq h^2$. In this case two different calibrations can be constructed in the following way: the projection of these calibrations onto the (x_1, t) -plane are given by (7) and (8), with $\lambda = \sqrt{\alpha}/\sqrt{a}$ and x replaced by x_1 , while all other components of these calibrations vanish.

If $\alpha > h^2$, it may happen that u is still a Dirichlet minimizer of F^α . For instance, if $n = 2$ and $V =]0, b[$, with $b\alpha\pi \leq 2h^2$, a different calibration has been constructed in [1]. Therefore u is a Dirichlet minimizer of F^α even if α is very large with respect to h^2 , provided that $b\alpha$ is small enough.

Arguing as in the last part of example 4.2 one can prove that for every a and V there exists an open set U of the form (4), containing Γ_u , such that u is a Dirichlet U -minimizer of F^α .

Example 4.5. (Triple junction) Let $n := 2$, let $\Omega := B(0, r)$ be the open ball with radius $r > 0$ centered at the origin, and let u be given, in polar coordinates, by $u(\rho, \theta) := a$ for $0 \leq \theta < \frac{2}{3}\pi$, $u(\rho, \theta) := b$ for $\frac{2}{3}\pi \leq \theta < \frac{4}{3}\pi$, and $u(\rho, \theta) := c$ for $\frac{4}{3}\pi \leq \theta < 2\pi$, where a, b , and c are distinct constants. Thus S_u is given by three line segments meeting at the origin with equal angles. If

$$2\alpha r \leq \min\{|a - b|^2, |b - c|^2, |c - a|^2\}, \quad (12)$$

then u is a Dirichlet minimizer of F^α . To construct a calibration, it is not restrictive to assume $a < b = 0 < c$. Inspired by the one dimensional case described in example 4.2, we take $e_\pm := (\pm\sqrt{3}/2, -1/2)$, and $\lambda > 0$ such that $\frac{\lambda r}{2} + \frac{\alpha}{\lambda} \leq \min\{-a, c\}$

(which is possible by (12)), and we define the calibration by

$$\phi(x, t) := \begin{cases} (\lambda e_+, \lambda^2/4), & \text{if } \frac{\lambda}{4}(r + x \cdot e_+) \leq t \leq \frac{\lambda}{4}(r + x \cdot e_+) + \frac{\alpha}{\lambda}, \\ (\lambda e_-, \lambda^2/4), & \text{if } \frac{\lambda}{4}(-r + x \cdot e_-) - \frac{\alpha}{\lambda} \leq t \leq \frac{\lambda}{4}(-r + x \cdot e_-), \\ (0, 0), & \text{otherwise.} \end{cases} \quad (13)$$

If αr is much larger than $\min\{|a-b|^2, |b-c|^2, |c-a|^2\}$, it is easy to construct a comparison function v with the same boundary values as u and such that $F^\alpha(v) < F^\alpha(u)$. This shows that in this case u is not a Dirichlet minimizer.

However, for every value of the parameters α, r, a, b, c , one can construct a suitable neighbourhood U of the graph Γ_u , of the form (4), such that a variant of (13) is a calibration in U , and therefore u is a Dirichlet U -minimizer of F^α . We refer to [1] for the details.

We consider now the functional $F_g^{\alpha, \beta}$, with $\beta > 0$.

Example 4.6. (Solution of the Neumann problem) Let Ω be a bounded open set in \mathbf{R}^n with boundary of class $C^{1, \varepsilon}$ for some $\varepsilon > 0$, and let u be the solution of the Neumann problem

$$\begin{cases} \Delta u = \beta(u - g) & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (14)$$

with $\beta > 0$ and $g \in L^\infty(\Omega)$. Assume that condition (9) of example 4.3 is satisfied. Then u is a minimizer of $F_g^{\alpha, \beta}$. If the strict inequality holds in (9), then u is the unique minimizer. A Neumann calibration $\phi(x, t)$ is given by

$$\begin{cases} (0, \beta|\frac{m}{2} - \frac{u(x)}{2}|^2 - \beta|\frac{m}{2} + \frac{u(x)}{2} - g(x)|^2), & \text{if } t - \frac{u(x)}{2} < \frac{m}{2}, \\ (2\nabla u(x), |\nabla u(x)|^2 - \beta|t - g(x)|^2 + \beta|t - u(x)|^2), & \text{if } \frac{m}{2} \leq t - \frac{u(x)}{2} \leq \frac{M}{2}, \\ (0, \beta|\frac{M}{2} - \frac{u(x)}{2}|^2 - \beta|\frac{M}{2} + \frac{u(x)}{2} - g(x)|^2), & \text{if } \frac{M}{2} < t - \frac{u(x)}{2}, \end{cases}$$

where $m := \inf_\Omega u$ and $M := \sup_\Omega u$.

If (9) is not satisfied, u is still a U -minimizer of $F_g^{\alpha, \beta}$, where U is the open set defined by (11). A Neumann calibration on U is given by

$$\phi(x, t) := (2\nabla u(x), |\nabla u(x)|^2 - \beta|t - g(x)|^2 + \beta|t - u(x)|^2).$$

The hypothesis that $\partial\Omega$ is of class $C^{1, \varepsilon}$ is used only to obtain the boundary condition (c2) of theorem 3.1, which, in this case, becomes

$$\lim_{y \rightarrow x} \nabla u(y) \cdot \nu(x) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega. \quad (15)$$

It is clear that (15) is still true if for \mathcal{H}^{n-1} -a.e. $x \in \partial\Omega$ there exists an open neighbourhood V_x of x in \mathbf{R}^n such that $V_x \cap \partial\Omega$ is a manifold of class $C^{1, \varepsilon}$ (see [2, Theorem 7.5.2]). Therefore the result of this example is true also when Ω is polyhedral.

In the next examples we construct a calibration for $F_g^{\alpha, \beta}$ when the parameter β is large enough.

Example 4.7. (Smooth g and large β) Let Ω be a bounded open set in \mathbf{R}^n with smooth boundary, and let $g \in C^2(\overline{\Omega})$. There exists a constant $\beta_0 \geq 0$, depending on g and α , such that for every $\beta > \beta_0$ the solution u of the Neumann problem (14) of example 4.6 is the unique minimizer of $F_g^{\alpha,\beta}$. A Neumann calibration is constructed in [1].

This shows that the minimizer of $F_g^{\alpha,\beta}$ is smooth, provided that g is smooth and β is large enough. Therefore the solution of the image segmentation problem ($n = 2$) based on the minimization of $F_g^{\alpha,\beta}$ has an empty set of segmentation lines if the “grey level” function g is smooth and the parameter β in the fidelity term $\beta \int_{\Omega} |u - g|^2 dx$ is large.

Example 4.8. (Function g with only two values) Let Ω be an open set in \mathbf{R}^n and let E be a compact set contained in Ω with boundary of class C^2 . Let $g(x) := a$ for $x \in E$ and $g(x) := b$ for $x \in \Omega \setminus E$, with $a \neq b$. There exists a constant $\beta_0 \geq 0$, depending on g and α , such that for every $\beta > \beta_0$ the function $u := g$ is the unique minimizer of $F_g^{\alpha,\beta}$. To construct a calibration, it is not restrictive to assume $a < b$. We take a C^1 vectorfield $v: \Omega \rightarrow \mathbf{R}^n$ with compact support in Ω such that $|v(x)| \leq 1$ for every $x \in \Omega$ and $v(x)$ is the outer unit normal to ∂E for every $x \in \partial E$. Then we set $\phi^x(x, t) = \sigma(t)v(x)$, where σ is a fixed positive smooth function with integral equal to α and support contained in $]a, b[$. We see that conditions (b1), (b2), and (c2) of theorem 3.1 are satisfied by construction. It remains to choose ϕ^t so that (a1), (a2), and (c1) hold. Condition (a2) forces us to set $\phi^t(x, t) = 0$ for $t = g(x)$, while (c1) gives $\partial_t \phi^t(x, t) = -\sigma(t) \operatorname{div}_x v(x)$. These two conditions determine $\phi^t(x, t)$ at every point (x, t) . It is then easy to see that (a1) holds if β is large enough. We refer to [1] for the details.

This example shows that, if $g \in SBV(\Omega)$ has only two values, and S_g is smooth enough, then the minimizer of the Mumford-Shah functional $F_g^{\alpha,\beta}$ reconstructs g exactly, when β is large enough.

Recently the following question has been studied by using the calibration method: is it true that a function u is a (Dirichlet) minimizer of $F_g^{\alpha,\beta}$, if it satisfies the Euler-Lagrange equations and the domain Ω is sufficiently small? For the moment we have only a partial answer. In [4] we have considered the case where $n := 2$ and S_u is a line segment joining two points of the boundary of Ω . If u satisfies the Euler-Lagrange equations for the “homogeneous functional” $F^\alpha := F_g^{\alpha,0}$, then for every $x_0 \in S_u$ there exists an open neighbourhood Ω_0 of x_0 , contained in Ω , such that u is a Dirichlet minimizer of F^α in Ω_0 . The minimality is proved by constructing a complicated calibration on $\Omega_0 \times \mathbf{R}$.

This result has been extended in [8] to the case where S_u is an analytic curve joining two points of $\partial\Omega$. The (more difficult) construction of the calibration presented in this paper shows that one can take the same set Ω_0 for every $x_0 \in S_u$; in other words, one can take as Ω_0 a suitable tubular neighbourhood of S_u . Moreover, it is proved in [8] that an additional condition on u and S_u implies that u is a Dirichlet U -minimizer for a suitable open neighbourhood U of the graph Γ_u . A counterexample (where S_u is a line segment joining two points of $\partial\Omega$) shows that

this is not always true when u is just a solution of the Euler-Lagrange equations with $S_u \neq \emptyset$, in contrast to the case $S_u = \emptyset$ (see example 4.6).

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