Geometry on Arc Spaces of Algebraic Varieties

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Abstract. This paper is a survey on arc spaces, a recent topic in algebraic geometry and singularity theory. The geometry of the arc space of an algebraic variety yields several new geometric invariants and brings new light to some classical invariants.

1. Introduction

For an algebraic variety X over the field $\mathbb{C}$ of complex numbers, one considers the arc space $\mathcal{L}(X)$, whose points are the $\mathbb{C}[[t]]$-rational points on X, and the truncated arc spaces $\mathcal{L}_n(X)$, whose points are the $\mathbb{C}[[t]]/t^{n+1}$-rational points on X. The geometry of these spaces yields several new geometric invariants of X and brings new light to some classical invariants. For example, Denef and Loeser [13] showed that the Hodge spectrum of a critical point of a polynomial can be expressed in terms of geometry on arc spaces, yielding a new proof and a generalization [15] of the Thom-Sebastiani Theorem for the Hodge spectrum due to Varchenko [42] and Saito [33, 34]. In a different direction, Batyrev [6] used arc spaces to prove a conjecture of Reid [30] on quotient singularities (the McKay correspondence), and to construct his stringy Hodge numbers [5] appearing in mirror symmetry. All these developments are based on Kontsevich's construction [23] of a measure on the arc space $\mathcal{L}(X)$, the motivic measure, which is an analogue of the p-adic measure on a p-adic variety.

In section 2 we define the arc spaces $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ of an algebraic variety over any field k of characteristic zero. The first question that appears is how $\mathcal{L}_n(X)$ and $\pi_n(\mathcal{L}_n(X))$ change with n, where π_n denotes the truncation map from $\mathcal{L}(X)$ to $\mathcal{L}_n(X)$. As a partial answer to this question we will see in theorem 2.1 that the power series

$$J(T,\chi) := \sum_{n\geq 0} \chi(\mathcal{L}_n(X))\, T^n, \quad P(T,\chi) = \sum_{n\geq 0} \chi(\pi_n(\mathcal{L}(X)))\, T^n$$

are rational (i.e. a quotient of two polynomials), for any reasonable generalized Euler characteristic χ. This is a direct consequence of results of Denef and Loeser [14]. Instead of working with particular generalized Euler characteristics, such as the topological Euler characteristic, the Hodge polynomial or the Hodge characteristic, it is more general to work with the universal Euler characteristic which associates to any algebraic variety X over k its class $[X]$ in the Grothendieck group $K_0(\mathrm{Var}_k)$ of algebraic varieties over k. This is the abelian group generated by symbols $[X]$, for X a variety over k, with the relations $[X] = [Y]$ if X and Y are isomorphic, and $[X] = [Y] + [X \setminus Y]$ if Y is Zariski closed in X. There is a natural ring structure on $K_0(\mathrm{Var}_k)$, the product of $[X]$ and $[Y]$ being equal to $[X \times Y]$. We denote by $\mathcal{M}_k$ the ring obtained from $K_0(\mathrm{Var}_k)$ by inverting the class of $\mathbb{A}^1_k$. The above rationality result applied to the universal Euler characteristic says that the power series

$$J(T) := \sum_{n \geq 0} [\mathcal{L}_n(X)]\, T^n, \quad P(T) := \sum_{n \geq 0} [\pi_n(\mathcal{L}(X))]\, T^n$$

in $\mathcal{M}_k[[T]]$ are rational.

Power series like $J(T)$ and $P(T)$, with coefficients in $\mathcal{M}_k[[T]]$, are called "motivic", because they specialize to power series over the Grothendieck group $K_0(\mathrm{Mot}_k)$ of the category of Chow motives over k. Actually in several of our papers on arcs we work over $K_0(\mathrm{Mot}_k)$ instead of over $\mathcal{M}_k[[T]]$.

In section 3 we introduce the motivic zeta function $Z(T)$ associated to a morphism f from an nonsingular algebraic variety X to the affine line, cf. [18]. A naive version of it is the power series over $\mathcal{M}_k$ defined by

$$Z^{\mathrm{naive}}(T) := \sum_{n \geq 1} [\mathfrak{X}_n]\, [\mathbb{A}^1_k]^{-nd}\, T^n \,.$$

Here $\mathfrak{X}_n$ denotes the set of arcs φ in $\mathcal{L}(X)$ with $f(\varphi)$ a power series of order n. The motivic zeta function of f contains a wealth of geometric information about f. For example the Hodge spectrum of any critical point of f can be expressed in terms of $\lim_{T \to \infty} Z(T)$. This limit is a well defined element of $\mathcal{M}_k$, and can be considered as the "virtual motivic incarnation" of the Milnor fibers of f. All this is explained in section 3.5. In section 3.4 we also show that the topological zeta functions of Denef and Loeser [12] can be expressed in terms of the motivic zeta function.

We explain in section 4 the notion of motivic integration on $\mathcal{L}(X)$, due to Kontsevich [23], and further developed by Batyrev [5, 6], Denef and Loeser [13]–[18], and Looijenga [27]. This notion plays a key role in the present paper. Kontsevich used it to prove that two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. This result, together with some other direct applications of motivic integration, is discussed in section 4.4.

One of the most striking applications of arc spaces and motivic integration is Batyrev's proof [6] of the conjecture of Reid on the generalized McKay correspondence. We will not treat this material in the present paper, but refer to the Bourbaki report of Reid [30], see also [16] and [27].

In section 5 we explain how the relation between the Hodge spectrum and the motivic zeta function yields a new proof of Varchenko's and Saito's Thom-Sebastiani Theorem which expresses the Hodge spectrum of $f(x) + g(y)$ in terms of the Hodge spectra of $f(x)$ and $g(y)$. Our method [15] actually yields a much stronger result which relates the "virtual motivic incarnations" of the Milnor fibers of these three functions. In our paper [15] we obtained this result only at the level of Chow motives, and is was Looijenga [27] who showed how to work at the level of the Grothendieck ring of algebraic varieties.

Finally in section 6 we briefly discuss the connections with the p-adic case. Considering motivic integration as an analogue of p-adic integration, several arithmetical results in the p-adic case find their natural counterpart in complex geometry and in the theory of motives.

In the present paper we have avoided to work with Chow motives (with one important exception in section 6). Indeed, the more recent material in [18] and [27] shows that this is possible except for the functional equation in section 3 of [13] and the results in [17].

2. The Arc Space of a Variety

We fix a base field k of characteristic zero. The reader may choose to only consider the case where k is the field $\mathbb{C}$ of complex numbers. Let X be an algebraic variety over k, not necessarily irreducible, i.e. X is a reduced separated scheme of finite type over k.

2.1. The arc space of X

For each natural number n we consider the *space* $\mathcal{L}_n(X)$ *of arcs modulo* t^{n+1} *on* X. This is an algebraic variety over k, whose K-rational points, for any field K containing k, are the $K[t]/t^{n+1}K[t]$-rational points of X. For example when X is an affine variety with equations $f_i(\vec{x}) = 0$, $i = 1, \dots, m$, $\vec{x} = (x_1, \dots, x_n)$, then $\mathcal{L}_n(X)$ is given by the equations, in the variables $\vec{a}_0, \dots, \vec{a}_n$, expressing that $f_i(\vec{a}_0 + \vec{a}_1 t + \dots + \vec{a}_n t^n) \equiv 0 \mod t^{n+1}, i = 1, \dots, m$.

Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$ we obtain the *arc space* $\mathcal{L}(X)$ *of* X, which is a reduced separated scheme over k. In general, $\mathcal{L}(X)$ is not of finite type over k (i.e. $\mathcal{L}(X)$ is an "algebraic variety of infinite dimension"). The K-rational points of $\mathcal{L}(X)$ are the $K[[t]]$-rational points of X. These are called K*-arcs on* X. For example when X is an affine complex variety with equations $f_i(\vec{x}) = 0, i = 1, \dots m, \vec{x} = (x_1, \dots, x_n)$, then the $\mathbb{C}$-rational points of $\mathcal{L}(X)$ are the sequences $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots) \in (\mathbb{C}^n)^{\mathbb{N}}$ satisfying $f_i(\vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \dots) = 0$, for $i = 1, \dots, m$. For any n, and for $m > n$, we have natural morphisms

$$\pi_n \colon \mathcal{L}(X) \to \mathcal{L}_n(X) \quad \text{and} \quad \pi_n^m \colon \mathcal{L}_m(X) \to \mathcal{L}_n(X)\,,$$

obtained by truncation. Note that $\mathcal{L}_0(X) = X$ and that $\mathcal{L}_1(X)$ is the tangent bundle of X. For any arc γ on X (i.e. a K-arc for some field K containing k), we call $\pi_0(\gamma)$ *the origin of the arc* γ.

By a theorem of Greenberg [20], given an algebraic variety X over k, there exists a number $c > 0$, such that for any n and for any field K containing k we have

$$\pi_n(\mathcal{L}(X)(K)) = \pi_n^{cn}(\mathcal{L}_{cn}(X)(K))\,,$$

writing $Y(K)$ to denote the set of K-rational points on any variety Y over k. This implies that $\pi_n(\mathcal{L}(X))$ is a constructible subset of the algebraic variety $\mathcal{L}_n(X)$. If X is smooth, then we can take $c = 1$ and π_n is surjective. Moreover, in that case, π_n^m is a locally trivial fibration with fiber $\mathbb{A}_k^{(m-n)\dim X}$. Here $\mathbb{A}_k^d$ denotes the affine space of dimension d over k.

Probably Nash [28] was the first to study arc spaces in a systematic way (his paper was written in 1968, but published only recently). For a singular point P on X, he considered the space $\mathcal{L}_{\{P\}}(X) := \pi_0^{-1}(P)$ of arcs on X with origin P, and its subspace $\mathcal{N}_{\{P\}}(X)$ of arcs with origin P which are not contained in the singular locus of X. He proved that the number of irreducible components of the Zariski closure of $\pi_n(\mathcal{N}_{\{P\}}(X))$ stabilizes for n big enough, and associated to each irreducible component of this closure (when $n \gg 0$), in a canonical and injective way, an irreducible component of the preimage of P in any resolution of singularities of X. Moreover he conjectured that any irreducible component of the preimage of P in a given resolution of X, which "appears" in all resolutions of X, is obtained in that way. For other results concerning arc spaces we refer to [24, 21, 25].

2.2. How do $\mathcal{L}_n(X)$ and $\pi_n(\mathcal{L}(X))$ change with n?

The work of Nash [28] is the first result towards the question of how the geometry of $\pi_n(\mathcal{L}_{\{P\}}(X))$ changes with n. Recently we investigated how the topological Euler characteristic χ_{top} (case $k = \mathbb{C}$), and generalized Euler characteristics of the spaces $\mathcal{L}_n(X), \pi_n(\mathcal{L}(X)), \pi_n(\mathcal{L}_{\{P\}}(X))$ change with n.

With a *generalized Euler characteristic* on the category Var_k of algebraic varieties over k, we mean a map χ from Var_k to some commutative ring R such that $\chi(X) = \chi(Y)$ when $X \cong Y$, $\chi(X) = \chi(Y) + \chi(X \setminus Y)$ when Y is a Zariski closed subvariety of X, and $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. Clearly $\chi = \chi_{\mathrm{top}}$ satisfies the above requirements with $R = \mathbb{Z}$, when $k = \mathbb{C}$. Another example of a generalized Euler characteristic, when $k = \mathbb{C}$, is given by $\chi_{hp} \colon \mathrm{Var}_{\mathbb{C}} \to \mathbb{Z}[u, v] \colon X \mapsto \sum_{i,p,q} (-1)^i h^i_{p,q} u^p v^q$, where $h^i_{p,q}$ is the dimension of the (p, q)-component of the mixed Hodge structure on $H^i_c(X, \mathbb{C})$. One calls $\chi_{hp}(X)$ the *Hodge polynomial of* X. For example, $\chi_{hp}(\mathbb{A}^1_{\mathbb{C}}) = uv$. We refer to 3.1.2 for the *Hodge characteristic* χ_h which takes values in the Grothendieck group $K_0(\mathrm{HS})$ of the abelian category of Hodge structures. There are many other examples of generalized Euler characteristics. For example, to mention an exotic one, when $k = \mathbb{Q}$, there is *the conductor* $c(X)$ of X, which yields a generalized Euler characteristic $c \colon Var_{\mathbb{Q}} \to \mathbb{Q}_{>0} \colon X \mapsto c(X) := \prod_i (c_i)^{(-1)^{i+1}}$, where c_i denotes the conductor of the ℓ-adic representation of $\mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$ on the étale cohomology $H^i_c(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$, where ℓ is a fixed prime and $\bar{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$; see e.g. [36]. Here $\mathbb{Q}_{>0}$ is the multiplicative group of positive rational numbers with ring structure inherited by $Q_{>0} \cong \oplus_{p\,\mathrm{prime}} \mathbb{Z} \colon x \mapsto (\mathrm{ord}_p x)_p$. For example, when X is an elliptic curve, $c(X)$ is the usual conductor of X, related to the primes at which X has had reduction.

Theorem 2.1. ([14]) *Let $\chi\colon \mathrm{Var}_k \to R$ be a generalized Euler characteristic, and suppose that $\chi(\mathbb{A}^1_k)$ is not a zero divisor in R. Then the power series*

$$J(T,\chi) := \sum_{n\geq 0} \chi(\mathcal{L}_n(X))\, T^n, \quad P(T,\chi) = \sum_{n\geq 0} \chi(\pi_n(\mathcal{L}(X)))\, T^n$$

are rational (i.e. a quotient of two polynomials). Actually the denominators are products of polynomials of the form $1-(\chi(\mathbb{A}^1_k))^a T^b$, with $b \in \mathbb{N}\setminus\{0\}, a \in \mathbb{Z}$.

In 2.3 below, we will construct the universal Euler characteristic $\mathrm{Var}_k \to K_0(\mathrm{Var}_k)\colon X \mapsto [X]$, where $K_0(\mathrm{Var}_k)$ denotes the Grothendieck group of varieties over k (see 2.3). Any generalized Euler characteristic on Var_k factorizes over this universal one. So it suffices to prove theorem 2.1 with R the ring $\mathcal{M}_k$ obtained from $K_0(\mathrm{Var}_k)$ by inverting $[\mathbb{A}^1_k]$. The rationality of $J(T,\chi)$ follows from the material we will discuss in section 3, when X is an affine hypersurface, see 3.3.1. The proof of the rationality of $P(T,\chi)$ is more complicated and uses a theorem of Pas [29] on quantifier elimination for power series rings. If $f\colon X \to Y$ is a morphism of algebraic varieties, then the image $f(A)$ in $\mathcal{L}(Y)$ of a constructible subset A (cf. 4.1 below) of $\mathcal{L}(X)$ is generally not constructible. The theorem of Pas implies that $f(A)$ still has a "simple description" and is in fact what is called a semi-algebraic subset of $\mathcal{L}(Y)$, cf. [14]. This plays a key role in the proof of the rationality of $P(T,\chi)$. For a survey on applications of quantifier elimination results for valued fields, see [11].

2.3. Grothendieck groups of varieties

Let S be an algebraic variety over k. By an S-variety we mean a variety X together with a morphism $X \to S$. The S-varieties from a category denoted by Var_S, the arrows are the morphisms that commute with the morphisms to S.

We denote by $K_0(\mathrm{Var}_S)$ *the Grothendieck group of S-varieties.* It is an abelian group generated by symbols $[X]$, for X an S-variety, with the relations $[X]=[Y]$ if X and Y are isomorphic in Var_S, and $[X]=[Y]+[X\setminus Y]$ if Y is Zariski closed in X. There is a natural ring structure on $K_0(\mathrm{Var}_S)$, the product of $[X]$ and $[Y]$ being equal to $[X\times_S Y]$. Sometimes we will also write $[X/S]$ instead of $[X]$, to emphasize the role of S. We write $\mathbb{L}$ to denote the class of $\mathbb{A}^1_k \times S$ in $K_0(\mathrm{Var}_S)$, where the morphism from $\mathbb{A}^1_k \times S$ to S is the natural projection. We denote by $\mathcal{M}_S$ the ring obtained from $K_0(\mathrm{Var}_S)$ by inverting $\mathbb{L}$. When A is a constructible subset of some S-variety, we define $[A/S]$ in the obvious way, writing A as a disjoint union of a finite number of locally closed subvarieties A_i. Indeed $[A/S] := \sum_i [A_i/S]$ does not depend on the choice of the subvarieties A_i.

When S consists of only one geometric point, i.e. $S=\mathrm{Spec}(k)$, then we will write $K_0(\mathrm{Var}_k)$ instead of $K_0(\mathrm{Var}_S)$ (to denote the Grothendieck group of algebraic varieties over k), and $\mathcal{M}_k$ instead of $\mathcal{M}_S$. Clearly the map $\mathrm{Var}_k \to K_0(\mathrm{Var}_k)$ is the universal generalized Euler characteristic, in the sense that any generalized Euler characteristic on Var_k factors through it.

In our papers [13]–[18], we always work with $K_0(\mathrm{Var}_k)$, but recently E. Looijenga, in his Bourbaki talk [27], introduced the relative Grothendieck ring

$K_0(\mathrm{Var}_S)$, stating some of our results in a stronger form. For example, considering $\mathcal{L}_n(X)$ as an X-variety through the morphism π_0^n, our proof of theorem 2.1 actually yields the slightly stronger

Theorem 2.2. *Let X be a variety over k, then the power series*

$$J(T) := \sum_{n\geq 0} [\mathcal{L}_n(X)/X]\, T^n, \quad P(T) := \sum_{n\geq 0} [\pi_n(\mathcal{L}(X))/X]\, T^n$$

in $\mathcal{M}_X[[T]]$ are rational, with denominator a product of polynomials of the form $1 - \mathbb{L}^a T^b$, with $b \in \mathbb{N} \setminus \{0\}, a \in \mathbb{Z}$.

2.4. Equivariant Grothendieck groups

We need some technical preparation in order to take care of the monodromy actions in the next section.

For any positive integer n, let μ_n be the group of all n-th roots of unity (in some fixed algebraic closure of k). Note that μ_n is actually an algebraic variety over k, namely $\mathrm{Spec}(k[x]/(x^n - 1))$. The μ_n form a projective system, with respect to the maps $\mu_{nd} \to \mu_n \colon x \mapsto x^d$. We denote by $\hat{\mu}$ the projective limit of the μ_n. Note that the group $\hat{\mu}$ is not an algebraic variety. It is called a pro-variety.

Let X be an S-variety. A *good μ_n-action on X* is a group action $\mu_n \times X \to X$ which is a morphism of S-varieties, such that each orbit is contained in an affine subvariety of X. This last condition is automatically satisfied when X is a quasi projective variety. A *good $\hat{\mu}$-action on X* is an action of $\hat{\mu}$ on X which factors through a good μ_n-action, for some n.

The *monodromic Grothendieck group* $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ is defined as the abelian group generated by symbols $[X, \hat{\mu}]$ (also denoted by $[X/S, \hat{\mu}]$, or simply $[X]$), for X an S-variety with good $\hat{\mu}$-action, with the relations $[X, \hat{\mu}] = [Y, \hat{\mu}]$ if X and Y are isomorphic as S-varieties with $\hat{\mu}$ -ction, and $[X, \hat{\mu}] = [Y, \hat{\mu}] + [X \setminus Y, \hat{\mu}]$ if Y is Zariski closed in X with the $\hat{\mu}$-action on Y induced by the one on X, and moreover $[X \times V, \hat{\mu}] = [X \times \mathbb{A}_k^n, \hat{\mu}]$ where V is the n-dimensional affine space over k with any good $\hat{\mu}$-action, and $\mathbb{A}_k^n$ is taken with the trivial $\hat{\mu}$-action. There is a natural ring structure on $K_0^{\hat{\mu}}(\mathrm{Var}_S)$, the product being induced by the fiber product over S. We write $\mathbb{L}$ to denote the class in $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ of $\mathbb{A}_k^1 \times S$ with the trivial $\hat{\mu}$-action.

We denote by $\mathcal{M}_S^{\hat{\mu}}$ the ring obtained from $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ by inverting $\mathbb{L}$. When A is a constructible subset of X which is stable under the $\hat{\mu}$-action, then we define $[A, \hat{\mu}]$ in the obvious way. When S consists of only one geometric point, i.e. $S = \mathrm{Spec}(k)$, then we will write $K_0^{\hat{\mu}}(\mathrm{Var}_k)$ instead of $K_0^{\hat{\mu}}(\mathrm{Var}_S)$. The group $K_0^{\hat{\mu}}(\mathrm{Var}_k)$ was first introduced in [18].

Note that for any $s \in S(k)$ we have natural maps $K_0^{\hat{\mu}}(\mathrm{Var}_S) \to K_0^{\hat{\mu}}(\mathrm{Var}_k)$ and $\mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_k^{\hat{\mu}}$ given by $[X, \hat{\mu}] \to [X_s, \hat{\mu}]$, where X_s denotes the fiber at s of $X \to S$.

Although $\mathcal{M}_k^{\hat{\mu}}$ is a very complicated ring, there are many interesting morphisms from it to simpler rings. For example when $k = \mathbb{C}$, for any character α of $\hat{\mu}$

(i.e. a group homomorphism $\alpha\colon \hat{\mu} \to \mathbb{C}^\times$), there is a natural ring homomorphism

$$\chi_{\rm top}(-,\alpha)\colon \mathcal{M}_k^{\hat{\mu}} \longrightarrow \mathbb{Z}\colon \quad X \longmapsto \sum_{q\geq 0}(-1)^q \dim H^q(X,\mathbb{C})_\alpha\,,$$

where $H^*(X,\mathbb{C})_\alpha$ is the part of $H^*(X,\mathbb{C})$ on which $\hat{\mu}$ acts by multiplication by α.

3. The Motivic Zeta Function of a Regular Function

Let X be a nonsingular irreducible algebraic variety over k of dimension d and $f\colon X \to \mathbb{A}^1_k$ a non constant morphism. In this section we introduce several new invariants of f. These are constructed using arc spaces. We first recall in 3.1 some classical invariants associated to f. In what follows we denote by X_0 the *locus of* $f=0$ in X.

3.1. The monodromy zeta function and the Hodge spectrum

In this subsection 3.1 we suppose that $k=\mathbb{C}$. Let x be a point of $X_0 = f^{-1}(0)$. We fix a smooth metric on X.

3.1.1. MONODROMY We set $X^\times_{\epsilon,\eta} := B(x,\epsilon)\cap f^{-1}(D^\times_\eta)$, with $B(x,\epsilon)$ the open ball of radius ϵ centered at x and $D^\times_\eta := D_\eta\setminus\{0\}$, with D_η the open disk of radius η centered at 0. For $0<\eta\ll\epsilon\ll 1$, the restriction of f to $X^\times_{\epsilon,\eta}$ is a locally trivial fibration, called the *Milnor fibration,* onto $D^\times_\eta$ with fiber F_x, *the Milnor fiber at* x. The action of a characteristic homeomorphism of this fibration on cohomology gives rise to the *monodromy operator*

$$M_x\colon H^\cdot(F_x,\mathbb{Q}) \to H^\cdot(F_x,\mathbb{Q})\,.$$

For any natural number n, we consider the *Lefschetz number*

$$\Lambda(M_x^n) := \sum_{q\geq 0}(-1)^q\,\mathrm{Trace}\,(M_x^n, H^q(F_x,\mathbb{Q}))\,,$$

of the n-th iterate of M_x. These numbers are related to the *monodromy zeta function of* f *at* x

$$Z_x^{\rm mon}(T) := \prod_{q\geq 0}\mathrm{Det}\,(Id - TM_x, H^q(F_x,\mathbb{Q}))^{(-1)^q}$$

as follows if one writes $\Lambda(M_x^n)=\sum_{i|n}s_i$ for $n\geq 1$, then $Z_x^{\rm mon}(T)=\prod_{i\geq 1}(1-t^i)^{s_i/i}$. The monodromy zeta function of f at x (or equivalently the Lefschetz numbers) is an important topological invariant of f which has been studied intensively.

3.1.2. Hodge structures A *Hodge structure* is a finite dimensional $\mathbb{Q}$-vector-space H together with a bigrading $H \otimes \mathbb{C} = \oplus_{p,q\in\mathbb{Z}} H^{p,q}$, such that $H^{q,p}$ is the complex conjugate of $H^{p,q}$ and each *weight m summand*, $\oplus_{p+q=m} H^{p,q}$, is defined over $\mathbb{Q}$. The Hodge structures, with the evident notion of morphism, form an abelian category HS with tensor product. The elements of the Grothendieck group $K_0(\mathrm{HS})$ of this abelian category are representable as a formal difference of Hodge structures $[H] - [H']$, and $[H] = [H']$ iff $H \cong H'$. Note that $K_0(\mathrm{HS})$ becomes a ring with respect to the tensor product.

A *mixed Hodge structure* is a finite dimensional $\mathbb{Q}$-vector space V with a finite increasing filtration $W_\bullet V$, called the *weight filtration*, such that the associated graded vector space $\mathrm{Gr}^W_\bullet(V)$ underlies a Hodge structure having $\mathrm{Gr}^W_m(V)$ as weight m summand. Note that V determines in a natural way an element [V] in $K_0(\mathrm{HS})$, namely $[V] := \sum_m [\mathrm{Gr}^W_m(V)]$.

When X is an algebraic variety over $k = \mathbb{C}$, the simplicial cohomology groups $H^i_c(X, \mathbb{Q})$ of X, with compact support, underly a natural mixed Hodge structure, and the *Hodge characteristic* $\chi_h(X)$ of X (with compact support) is defined by

$$\chi_h(X) := \sum_i (-1)^i [H^i_c(X, \mathbb{Q})] \in K_0(\mathrm{HS}) .$$

This yields a map $\chi_h \colon \mathrm{Var}_{\mathbb{C}} \to K_0(\mathrm{HS})$, which is a generalized Euler characteristic, and which factors through $\mathcal{M}_k$, because $\chi_k(\mathbb{A}^1_k)$ is actually invertible in the ring $K_0(\mathrm{HS})$. When X is proper an smooth, the mixed Hodge structure on $H^i_c(X, \mathbb{Q})$ is in fact a Hodge structure, the weight filtration being concentrated in weight i. We refer to [41] for an introduction to Hodge structures.

3.1.3. The Hodge spectrum The cohomology groups $H^i_c(F_x, \mathbb{Q})$ of the Milnor fiber F_x carry a natural mixed Hodge structure ([39, 31, 32]), which is compatible with the monodromy operator M_x. Hence we can define the Hodge characteristic $\chi_h(F_x)$ of F_x by

$$\chi_h(F_x) := \sum_i (-1)^i [H^i(F_x, \mathbb{Q})] \in K_0(\mathrm{HS}) .$$

Actually by taking into account the monodromy action we can consider $\chi_h(F_x)$ as an element of the Grothendieck group $K_0(\mathrm{HS}^{\mathrm{mon}})$ of the abelian category $\mathrm{HS}^{\mathrm{mon}}$ of Hodge structures with a quasi-unipotent endomorphism. (Quasi-unipotent means that some power of it is unipotent.) Again $K_0(\mathrm{HS}^{\mathrm{mon}})$ is a ring by the tensor product.

There is a natural linear map, called the Hodge spectrum

$$\mathrm{hsp} \colon K_0(\mathrm{HS}^{\mathrm{mon}}) \to \mathbb{Z}[t^{1/\mathbb{N}}] := \cup_{n\geq 1} \mathbb{Z}[t^{1/n}, t^{-1/n}] ,$$

with $\mathrm{hsp}([H]) := \sum_{\alpha\in\mathbb{Q}\cap[0,1[} t^\alpha (\sum_{p,q\in\mathbb{Z}} \dim(H^{p,q})_\alpha) t^p$, for any Hodge structure H with a quasi-unipotent endomorphism, where $H^{p,q}_\alpha$ is the generalized eigenspace of $H^{p,q}$ with respect to the eigenvalue $e^{2\pi\sqrt{-1}\alpha}$. Note that hsp is not a ring homomorphism, although it becomes one when we endow $K_0(\mathrm{HS}^{\mathrm{mon}})$ with a different ring multiplication, namely the one induced by the operation $*$ in section 5.

We recall that $\mathrm{hsp}(f,x) := (-1)^{d-1}\mathrm{hsp}(\chi_h(F_x) - 1)$ is called the Hodge spectrum of f at x. It is a very important invariant, with remarkable properties, see [39, 40, 42].

3.2. The motivic zeta function

Let $n \geq 1$ be an integer. The morphism $f\colon X \to \mathbb{A}^1_k$ induces a morphism $f_n\colon \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}^1_k)$.

Any point α of $\mathcal{L}(\mathbb{A}^1_k)$, resp. $\mathcal{L}_n(\mathbb{A}^1_k)$, yields a K-rational point, for some field K containing k, and hence a power series $\alpha(t) \in K[[t]]$, resp. $\alpha(t) \in K[[t]]/t^{n+1}$. This yields maps

$$\mathrm{ord}_t\colon \mathcal{L}(\mathbb{A}^1_k) \to \mathbb{N} \cup \{\infty\}, \quad \mathrm{ord}_t\colon \mathcal{L}_n(\mathbb{A}^1_k) \to \{0,1,\ldots,n,\infty\},,$$

with $\mathrm{ord}_t\alpha$ the largest e such that t^e divides $\alpha(t)$.

We set

$$\mathfrak{X}_n := \{\varphi \in \mathcal{L}_n(X) \,|\, \mathrm{ord}_t f_n(\varphi) = n\},.$$

This is a locally closed subvariety of $\mathcal{L}_n(X)$. Note that $\mathfrak{X}_n$ is actually an X_0-variety, through the morphism $\pi^n_0\colon \mathcal{L}_n(X) \to X$. Indeed $\pi^n_0(\mathfrak{X}_n) \subset X_0$, since $n \geq 1$. We consider the morphism

$$\bar{f}_n\colon \mathfrak{X}_n \to \mathbb{G}_{m,k} := \mathbb{A}^1_k \setminus \{0\},,$$

sending a point φ in $\mathfrak{X}_n$ to the coefficient of t^n in $f_n(\varphi)$. There is a natural action of $\mathbb{G}_{m,k}$ on $\mathfrak{X}_n$ given by $a \cdot \varphi(t) = \varphi(at)$, where $\varphi(t)$ is the vector of power series corresponding to φ (in some local coordinate system). Since $\bar{f}_n(a \cdot \varphi) = a^n \bar{f}_n(\varphi)$ it follows that $\bar{f}_n$ is a locally trivial fibration.

We denote by $\mathfrak{X}_{n,1}$ the fiber $\bar{f}_n^{-1}(1)$. Note that the action of $\mathbb{G}_{m,k}$ on $\mathfrak{X}_n$ induces a good action of μ_n (and hence of $\hat{\mu}$) on $\mathfrak{X}_{n,1}$. Since $\bar{f}_n$ is a locally trivial fibration, the X_0-variety $\mathfrak{X}_{n,1}$ and the action of μ_n on it, completely determines both the variety $\mathfrak{X}_n$ and the morphism

$$(\bar{f}_n, \pi^n_0)\colon \mathfrak{X}_n \to \mathbb{G}_{m,k} \times X_0\,.$$

Indeed it is easy to verify that $\mathfrak{X}_n$, as a $(\mathbb{G}_{m,k} \times X_0)$-variety, is isomorphic to the quotient of $\mathfrak{X}_{n,1} \times \mathbb{G}_{m,k}$ under the μ_n-action defined by $a(\varphi,b) = (a\varphi, a^{-1}b)$.

Definition 3.1. *The motivic zeta function of $f\colon X \to \mathbb{A}^1_k$, is the power series over $\mathcal{M}^{\hat{\mu}}_{X_0}$ defined by*

$$Z(T) := \sum_{n \geq 1} [\mathfrak{X}_{n,1}/X_0, \hat{\mu}]\, \mathbb{L}^{-nd}\, T^n\,.$$

Moreover we define the naive motivic zeta function of f as the power series over $\mathcal{M}_{X_0}$ defined by

$$Z^{\mathrm{naive}}(T) := \sum_{n \geq 1} [\mathfrak{X}_n/X_0]\, \mathbb{L}^{-nd}\, T^n\,.$$

Theorem 3.2 and corollary 3.3 below show that $Z(T)$ and $Z^{\mathrm{naive}}(T)$ are rational. In 3.4 and 3.5 we will see that $Z(T)$ and $Z^{\mathrm{naive}}(T)$ give rise to interesting new invariants of f. The definition of $Z(T)$ goes back to [18] (in the non-relative version working in $\mathcal{M}_k^{\hat{\mu}}$). Many related motivic zeta functions, and $Z^{\mathrm{naive}}(T)$, were first introduced in [13], inspired by work of Kontsevich [23]. The idea of rather working in the relative Grothendieck group was introduced by Looijenga [27].

3.3. A formula for the motivic zeta function

We recall that X_0 denotes the locus of $f = 0$ in X. Let (Y, h) be a resolution of f. By this, we mean that Y is a nonsingular and irreducible algebraic variety over k, $h\colon Y \to X$ is a proper morphism, that the restriction $h\colon Y \setminus h^{-1}(X_0) \to X \setminus X_0$ is an isomorphism, and that $h^{-1}(X_0)$ has only normal crossings as a subvariety of Y.

We denote by $E_i, i \in J$, the irreducible components (over k) of $h^{-1}(X_0)$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y, and by $\nu_i - 1$ the multiplicity of E_i in the divisor of $h^* dx$, where dx is a local non vanishing volume form at any point of $h(E_i)$, i.e. a local generator of the sheaf of differential forms of maximal degree. For $i \in J$ and $I \subset J$, we consider the nonsingular varieties $E_i^\circ := E_i \setminus \cup_{j \neq i} E_j$, $E_I = \cap_{i \in I} E_i$, and $E_I^\circ := E_I \setminus \cup_{j \in J \setminus I} E_j$.

Let $m_I = \gcd(N_i)_{i \in I}$. We introduce an unramified Galois cover $\tilde{E}_I^\circ$ of E_I°, with Galois group μ_{m_I}, as follows. Let U be an affine Zariski open subset of Y, such that, on U, $f \circ h = u v^{m_I}$, with u a unit on U and v a morphism from U to $\mathbb{A}_k^1$. Then the restriction of $\tilde{E}_I^\circ$ above $E_I^\circ \cap U$, denoted by $\tilde{E}_I^\circ \cap U$, is defined as

$$\{(z, y) \in \mathbb{A}_k^1 \times (E_I^\circ \cap U) | z^{m_I} = u^{-1}\}.$$

Note that E_I° can be covered by such affine open subsets U of Y. Gluing together the covers $\tilde{E}_I^\circ \cap U$, in the obvious way, we obtain the cover $\tilde{E}_I^\circ$ of E_I° which has a natural μ_{m_I}-action (obtained by multiplying the z-coordinate with the elements of μ_{m_I}). This μ_{m_I}-action on $\tilde{E}_I^\circ$ induces an $\hat{\mu}$-action on $\tilde{E}_I^\circ$ in the obvious way.

Theorem 3.2. ([18, 27]) *With the previous notations, the following relation holds in* $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$

$$Z(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ / X_0, \hat{\mu}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

From the above theorem one easily deduces (using e.g. Lemma 5.1 in [27]) the following corollary, which is basically a special case of Theorem 2.2.1 in [13].

Corollary 3.3. *With the previous notations, the following relation holds in* $\mathcal{M}_{X_0}[[T]]$

$$Z^{\mathrm{naive}}(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L} - 1)^{|I|} [E_I^\circ / X_0] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

3.3.1. PROOF OF THE RATIONALITY OF $J(T)$ We defined $J(T)$ in theorem 2.2 above. We will now discuss the proof of the rationality of $J(T)$ in the special case when the variety X in theorem 2.2 is the locus X_0 of a polynomial f in the affine space $\mathbb{A}^d_k$. Let $Z^{\text{naive}}(T)$ be the naive motivic zeta function of $f\colon \mathbb{A}^d_k \to \mathbb{A}^1_k$. It is straightforward to verify that $J(T\mathbb{L}^{-d}) = \frac{[X_0]-Z^{\text{naive}}(T)}{1-T}$. Hence the rationality of $J(T)$ is a direct consequence of corollary 3.3.

3.4. The topological zeta functions

Let $\mathcal{M}_{S,\text{loc}}$ resp. $\mathcal{M}^{\hat{\mu}}_{S,\text{loc}}$, be the ring obtained from $\mathcal{M}_S$, resp. $\mathcal{M}^{\hat{\mu}}_S$, by inverting the elements $[\mathbb{P}^i_k] = 1+\mathbb{L}+\mathbb{L}^2+\cdots+\mathbb{L}^i$, for $i = 1,2,3,\ldots$, where $\mathbb{P}^i_k$ denotes the i-dimensional projective space over k.

We keep the notations of 3.3, but take $k = \mathbb{C}$. For any integer $s \geq 1$, evaluating $Z^{\text{naive}}(T)$ at $T = \mathbb{L}^{-s}$ yields a well-defined element of $\mathcal{M}_{X_0,\text{loc}}$, namely $\sum_{\emptyset\neq I\subset J}[E_I^\circ/X_0]\prod_{i\in I}[\mathbb{P}^{sN_i+\nu_i-1}]^{-1}$. Applying the topological Euler characteristic χ_{top} we obtain

$$Z_{\text{top}}(s) := \chi_{\text{top}}(Z^{\text{naive}}(\mathbb{L}^{-s})) := \sum_{\emptyset\neq I\subset J}\chi_{\text{top}}(E_I^\circ)\prod_{i\in I}\frac{1}{sN_i+\nu_i}. \qquad (*)$$

We call $Z_{\text{top}}(s)$, considered as a rational function in the variable s, the *untwisted topological zeta function* of $f\colon X \to \mathbb{A}^1_k$.

Evaluating $(\mathbb{L}-1)Z(T)$, instead of $Z^{\text{naive}}(T)$, at $T = \mathbb{L}^{-s}$, and applying the equivariant topological Euler characteristic $\chi_{\text{top}}(-,\alpha)$, with $\alpha\colon \hat{\mu} \to \mathbb{C}$ a character of order e, we obtain the *twisted topological zeta function* (for any integer $e \geq 1$)

$$\begin{aligned} Z^{(e)}_{\text{top}}(s) :=& \chi_{\text{top}}((\mathbb{L}-1)Z(\mathbb{L}^{-s}),\alpha) \\ :=& \sum_{\emptyset\neq I\subset J,\, e|m_I}\chi_{\text{top}}(E_I^\circ)\prod_{i\in I}\frac{1}{sN_i+\nu_i} \end{aligned} \qquad (**)$$

Note that, if we would define the topological zeta functions by the right-hand-side of $(*)$ and $(**)$, then it would be not at all clear that this is independent of the choosen resolution. It is the intrinsic definition using the motivic zeta function (which is based on the notion of arc spaces) that makes this independence obvious. The topological zeta functions were first introduced by Denef and Loeser in [12] using p-adic integration and the Grothendieck-Lefschetz trace formula to prove their independence of the choosen resolution. Our approach using arc spaces first appeared in [13].

The topological zeta functions are quite subtle invariants of f, and have been further investigated by Veys [45, 46]. There are some fascinating conjectures about them.

Conjecture 3.4. (Monodromy conjecture for $\boldsymbol{Z^{(e)}_{\text{top}}}$) *If s is a pole of $Z^{(e)}_{\text{top}}(s)$ then $e^{2\pi\sqrt{-1}s}$ is an eigenvalue of the monodromy action on the cohomology of the Milnor fiber at some point of the locus of f.*

Conjecture 3.5. Holomorphy conjecture for $Z^{(e)}_{\mathrm{top}}$ *The function $Z^{(e)}_{\mathrm{top}}(s)$ is a polynomial in s, unless there is an eigenvalue with order divisible by e of the monodromy action on the cohomology of the Milnor fiber at some point of the locus of f.*

Loeser [26] and Veys [44] proved that these conjectures are true when $X = \mathbb{A}^2_{\mathbb{C}}$. A lot of experimental evidence has been obtained by Veys [43] when $X = \mathbb{A}^3_{\mathbb{C}}$. We refer to [46] and [47] for very interesting generalizations.

3.5. Relations with monodromy and the motivic Milnor fiber

The Lefschetz numbers $\Lambda(M^n_x)$ of f at x, which we recalled in 3.1.1 can be expressed in terms of a resolution of f, by the following formula of A'Campo.

Theorem 3.6. (A'Campo, [1]) *Let $k = \mathbb{C}$. Assume the notations of 3.1 and 3.3. Then for any integer $n \geq 0$ we have*

$$\Lambda(M^n_x) = \sum_{N_i | n} N_i \, \chi_{\mathrm{top}}(E^\circ_i \cap h^{-1}(x)) \,.$$

In particular we see that the right-hand-side of the above formula is independent of the choosen resolution h. Note that the material in 3.3 and 3.4 yields many other expressions which are independent of the chosen resolution, but A'Campo's result was probably the first in this direction.

Applying the natural map $\mathrm{Fiber}_x \colon \mathcal{M}^{\hat{\mu}}_{X_0} \to \mathcal{M}^{\hat{\mu}}_{k} \colon [A/X_0, \hat{\mu}] \mapsto [A \times_{X_0} \{x\}, \hat{\mu}]$, followed by the equivariant topological Euler characteristic $\chi_{\mathrm{top}}(-,1)$, on the coefficients of $Z(T)$ and using theorem 3.2, we obtain the following theorem.

Theorem 3.7. ([18]) *Let $k = \mathbb{C}$, then for any integer $n \geq 1$ we have $\Lambda(M^n_x) = \chi_{\mathrm{top}}(\mathfrak{X}_{n,1,x})$, where*

$$\mathfrak{X}_{n,1,x} := \mathfrak{X}_{n,1} \times_{X_0} \{x\} \,.$$

Thus we see that the monodromy zeta function of f at x is completely determined by the motivic zeta function $Z(T)$. Next, we will see that also the Hodge spectrum of f at x is determined by $Z(T)$.

Definition 3.8. ([13, 18]) *Expanding the rational function $Z(T)$ as a power series in T^{-1} and taking minus its constant term, yields a well defined element of $\mathcal{M}^{\hat{\mu}}_{X_0}$, namely*

$$\mathcal{S} := -\lim_{T \to \infty} Z(T) := \sum_{\emptyset \neq I \subset J} (1 - \mathbb{L})^{|I|-1} [\tilde{E}^\circ_I] \,.$$

Moreover we set $\mathcal{S}_x := \mathrm{Fiber}_x(\mathcal{S}) \in \mathcal{M}^{\hat{\mu}}_k$. Instead of $\mathcal{S}$ and $\mathcal{S}_x$ we will also write $\mathcal{S}_f$ and $\mathcal{S}_{f,x}$. These definitions hold for any field k of characteristic zero.

Note again that the most right-hand-side of the above formula is independent of the choosen resolution (because of its relation to $Z(T)$), although a priori this is not at all evident.

We strongly believe that $\mathcal{S}_x$ is the correct virtual motivic incarnation of the Milnor fiber F_x of f at x (which is in itself not at all motivic). We will see below (theorem 3.10) that this is indeed true for the Hodge realization. A similar result holds for ℓ-adic cohomology, see [10]. Moreover we strongly believe that $\mathcal{S}$ is the virtual motivic incarnation of the so called complex of nearby cycles ψ_f of f, which is a complex of sheaves on X_0. For the definition of ψ_f and the complex ϕ_f of vanishing cycles, we refer to [35], Exp. XIII; but we will not need these notions in the present survey. Inspired by the notation ϕ_f from the theory of vanishing cycles, we introduce the following

Notation 3.9. *We set* $\mathcal{S}_f^{\phi} := (-1)^{d-1}(\mathcal{S}_f - [X_0]) \in \mathcal{M}_{X_0}^{\hat{\mu}}$ *and* $\mathcal{S}_{f,x}^{\phi} := (-1)^{d-1}(\mathcal{S}_{f,x} - 1) \in \mathcal{M}_k^{\hat{\mu}}$.

We regard $\mathcal{S}_f^{\phi}$ as the virtual motivic incarnation of the complex $\phi_f[d-1]$.

Assume now again that $k = \mathbb{C}$. We denote by χ_h the canonical ring homomorphism (called the Hodge characteristic)

$$\chi_h \colon \mathcal{M}_k^{\hat{\mu}} \to K_0(\mathrm{HS}^{\mathrm{mon}}) \,,$$

which associates to any complex algebraic variety Z, with a good μ_n-action, its Hodge characteristic as defined in 3.1.2, together with the endomorphism induced by $Z \to Z \colon z \mapsto e^{2\pi\sqrt{-1}/n} z$. (For the definition of $K_0(\mathrm{HS}^{\mathrm{mon}})$, see 3.1.3.)

Theorem 3.10. ([13]) *Assume the above notation with $k = \mathbb{C}$, and the notation of 3.1. Then we have the following equality in* $K_0(\mathrm{HS}^{\mathrm{mon}})$

$$\chi_h(F_x) = \chi_h(\mathcal{S}_x) \,.$$

Moreover this theorem can be enhanced as an equality in the Grothendieck group of the abelian category of variations of Hodge structures with a quasi-unipotent endomorphism, when we replace $\mathcal{S}_x$ by $\mathcal{S}$, and F_x by ψ_f.

Theorem 3.10 yields that $\mathrm{hsp}(f, x) = \mathrm{hsp}(\chi_h(\mathcal{S}_{f,x}^{\phi}))$. Thus the motivic zeta function $Z(T)$ completely determines the Hodge spectrum of f at x.

4. Motivic Integration and the Proof of Theorem 3.2

The notion of motivic integration on $\mathcal{L}(X)$ is due to Kontsevich [23], who discovered its basic properties when X is nonsingular. This subject has been further developed by Batyrev [5, 6] and Denef-Loeser [13, 14, 15, 16, 17, 18]. See also the recent report by Looijenga [27] which contains some substantial improvements. Actually the best way to understand motivic integration is to consider it as being an analogue of p-adic integration, cf. section 6.

Let X be an algebraic variety over k of pure dimension d, not necessarily nonsingular. Let X_{sing} denote the singular locus of X.

4.1. Naive motivic integration

A subset of A of $\mathcal{L}(X)$ is called *constructible* if $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$ for some integer $n \geq 0$. A subset A of $\mathcal{L}(X)$ is called *stable* if it is constructible and $A \cap \mathcal{L}(X_{\mathrm{sing}}) = \emptyset$. If $A \subset \mathcal{L}(X)$ is stable, then $[\pi_n(A)]\mathbb{L}^{-(n+1)d}$, considered as an element of $\mathcal{M}_k$, stabilizes for n big enough, and

$$\tilde{\mu}(A) := \lim_{n\to\infty} [\pi_n(A)]\mathbb{L}^{-(n+1)d} \in \mathcal{M}_k$$

is called the *naive motivic measure* of A. When X is nonsingular, this claim follows from the fact that the natural maps $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are locally trivial fibrations with fiber $\mathbb{A}_k^d$. In the general case, the claim follows from [14], Lemma 4.1.

When $\theta\colon A \to \mathcal{M}_k$ is a map with finite image whose fibers are stable subsets of $\mathcal{L}(X)$, we define the integral $\int_A \theta d\tilde{\mu} := \sum_{c\in \mathrm{Image}\,\theta} c\tilde{\mu}(\theta^{-1}(c))$. The most fundamental result in the theory of arc spaces is the following change of variables formula, which was first obtained by Kontsevich [23] when X is nonsingular.

Theorem 4.1. ([23, 14, 16]) *Let $h\colon Y \to X$ be a morphism of algebraic varieties over k. Suppose that h is birational and proper. Let $A \subset \mathcal{L}(X)$ be stable and suppose that* $\mathrm{ord}_t\mathrm{Jac}_h$ *is bounded on $h^{-1}(A) \subset \mathcal{L}(Y)$. Then*

$$\tilde{\mu}(A) = \int_{h^{-1}(A)} \mathbb{L}^{-\mathrm{ord}_t\mathrm{Jac}_h} d\tilde{\mu}\,.$$

In the above theorem, $\mathrm{ord}_t\mathrm{Jac}_h$, for $y \in \mathcal{L}(Y)$, denotes the t-order of the Jacobian of h at y. When X and Y are nonsingular this is the ord_t of the determinant of the Jacobian matrix of h at y with respect to any system of local coordinates on X and on Y. For the definition of $\mathrm{ord}_t\mathrm{Jac}_h$, in the general case, we refer to [14] and [16].

4.2. About the proof of theorem 3.2

The proof of theorem 3.2 consists of an explicit calculation of $[\mathfrak{X}_{n,1}/X_0, \hat{\mu}] \in \mathcal{M}_{X_0}^{\hat{\mu}}$ for each n. Note that in $\mathcal{M}_k$ we have the equality

$$[\mathfrak{X}_{n,1}] = \mathbb{L}^{(n+1)d}\tilde{\mu}(\pi_n^{-1}(\mathfrak{X}_{n,1}))\,.$$

Thus using the change of variables formula (theorem 4.1), we see that $[\mathfrak{X}_{n,1}]$ is equal to an integral over a stable subset of $\mathcal{L}(Y)$, where $h\colon Y \to X$ is a resolution of f as in 3.3. Because $f \circ h$ is locally a monomial, that integral can be explicitely calculated and yields an explicit expression for $[\mathfrak{X}_{n,1}]$ as an element of $\mathcal{M}_k$. Taking into account the μ_n-action on $\mathfrak{X}_{n,1}$ and the natural map $\mathfrak{X}_{n,1} \to X_0$, one actually obtains a similar formula for $[\mathfrak{X}_{n,1}/X_0, \hat{\mu}]$, which yields theorem 3.2.

4.3. Motivic integration

Let A be a constructible subset of $\mathcal{L}(X)$. When A is not stable, $[\pi_n(A)]\mathbb{L}^{-(n+1)d}$ will not always stabilize. However it is easy to prove (see [14]) that the limit

$$\mu(A) := \lim_{n\to\infty} [\pi_n(A)]\mathbb{L}^{-(n+1)d}$$

exists in the *completed Grothendieck group* $\hat{\mathcal{M}}_k$, which is the completion of $\mathcal{M}_k$ with respect to the filtration $F^m\mathcal{M}_k, m \in \mathbb{Z}$, where $F^m\mathcal{M}_k$ is the subgroup of $\mathcal{M}_k$ generated by the elements $[S]\mathbb{L}^{-i}$, with $S \in \mathrm{Var}_k$, $i - \dim S \geq m$. The completed Grothendieck ring $\mathcal{M}_k$ was first introduced by Kontsevich. (In a similar way one can define the completions $\hat{\mathcal{M}}_S$ and $\hat{\mathcal{M}}_S^{\hat{\mu}}$ of $\mathcal{M}_S$ and $\mathcal{M}_S^{\hat{\mu}}$.) The element $\mu(A)$ of $\hat{\mathcal{M}}_k$ is called the *motivic measure of* A. This yields a σ-additive measure μ on the Boolean algebra of constructible subsets of $\mathcal{L}(X)$. Actually all the above still works when A is a semi-algebraic subset of $\mathcal{L}(X)$, cf. [14]. It is even possible to define the notion of a measurable subset of $\mathcal{L}(X)$ and to integrate measurable functions on $\mathcal{L}(X)$, see [5, 16].

The change of variables formula (theorem 4.1) remains true with $\tilde{\mu}$ replaced by μ, for any constructible (or measurable) subset of $\mathcal{L}(X)$, without assuming that $\mathrm{ord}_t \mathrm{Jac}_h$ is bounded on $h^{-1}(A)$.

It is not known whether the natural map $\mathcal{M}_k \to \hat{\mathcal{M}}_k$ is injective, but the topological Euler characteristic, the Hodge-Deligne polynomial, the Hodge characteristic, and many other important generalized Euler characteristics all factor through the image $\bar{\mathcal{M}}_k$ of $\mathcal{M}_k$ in $\hat{\mathcal{M}}_k$ (after inverting the image of $\mathbb{L}$ in the target ring).

We can consider the motivic volume of the whole arc space $\mathcal{L}(X)$, namely $\mu(\mathcal{L}(X))$. Clearly, when X is nonsingular, $\mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}$ in $\hat{\mathcal{M}}_k$. Here and in what follows, we denote the image of $[X]$, resp. $\mathbb{L}$, in $\hat{\mathcal{M}}_k$ again by $[X]$, resp. $\mathbb{L}$. When X is not necessarily nonsingular, we can calculate $\mu(\mathcal{L}(X))$ using a suitable resolution of singularities $h\colon Y \to X$ of X. More precisely we have the following

Theorem 4.2. *Let $h\colon Y \to X$ be a proper birational morphism with Y nonsingular. Assume that the exceptional locus of h has normal crossings and that the image of $h^*(\Omega_X^d)$ in Ω_Y^d is an invertible sheaf, where Ω_X^d and Ω_Y^d denote the sheaf of differential forms of maximal degree. Let $E_j, j \in J$, be the k-irreducible components of the exceptional locus of h. For any subset I of J, set $E_I^\circ = (\cap_{i\in I} E_i) \setminus \cup_{j \in J\setminus I} E_j$. For $i \in I$, let $\nu_i - 1$ be the multiplicity along E_i of the divisor associated to $h^*(\Omega_X^d)$. Then, in $\hat{\mathcal{M}}_k$, we have*

$$\mu(\mathcal{L}(X)) = \mathbb{L}^{-d} \sum_{I \subset J} [E_I^\circ] \prod_{i \in I} [\mathbb{P}^{\nu_i - 1}]^{-1} .$$

In particular we see that $\mu(\mathcal{L}(X)) \in \bar{\mathcal{M}}_{k,\mathrm{loc}} \subset \hat{\mathcal{M}}_k$, where $\bar{\mathcal{M}}_{k,\mathrm{loc}}$ denotes the ring obtained from $\bar{\mathcal{M}}_k$ by inverting the elements $1 + \mathbb{L} + \cdots + \mathbb{L}^i$, for all $i = 1, 2, 3, \ldots$.

About the proof of this theorem, we remark that $\mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_t \mathrm{Jac}_h} d\mu$, by the change of variables formula. Because Jac_h is locally a monomial, this integral can be easily calculated, which yields the theorem.

4.4. Applications

4.4.1. New invariants of singular varieties Suppose $k = \mathbb{C}$. Since χ_{top} and χ_{hp} factor through $\bar{\mathcal{M}}_k$, we have natural maps $\chi_{\mathrm{top}}\colon \bar{\mathcal{M}}_{k,\mathrm{loc}} \to \mathbb{Q}$ and

$\chi_{hp}\colon \bar{\mathcal{M}}_{k,\mathrm{loc}} \to \mathbb{Z}[[u,v]][u^{-1},v^{-1}]$. Hence we can consider $\chi_{\mathrm{top}}(\mathcal{L}(X)) \in \mathbb{Q}$ and $(uv)^d\chi_{\mathrm{hp}}(\mathcal{L}(X)) \in \mathbb{Z}[[u,v]]$, which are new invariants of X when X is singular. When X is nonsingular, these invariants equal $\chi_{\mathrm{top}}(X)$, resp. $\chi_{\mathrm{hp}}(X)$. We call the coefficients of $(uv)^d\chi_{\mathrm{hp}}(\mathcal{L}(X))$ (with an appropriate sign change) the arc-Hodge numbers of X. When X has only canonical Gorenstein singularities, Batyrev [5] introduced the so called *stringy Hodge numbers* of X, which are obtained in a similar way, replacing $\mu(\mathcal{L}(X))$ by $\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t\omega_X}\,d\mu$, where ω_X denotes the canonical class of X. The stringy Hodge numbers play an important role in the work of Batyrev on mirror symmetry, see [5, 7, 3, 2]. Other fascinating related invariants were obtained by Veys [46, 47].

4.4.2. Calabi-Yau manifolds Let X and Y be two Calabi-Yau manifolds, i.e. nonsingular proper complex algebraic varieties which admit a nonvanishing differential form of maximal degree, which we denote respectively by ω_X and ω_Y. Kontsevich [23] proved that X and Y have the same Hodge numbers and the same Hodge structure on their cohomology, when X and Y are birationally equivalent. The proof goes as follows There exists a nonsingular proper complex algebraic variety Z and birational morphisms $h_X\colon Z \to X$ and $h_Y\colon Z \to Y$. Note that $(h_Y \circ h_X^{-1})^*(\omega_Y)$ equals $c\,\omega_X$ for some $c \in \mathbb{C}^\times$ because ω_X has no zeroes. Hence $c\ h_X^*(\omega_X) = h_Y^*(\omega_Y)$. Thus $\mathrm{ord}_t\mathrm{Jac}_{h_X} = \mathrm{ord}_t\mathrm{Jac}_{h_Y}$ on $\mathcal{L}(Z)$, and by the change of variables formula both $\mu(\mathcal{L}(X))$ and $\mu(\mathcal{L}(Y))$ equal the same integral on $\mathcal{L}(Z)$. Because $\mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}$ and $\mu(\mathcal{L}(Y)) = [Y]\mathbb{L}^{-d}$, this implies that $[X]=[Y]$ in $\bar{\mathcal{M}}_k$, which finishes the proof.

Actually Batyrev [4] first proved that X and Y have the same Betti numbers using p-adic integration and the Weil conjectures, and Kontsevich invented motivic integration to prove that X and Y have the same Hodge numbers.

4.4.3. Euler characteristics and modifications Let $h\colon Y \to X$ be a modification of nonsingular algebraic varieties over k, meaning that h is a proper birational morphism. Assume that the exceptional locus of h has normal crossings, and let J, E_i, E_I° and ν_i be as in theorem 4.2. Because X is nonsingular, $\mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}$ and theorem 4.2 yields the following equality in $\bar{\mathcal{M}}_{k,\mathrm{loc}}$:

$$[X] = \sum_{I\subset J} [E_I^\circ] \prod_{i\in I} [\mathbb{P}^{\nu_i-1}]^{-1}. \tag{$*$}$$

a) When $k=\mathbb{C}$, applying the topological Euler characteristic on $(*)$ yields $\chi_{\mathrm{top}}(X) = \sum_{I\subset J}\chi(E_I^\circ)/\prod_{i\in I}\nu_i$. This surprising formula about the Euler characteristic of modifications was first obtained in [12] using p-adic integration and the Grothendieck-Lefschetz trace formula.

b) When $k=\mathbb{Q}$, applying the conductor (with respect to ℓ-adic cohomology, see section 2.2 yields the following remarkable formula for the conductor $c(X)$ of X

$$c(X) = \prod_{I\subset J} c(E_I^\circ)^{1/\prod_{i\in I}\nu_i}.$$

5. The Motivic Thom-Sebastiani Theorem

Let k be a field of characteristic zero, X and Y nonsingular irreducible algebraic varieties over k, $f : X \to \mathbb{A}^1_k$, $g : Y \to \mathbb{A}^1_k$ non constant morphisms, and $x \in X(k), y \in Y(k)$. We denote by $f * g$ the morphism

$$f * g \colon X \times Y \to \mathbb{A}^1_k \colon (x,y) \mapsto f(x) + g(y)\,.$$

The following theorem was first proved by A. Varchenko [42], when x and y are isolated singular points of f and g, and by M. Saito [33, 34] in the general case. A similar but much weaker result for the eigenvalues of monodromy was first proved by Thom and Sebastiani [37].

Theorem 5.1. (Thom-Sebastiani Theorem for the Hodge spectrum) *Assume the notation of 3.1.3, with $k = \mathbb{C}$. We have the following equality in $\mathbb{Z}[t^{1/\mathbb{N}}]$:*

$$\mathrm{hsp}(f * g, (x,y)) = \mathrm{hsp}(f,x)\,\mathrm{hsp}(g,y)\,.$$

We recall that $\mathrm{hsp}(f,x) = \mathrm{hsp}(\chi_h(\mathcal{S}^{\phi}_{f,x}))$, with the notation of 3.9. We will see next that the above theorem is a direct consequence of a much stronger result which expresses $\mathcal{S}^{\phi}_{f*g,(x,y)}$ in terms of $\mathcal{S}^{\phi}_{f,x}$ and $\mathcal{S}^{\phi}_{g,y}$. Below, we define a binary operation $*$ on $\mathcal{M}^{\hat{\mu}}_k$ which yields an alternative ring structure on $\mathcal{M}^{\hat{\mu}}_k$, such that $\mathrm{hsp} \circ \chi_h$ becomes a homomorphism of rings (which is not true for the usual multiplication on $\mathcal{M}^{\hat{\mu}}_k$).

Using the theory of arc spaces and the definition of $\mathcal{S}_f$ in terms of the motivic zeta function $Z(T)$, work of Denef, Loeser and Looijenga yields the following theorem

Theorem 5.2. (Motivic Thom-Sebastiani Theorem) *Let k be a field of characteristic zero. Then $\mathcal{S}^{\phi}_{f*g,(x,y)}$ and $\mathcal{S}^{\phi}_{f,x} * \mathcal{S}^{\phi}_{g,y}$ are equal in $\mathcal{M}^{\hat{\mu}}_k$, where the operation $*$ is defined below.*

Actually Denef and Loeser [15] first proved the above equality in the completed Grothendieck group of Chow motives. Later Looijenga [27] introduced the operation $*$ and proved, using basically the same method, an equality which is similar to theorem 5.2. The proof of theorem 5.2 uses arc spaces in a very essential way by deriving first a formula relating the motivic zeta functions of $f * g$, f and g, and taking afterwards the limit for $T \to \infty$. More precisely, set

$$Z^{\phi}_f(T) := (-1)^{d-1}\Big[Z_f(T) + [X_0] + \frac{Z^{\mathrm{naive}}_f(T) - [X_0]}{1-T}\Big]\,,$$

where $Z_f(T)$, resp. $Z^{\mathrm{naive}}_f(T)$, is the motivic, resp. naive motivic, zeta function of f, X_0 is the locus of $f = 0$ in X, and d is the dimension of X. Let $Z^{\phi}_{f,x}(T)$ be obtained from $Z^{\phi}_f(T)$ by applying the map fiber_x to its coefficients. Clearly $-\lim_{T\to\infty} Z^{\phi}_f(T)$ is $\mathcal{S}^{\phi}_f$. One proves that

$$Z^{\phi}_{f*g,(x,y)}(T) = Z^{\phi}_{f,x}(T) * Z^{\phi}_{g,y}(T)$$

in $\mathcal{M}_k^{\hat{\mu}}$, where $*$ is defined coefficientswise. This implies theorem 5.2 because $-\lim_{T\to\infty}$ commutes with $*$ on such power series without constant terms.

Finally, we explain the definition of the operation $*$ on $\mathcal{M}_k^{\hat{\mu}}$. Let X and Y be algebraic varieties over k with good μ_n-action, for some integer $n \geq 1$. Let J_n be the Fermat curve in $(\mathbb{A}_k^1 \setminus \{0\})^2$ defined by $u^n + v^n = 1$. There is an action of $\mu_n \times \mu_n$ on J_n given by $(\xi, \xi') \cdot (u, v) := (\xi u, \xi' v)$. We define $J_n(X, Y)$ in Var_k as the quotient of $J_n \times X \times Y$ under the equivalence relation given by $(\xi u, \xi' v, x, y) = (u, v, \xi x, \xi' y)$ for all $\xi, \xi' \in \mu_n$. We let μ_n act on $J_n(X, Y)$ by $\xi \cdot (u, v, x, y) := (\xi u, \xi v, x, y)$. This yields an element $[J_n(X, Y)]$ in $\mathcal{M}_k^{\hat{\mu}}$. If m is a divisor of n, and the action of μ_n on X and Y factors through μ_m, then $J_m(X, Y) = J_n(X, Y)$. Thus, in this way, we obtain a binary operation $J\colon \mathcal{M}_k^{\hat{\mu}} \times \mathcal{M}_k^{\hat{\mu}} \to \mathcal{M}_k^{\hat{\mu}}$, which was first introduced by Looijenga [27]. The operation J is commutative and bilinear over $\mathcal{M}_k$, considering $\mathcal{M}_k^{\hat{\mu}}$ as a module over $\mathcal{M}_k$ through the natural map $\mathcal{M}_k \to \mathcal{M}_k^{\hat{\mu}}$. One verifies that $J(a, 1) = (\mathbb{L} - 1)\bar{a} - a$, where $a \mapsto \bar{a}\colon \mathcal{M}_k^{\hat{\mu}} \to \mathcal{M}_k$ is the morphism induced by $[Z] \to$ [space of $\hat{\mu}$-orbits of Z], for any k-variety Z with good $\hat{\mu}$-action, cf. [27]. In particular we see that 1 is not a neutral element for the operation J. For this reason it is natural to introduce the operation $*$ on $\mathcal{M}_k^{\hat{\mu}}$ given by

$$a * b = -J(a, b) + (\mathbb{L} - 1)\overline{ab}\,,$$

for a and b in $\mathcal{M}_k^{\hat{\mu}}$. Clearly the operation $*$ is commutative and bilinear over $\mathcal{M}_k$, and $a * 1 = a$ for all a in $\mathcal{M}_k^{\hat{\mu}}$. Moreover one easily verifies that $\mathrm{hsp} \circ \chi_h$ is a ring homomorphism with respect to the alternative ring structure on $\mathcal{M}_k^{\hat{\mu}}$ given by $*$.

6. The Arithmetic Motivic Poincaré Series $P_{\mathrm{arith}}(T)$

6.1. The p-adic case

Assume that X is an algebraic variety over $\mathbb{Z}$, i.e. a reduced separated scheme of finite type over $\mathbb{Z}$. Let p be a prime number. We consider the Poincaré series

$$J_p(T) = \sum_{n \in \mathbb{N}} \#X(\mathbb{Z}/p^{n+1}\mathbb{Z})\, T^n, \quad P_p(T) = \sum_{n \in \mathbb{N}} \#(\pi_n(X(\mathbb{Z}_p)))\, T^n\,,$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers and π_n is the natural projection $\pi_n\colon \mathbb{Z}_p \to \mathbb{Z}/p^{n+1}\mathbb{Z}$. Igusa [22], resp. Denef [8], proved that $J_p(T)$, resp. $P_p(T)$, is a rational function of T. The proofs are based on p-adic integration, resolution of singularities, and for $P_p(T)$ also the theory of p-adic semi-algebraic sets. Actually the proof of theorem 2.2 about the rationality of $J(T)$ and $P(T)$ was very much inspired by the proofs of the rationality of $J_p(T)$ and $P_p(T)$, replacing p-adic integration by motivic integration. As a matter of fact, for all the material discussed in the previous sections, p-adic counterparts exist which were discovered first, see [9] for a survey.

6.2. Comparing $J(T)$ and $J_p(T)$

For any rational power series $G(T)$ over $K_0(\mathrm{Var}_{\mathbb{Q}})$ (with denominator a product of polynomials of the form $1-\mathbb{L}^aT^b, b \in \mathbb{N}\setminus\{0\}, a \in \mathbb{Z}$) we choose representatives in $K_0(\mathrm{Var}_{\mathbb{Z}})$ for the coefficients in $K_0(\mathrm{Var}_k)$ of numerator and denominator. In this way we find a power series over $K_0(\mathrm{Var}_{\mathbb{Z}})$, and, for any prime number p, we can apply to each coefficient the operation $N_p\colon K_0(\mathrm{Var}_{\mathbb{Z}}) \to \mathbb{Z}\colon [X] \mapsto \#X(\mathbb{Z}/p\mathbb{Z})$. This yields a power series over $\mathbb{Z}$ which we will denote by $N_p(G(T))$. If we choose other representatives in $K_0(\mathrm{Var}_{\mathbb{Z}})$, the resulting power series $N_p(G(T))$ will be the same for almost all p (i.e. for all but finitely many prime numbers p).

Comparing the proof of the rationality of $J(T)$ and $J_p(T)$ actually yields the following

Theorem 6.1. *Assume the notation of 6.1 and 6.2. For almost all p we have $J_p(T) = N_p(J(T))$.*

Also the motivic zeta functions $Z(T)$ and $Z^{\mathrm{naive}}(T)$, have similar arithmetic interpretations, related to Igusa's local zeta functions, see [13]. However it is not true in general that $N_p(P(T)) = P_p(T)$ for almost all p. Indeed $N_p(P(T))$ does not count the elements of $X(\mathbb{Z}/p^{n+1}\mathbb{Z})$ which can be lifted to $X(\mathbb{Z}_p)$, but counts (for almost all p) the elements of $X(\mathbb{Z}/p^{n+1}\mathbb{Z})$ which can be lifted to $X(\mathbb{Z}_p^{\mathrm{unram}})$, where $\mathbb{Z}_p^{\mathrm{unram}}$ is the maximal unramified extension of $\mathbb{Z}_p$. Note that the residue field of $\mathbb{Z}_p^{\mathrm{unram}}$ is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$.

6.3. The motivic Poincaré series $P_{\mathrm{arith}}(T)$

The above discussion leads to the question of defining in a canonical way a power series $P_{\mathrm{arith}}(T)$ over (some localization of) $K_0(\mathrm{Var}_Q)$ such that $N_p(P_{\mathrm{arith}})(T) = P_p(T)$ for almost all p.

In our recent paper [17], we construct, for any algebraic variety over k, in a canonical way, a rational power series $P_{\mathrm{arith}}(T)$ over $K_0(\mathrm{Mot}_k)\otimes\mathbb{Q}$, such that if $k=\mathbb{Q}$ then $N_p(P_{\mathrm{arith}}(T)) = P_p(T)$ for almost all p. Here Mot_k denotes the category of Chow motives over k. We refer to [38] for the definition of this important category, and we only remark here that there is a natural ring morphism $K_0(\mathrm{Var}_k) \to K_0(\mathrm{Mot}_k)$, see [19]. Actually the coefficients of $P_{\mathrm{arith}}(T)$ are in the image of $K_0(\mathrm{Var}_k)\otimes\mathbb{Q}$. We need to work at the level of Chow motives, to make our construction canonical.

The proof of our result is rather complicated and uses several results from mathematical logic (quantifier elimination for valued fields and finite fields). For a survey on such relations between logic, geometry and arithmetic, we refer to [11]. Examples seem to suggest that $P_{\mathrm{arith}}(T)$ captures more geometric information than $P(T)$, but very little is presently known about it!

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