Evolution of a Closed Interface between Two Liquids of Different Types

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Abstract. We study a free boundary problem governing the motion of two immiscible viscous capillary fluids. The fluids occupy the whole space $\mathbb{R}^3$ but one of them should have a finite volume. Every liquid may be of both types: compressible and incompressible.

Local (with respect to time) unique solvability of the problem is obtained in the Sobolev-Slobodetskiĭ spaces. After the passage to Lagrangian coordinates, one obtains a nonlinear, noncoercive initial boundary-value problem the proof of the existence theorem for which is based on the method of successive approximations and on an explicit solution of a model linear problem with a plane interface between the liquids.

Some restrictions to the fluid viscosities appear in the case when at least, one of the liquids is compressible.

1. Introduction

In this paper, we summarize of the study in the Sobolev spaces of the solvability of problems governing the motion of two viscous liquids separated by an unknown closed interface. Every fluid may be of both types: compressible and incompressible. They occupy the whole space $\mathbb{R}^3$. On the interface, we take the capillary forces into account. All results remain valid for noncapillary fluids too.

The main result of this investigation is the unique solvability of the problems mentioned above in sufficient small time intervals. As auxiliary results, one can consider the proof of existence of unique solutions for linearized problems in any finite time interval and global unique solvability in the weighted Sobolev spaces of linear model problems with a plane interface between the liquids. We compare explicit solutions in the dual spaces of these linear problems. We make passage to the limit from the solution of the problem for two compressible fluids through the solution of the "mixed type" system to the solution of the problem for two incompressible liquids. We remark that there is no restriction to the fluid viscosities for the last problem whereas for the "mixed type" problem the stated results are proved only if the dynamic viscosities of the fluids are not different more than in two times. As for two compressible liquids, these results are valid only for fluids with low viscosity.

2. Statement of the Problems and Formulation of the Main Results

First, we formulate the most complicated problem, for the case of two liquids of different types. A study of this problem is made in [2].

Let, for definitness, at the initial moment $t = 0$, the compressible fluid have a finite volume and be situated in a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$ inside incompressible one occupying the domain $\Omega_0^- \equiv \mathbb{R}^3 \setminus \overline{\Omega_0^+}$.

Let $\mu^+ > 0$, $\lambda^+ > 0$ be the dynamic viscosities of the compressible liquid. We denote the kinematic viscosity of the incompressible fluid by constant $\nu^- > 0$ and its density coefficient by $\rho^- > 0$. We consider that the compressible fluid is barotrpic. We note that we could also suppose the compressible fluid to be exterior to the incompressible one.

For $t > 0$, it is necessary to find Γ_t, the free interface between the liquids evolving in the domains Ω_t^- and Ω_t^+. Besides, it is required to find the density function $\rho^+(x,t) > 0$ of the compressible fluid, the pressure function $p^-(x,t)$ of the incompressible fluid, as well as the velocity vector field of both liquids $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ satisfying the initial-boundary value problem for the Navier-Stokes system:

$$\begin{aligned} \rho^{\pm}(\mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \nabla\mathbb{T} &= \rho^{\pm}\boldsymbol{f},\ \ \mathcal{D}_t\rho^{\pm} + \nabla\cdot(\rho^{\pm}\boldsymbol{v}) = 0 \text{ in } \Omega_t^- \cup \Omega_t^+\,, \\ \boldsymbol{v}|_{t=0} &= \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+\,, \end{aligned} \tag{1}$$

$$\rho^+|_{t=0} = \rho_0^+\,, \quad \text{in} \quad \Omega_0^+\,; \quad \boldsymbol{v} \xrightarrow[|x|\to\infty]{} 0\,, \quad p^- \xrightarrow[|x|\to\infty]{} 0\,; \tag{2}$$

$$[\boldsymbol{v}]\Big|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \boldsymbol{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \boldsymbol{v}(x) = 0\,, \quad [\mathbb{T}\boldsymbol{n}]\Big|_{\Gamma_t} = \sigma H\boldsymbol{n} \quad \text{on} \quad \Gamma_t\,. \tag{3}$$

Here $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, function $\rho^{\pm}$ is equal to $\rho^+(x)$ in Ω_t^+ and to the constant ρ^- in Ω_t^-; the stress tensor is

$$\mathbb{T} = \begin{cases} (-p^+(\rho^+) + \lambda\nabla\cdot\boldsymbol{v})\,\mathbb{I} + \mu^+\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^+\,, \\ -p^-\mathbb{I} + \mu^-\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^-\,, \end{cases} \tag{4}$$

$(\mathbb{S}(\boldsymbol{v}))_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i$, $i,k = 1,2,3$; $\mathbb{I}$ is the unit matrix; $\mu^- = \nu^-\rho^-$; $p^+(\rho^+)$ is the pressure of the compressible fluid given by a smooth function of its density; $\boldsymbol{f}$ is the given vector field of mass forces; $\boldsymbol{v}_0$ is the initial value of the velocity vector field; ρ_0^+ is the initial density distribution of the compressible fluid; $\sigma \geqslant 0$ is the surface tension coefficient, $\boldsymbol{n}$ is the outward normal vector to Ω_t^+, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-); $\nabla\mathbb{T}$ means the vector with the components $(\nabla\mathbb{T})_j = \frac{\partial T_{ij}}{\partial x_i}$, $T_{ij} = (\mathbb{T})_{ij}$, $j = 1,2,3$. We imply the summation from 1 to 3 with respect to repeated indices. A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The central dot denotes the scalar product. We mark the vectors and the vector spaces by boldface letters.

Since we suppose the liquids to be immiscible it is natural to impose on Γ_t a condition excluding the mass transportation through this surface. Mathematically, this condition means that Γ_t consists of the points $x(\xi,t)$ whose radius vector $\boldsymbol{x}(\xi,t)$ is a solution of the Cauchy problem

$$\mathcal{D}_t\boldsymbol{x} = \boldsymbol{v}(x(\xi,t),t)\,, \quad \boldsymbol{x}(\xi,0) = \boldsymbol{\xi}\,, \quad \xi\in\Gamma\,, \quad t>0\,, \tag{5}$$

where $\Gamma\equiv\Gamma_0 = \partial\Omega_0^+$ is a surface given at the initial moment. Hence, $\Omega_t^\pm = \{x = x(\xi,t)|\xi\in\Omega_0^\pm\}$.

Condition (5) completes system (1)–(3).

For two compressible fluids, the problem formulation differs from (1)–(5) by conditions (2) that look as follows

$$\rho^-|_{t=0} = \rho_0^- \quad \text{in} \quad \Omega_0^-\,; \qquad \rho^+|_{t=0} = \rho_0^+ \quad \text{in} \quad \Omega_0^+\,; \quad \boldsymbol{v}\xrightarrow[|x|\to\infty]{}0\,. \tag{6}$$

The stress tensor in this case is

$$\mathbb{T} = \begin{cases} (-p^+(\rho^+) + \lambda\nabla\cdot\boldsymbol{v})\,\mathbb{I} + \mu^+\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^+\,, \\ (-p^-(\rho^- - +\lambda\nabla\cdot\boldsymbol{v})\,\mathbb{I} + \mu^-\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^-\,. \end{cases} \tag{7}$$

This problem has been studied in [3].

In the case of two incompressible liquids, we would have (2) in the form

$$\boldsymbol{v}\xrightarrow[|x|\to\infty]{}0, \quad p\xrightarrow[|x|\to\infty]{}0\,. \tag{8}$$

The stress tensor would be given by

$$\mathbb{T} = \begin{cases} -p\mathbb{I} + \mu^+\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^+\,, \\ -p\mathbb{I} + \mu^-\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^-\,. \end{cases} \tag{9}$$

One has analysed the latter problem in [1, 5, 4].

We present an investigation scheme common for all three problems considering the example of (1)–(5). This technique was proposed by V. A. Solonnikov in [6, 7] for the study of the drop evolution in vacuum and it was modified by him and A. Tani in [8] for the case of the bubble motion in vacuum.

We transform the Eulerian coordinates $\{x\}$ into the Lagrangian ones $\{\xi\}$ by the formula

$$\boldsymbol{x}(\xi,t) = \boldsymbol{\xi} + \int_0^t \boldsymbol{u}(\xi,\tau)d\tau \equiv \boldsymbol{X}_{\boldsymbol{u}}(\xi,t) \tag{10}$$

where $\boldsymbol{u}(\xi,t)$ is the velocity vector field in the Lagrangian coordinates.

The Jacobian of transformation (10) $\mathcal{J}_{\boldsymbol{u}}(\xi,t) = \det\{a_{ij}\}_{i,j=1}^3$, $a_{ij}(\xi,t) = \delta_j^i + \int_0^t \frac{\partial u_i}{\partial\xi_j}\,d\tau$, being a solution of the Cauchy problem

$$\mathcal{D}_t\mathcal{J}_{\boldsymbol{u}}(\xi,t) = A_{ij}\frac{\partial u_i}{\partial\xi_j} \equiv \mathcal{J}_{\boldsymbol{u}}(\xi,t)(\nabla\cdot\boldsymbol{v}|_{x=X_{\boldsymbol{v}}})\,, \qquad \mathcal{J}_{\boldsymbol{u}}(\xi,0) = 1\,,$$

is expressed by the formula

$$\mathcal{J}_{\boldsymbol{u}}(\xi,t)=\exp(\int\limits_0^t \nabla\cdot\boldsymbol{v}|_{x=X_{\boldsymbol{u}}}d\tau)\equiv\exp(\int\limits_0^t \nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}\,d\tau)\,. \tag{11}$$

Here we use the standard notation $\{\delta^i_j\}^3_{i,j=1}$ for the Kronecker symbols and also $\nabla_{\boldsymbol{u}}\equiv\left\{\frac{\partial\xi_i}{\partial x_k}\frac{\partial}{\partial\xi_i}\right\}^3_{k=1}=\mathcal{J}^{-1}_{\boldsymbol{u}}\mathbb{A}\nabla$; $\mathbb{A}\equiv\{A_{ij}\}^3_{i,j=1}$ is the cofactors matrix of the Jacobi matrix $\{a_{ij}\}$ in (10). We note that $\mathcal{J}_{\boldsymbol{u}}(\xi,t)\equiv1$ in the domains with incompressible fluid.

After tranformation (10), we can integrate the second equation in (1) for the compressible liquid. Then we obtain the following expression for the density ρ^+ in the Lagrangean coordinates:

$$\widehat{\rho^+}(\xi,t)=\rho^+_0(\xi)\exp(-\int\limits_0^t\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}\,d\tau)=\rho^+_0(\xi)\mathcal{J}^{-1}_{\boldsymbol{u}}(\xi,t)\,.$$

Next, we use the well-known formula for twice the surface mean curvature:

$$H\boldsymbol{n}=\Delta(t)\boldsymbol{x}=\Delta(t)\boldsymbol{X}_{\boldsymbol{u}}$$

where $\Delta(t)$ is the Beltrami-Laplace operator on Γ_t. Moreover, we separate the last boundary condition in (3) on the tangential and normal components. To this end, we project it first onto the tangent plane of Γ_t and then onto that of Γ by means of projectors Π and Π_0, respectively.

Let $\boldsymbol{n}_0$ be the outward normal to Γ. It is connected with $\boldsymbol{n}$ by the relation $\boldsymbol{n}=\frac{\mathcal{J}^{-1}_{\boldsymbol{u}}\mathbb{A}\boldsymbol{n}_0}{|\mathcal{J}^{-1}_{\boldsymbol{u}}\mathbb{A}\boldsymbol{n}_0|}=\frac{\mathbb{A}\boldsymbol{n}_0}{|\mathbb{A}\boldsymbol{n}_0|}$.

As a result of the above transformation, we obtain the system:

$$\mathcal{D}_t\boldsymbol{u}-\frac{1}{\rho^+_0(\xi)}\mathbb{A}\nabla\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{u})=\boldsymbol{f}(X_{\boldsymbol{u}},t)-\frac{1}{\rho^+_0(\xi)}\mathbb{A}\nabla p^+(\rho^+_0\mathcal{J}^{-1}_{\boldsymbol{u}})\text{ in }Q^+_T\equiv\Omega^+_0\times(0,T)\,,$$

$$\mathcal{D}_t\boldsymbol{u}-\nu^-\nabla^2_{\boldsymbol{u}}\boldsymbol{u}+\frac{1}{\rho^-}\nabla_{\boldsymbol{u}}q=\boldsymbol{f}(X_{\boldsymbol{u}},t)\,,\quad\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}=0\quad\text{in}\quad Q^-_T\equiv\Omega^-_0\times(0,T)\,,$$

$$\boldsymbol{u}\Big|_{t=0}=\boldsymbol{v}_0\quad\text{in}\quad\Omega^-_0\cup\Omega^+_0,\quad\boldsymbol{u}\xrightarrow[|\xi|\to\infty]{}0,\quad q\xrightarrow[|\xi|\to\infty]{}0\,, \tag{12}$$

$$[\boldsymbol{u}]\Big|_{G_T}=0,\quad[\mu^\pm\Pi_0\Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}]\Big|_{G_T}=0\quad(G_T\equiv\Gamma\times(0,T))\,,$$

$$[\boldsymbol{n}_0\cdot\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{u},q)\boldsymbol{n}]\Big|_{G_T}-\sigma\boldsymbol{n}_0\cdot\Delta(t)\boldsymbol{X}_{\boldsymbol{u}}\Big|_{G_T}=(\boldsymbol{n}_0\cdot\boldsymbol{n})p^+(\rho^+_0\mathcal{J}^{-1}_{\boldsymbol{u}})\Big|_{G_T}\,,$$

which is equivalent to (1)–(5) provided that $\boldsymbol{n}\cdot\boldsymbol{n}_0 > 0$. In (12) we used the notation: $q(\xi,t)$ was the pressure function in the Lagrangian coordinates;

$$(\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w},q))_{i,j} = \begin{cases} (\lambda^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{w})\delta^i_j + \mu^+(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} & \text{in } Q^+_T\,, \\ -\delta^i_j q + \mu^-(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} & \text{in } Q^-_T\,; \end{cases}$$

$$(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} = \mathcal{J}^{-1}_{\boldsymbol{u}}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right);$$

$$\Pi_0\boldsymbol{\omega} = \boldsymbol{\omega} - (\boldsymbol{n}_0\cdot\boldsymbol{\omega})\boldsymbol{n}_0, \quad \Pi\boldsymbol{\omega} = \boldsymbol{\omega} - (\boldsymbol{n}\cdot\boldsymbol{\omega})\boldsymbol{n}\,.$$

We shall use the ordinary normalization for the Sobolev-Slobodetskiĭ spaces $W^m_2(\Omega)$ for $m>0$, Ω being a domain in $\mathbb{R}^n$, $n\in\mathbb{N}$. $\|\cdot\|_\Omega$ is the norm of $L_2(\Omega)$.

The anisotropic space $W^{m,m/2}_2(Q_T)$ consists of functions defined in the cylinder $Q_T = \Omega\times(0,T)$, $0<T\leqslant\infty$, and having finite norm

$$\|u\|_{W^{m,m/2}_2(Q_T)} = \left(\int_0^T \|u\|^2_{W^m_2(\Omega)}dt + \int_\Omega \|u\|^2_{W^{m/2}_2(0,T)}dx\right)^{1/2}.$$

Now we define three norms necessary for formulating the main result of this paper. The first of them is

$$\|u\|^{(m,m/2)}_{Q^-_T\cup Q^+_T} = \left(\|u\|^2_{\bigcup\limits_{i=-,+} W^{m,m/2}_2(Q^i_T)} + T^{-m}\|u\|^2_{\mathbb{R}^3_T}\right)^{1/2}.$$

It is equivalent to $\|u\|^2_{\bigcup\limits_i W^{m,m/2}_2(Q^i_T)}$ for $\forall T<\infty$. The square of the second one is determined by the formula

$$\left(\|u\|^{(2+l,1+l/2)}_{Q^-_T\cup Q^+_T}\right)^2 = \|u\|^2_{\bigcup\limits_{i=-,+} W^{2+l,1+l/2}_2(Q^i_T)} + T^{-l}\Big\{\|\mathcal{D}_t u\|^2_{Q^-_T\cup Q^+_T} + $$
$$+\sum_{|\boldsymbol{\alpha}|=2}\|\mathcal{D}^{\boldsymbol{\alpha}}_x u\|^2_{Q^-_T\cup Q^+_T}\Big\} + \sup_{t\le T}\|u(\cdot,t)\|^2_{\bigcup\limits_i W^{1+l}_2(\Omega^i_0)}.$$

For $\beta\in(0,1)$ we will consider the following Hoelder norm of $u\in\mathbb{R}^3_T\equiv\mathbb{R}^3\times(0,T)$

$$|||u|||_{\mathbb{R}^3_T} = \sup_{\mathbb{R}^3_T}|u| + \max_k \sup_{(x,t)\in\mathbb{R}^3_T}|\mathcal{D}_{x_k}u(x,t)| + \sup_{(x,t),\tau\le T}\frac{|u(x,t)-u(x,\tau)|}{|\tau-t|^\beta}.$$

Let B_d be the ball $\{x:|x|<d\}$. We choose a coordinate system $\{x\}$ so that Ω^+_0 is contained in the ball B_d, $d<\infty$, and we set $B^-_{dT}\equiv(B_d\setminus\overline{\Omega^+_0})\times(0,T)$.

Theorem 2.1. *Assume that for some* $l\in(1/2,1)$ *we have* $\Gamma\in W^{5/2+l}_2$, $\rho^+_0\in W^{1+l}_2(\Omega^+_0)$, $0<R_0\leqslant\rho^+_0(\xi)\leqslant R_\infty<\infty$, $\xi\in\Omega^+_0$, $p^+\in C^3(\mathbb{R}_+)$, $\boldsymbol{f}\in\boldsymbol{W}^{l,l/2}_2(\mathbb{R}^3_T)$, $0<T<\infty$, $\boldsymbol{f}(\cdot,t)\in\boldsymbol{C}^2(\mathbb{R}^3)$ *for* $\forall t\in[0,T]$, $\boldsymbol{f}(\xi,\cdot)$, $\nabla\boldsymbol{f}(\xi,\cdot)\in\boldsymbol{C}^\beta(0,T)$ *for*

$\forall\xi \in \mathbb{R}^3$ with some $\beta \in (1/2, 1)$. In addition, let the initial velocity vector $\boldsymbol{v}_0 \in \bigcup\limits_{i=-,+} \boldsymbol{W}_2^{1+l}(\Omega_0^i)$ satisfy the compatibility conditions

$$\nabla \cdot \boldsymbol{v}_0 = 0 \quad \text{in} \quad \Omega_0^-, \qquad [\boldsymbol{v}_0]\Big|_\Gamma = 0, \quad [\Pi_0 \mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]\Big|_\Gamma = 0\,,$$

and for the viscosities of the liquids, the inequalities

$$\mu^- > \mu^+, \quad \nu^- < \mu^+/R_\infty \tag{13}$$

hold.

Under these hypotheses, there exists a constant $T_0 \in (0, T]$ such that problem (12) is uniquely solvable on the interval $(0, T_0)$ and its solution $(\boldsymbol{u}, q)$ has the properties: $\boldsymbol{u} \in \bigcup\limits_{i=-,+} \boldsymbol{W}_2^{2+l,1+l/2}(Q_{T_0}^i)$, $q \in W_{2,loc}^{l,l/2}(Q_{T_0}^-)$, $\nabla q \in \boldsymbol{W}_2^{l,l/2}(Q_{T_0}^-)$, $q|_{G_{T_0}} \in W_2^{l+1/2,l/2+1/4}(G_{T_0})$ and

$$\begin{aligned} \|\boldsymbol{u}\|^{(2+l,1+l/2)}_{Q^-_{T_0}\cup Q^+_{T_0}} + \|\nabla q\|^{(l,l/2)}_{Q^-_{T_0}} + \|q\|^{(l,l/2)}_{B^-_{dT_0}} + \|q\|_{W_2^{l+1/2,l/2+1/4}(G_{T_0})} \leqslant \\ \leqslant c_1(c_2 + c_3 T_0^{\frac{1-l}{2}} \|\boldsymbol{v}_0\|_{\bigcup\limits_i \boldsymbol{W}_2^{1+l}(\Omega_0^i)}) \Big\{ |||\boldsymbol{f}|||_{\mathbb{R}^3_{T_0}} + \|\boldsymbol{v}_0\|_{\bigcup\limits_i \boldsymbol{W}_2^{1+l}(\Omega_0^i)} + \\ + \sigma \|H_0\|_{W_2^{l+1/2}(\Gamma)} + \|\frac{1}{\rho_0^+}\nabla p^+(\rho_0^+)\|_{W_2^l(\Omega_0^+)} + \|p^+(\rho_0^+)\|_{W_2^{1+l}(\Omega_0^+)} \Big\}\,. \end{aligned}$$

The value T_0 depends on the norms of $\boldsymbol{f}$, $\boldsymbol{v}_0$, ρ_0, p^+ and on the curvature value of Γ.

This theorem is proved by successive approximations in the same way as the analogous theorems for the case of a single incompressible fluid [7] or for the case of a single compressible one [8]. The role of the successive approximations is played by the solutions of the following linearized problems:

$$\begin{gathered} \mathcal{D}_t \boldsymbol{w} - \frac{1}{\rho_0^+(\xi)} \mathbb{A}\nabla \mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w}) = \boldsymbol{f} \quad \text{in} \quad Q_T^+\,, \\ \mathcal{D}_t \boldsymbol{w} - \nu^- \nabla^2_{\boldsymbol{u}} \boldsymbol{w} + \frac{1}{\rho_0^-} \nabla_{\boldsymbol{u}} s = \boldsymbol{f}, \quad \nabla_{\boldsymbol{u}} \cdot \boldsymbol{w} = r \quad \text{in} \quad Q_T^-\,, \\ \boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+, \quad \boldsymbol{w} \xrightarrow[|\xi|\to\infty]{} 0, \quad s \xrightarrow[|\xi|\to\infty]{} 0\,, \\ [\boldsymbol{w}]\Big|_{G_T} = 0, \quad [\mu^\pm \Pi_0 \Pi \mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w})\boldsymbol{n}]\Big|_{G_T} = \Pi_0 a\,, \\ [\boldsymbol{n}_0 \cdot \mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w}, s)\boldsymbol{n}]\Big|_\Gamma - \sigma \boldsymbol{n}_0 \cdot \Delta(t) \int_0^t \boldsymbol{w}\Big|_\Gamma d\tau = b + \sigma \int_0^t B\, d\tau, \quad t \in (0, T)\,. \end{gathered} \tag{14}$$

The proof of the existence theorem for problem (14) is also based on the successive approximation method, the solution of system (14) with $\boldsymbol{u} = 0$ being taken as the first approximation.

Existence of a unique solution of the latter problem and its smoothness may be proved by constructing a regularizer [7, 8] or by means of generalized solution [5]. Both these methods are based on the Schauder estimates of the solution and considering a model problem with plane interface Γ.

For problem (14), a theorem on the unique solvability takes place for an arbitrary finite time interval $(0, T)$ and for a vector $\boldsymbol{u} \in \bigcup\limits_{i=-,+} \boldsymbol{W}_2^{2+l,1+l/2}(Q_T^i)$ being continuous across the boundary Γ, such that the inequality

$$T^{1/2}\|\boldsymbol{u}\|^{(2+l,1+l/2)}_{Q_T^-\cup Q_T^+} \leqslant \delta$$

holds with a sufficiently small number δ.

For one-type liquid problems, similar results take place too. We note that in the case of two incompressible fluids, they are formulated without any restrictions to the liquid viscosities [4]. As for two compressible fluids, we have the inequalities

$$\frac{\mu^-}{2} \leqslant \mu^+ \leqslant 2\mu^-, \ 0 < \lambda^\pm \leqslant \mu^\pm \tag{15}$$

instead of (13). A theorem similar to theorem 2.1 can be found in [3].

3. The Model Problem with a Plane Interface between the Fluids

In this section, we consider the problem

$$\begin{gathered}
\mathcal{D}_t\boldsymbol{v} - \nu^+\nabla^2\boldsymbol{v} + \frac{1}{\rho_0^+}\nabla p = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \quad \text{in} \quad D_\infty^+ = \mathbb{R}_+^3 \times (0,\infty)\,, \\
\mathcal{D}_t\boldsymbol{v} - \nu^-\nabla^2\boldsymbol{v} - (\nu^- + \kappa^-)\nabla(\nabla\cdot\boldsymbol{v}) = 0 \quad \text{in} \quad D_\infty^- = \mathbb{R}_-^3 \times (0,\infty)\,, \\
\boldsymbol{v}\Big|_{t=0} = 0 \quad \text{on} \quad \mathbb{R}_-^3 \cup \mathbb{R}_+^3, \quad \boldsymbol{v} \xrightarrow[|x|\to\infty]{} 0, \quad p \xrightarrow[|x|\to\infty]{} 0\,, \\
[\boldsymbol{v}]\Big|_{x_3=0} = 0, \quad -\left[\mu^\pm\left(\frac{\partial v_\alpha}{\partial x_3} + \frac{\partial v_3}{\partial x_\alpha}\right)\right]\Big|_{x_3=0} = b_\alpha(x',t), \quad \alpha = 1,2\,; \\
-p - \lambda^-\nabla\cdot\boldsymbol{v} + \left[2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\Big|_{x_3=0} + \sigma\Delta'\int_0^t v_3 d\tau\Big|_{x_3=0} = b_3 + \sigma\int_0^t B\,d\tau \equiv b_3' \text{ on } \mathbb{R}_\infty^2\,.
\end{gathered} \tag{16}$$

Here we have used the notation: $\mathbb{R}_\pm^3 = \{\pm x_3 > 0\}$, $\mathbb{R}_\infty^2 = \mathbb{R}^2\times(0,\infty)$, $\kappa^- = \lambda^-/\rho_0^-$, $\rho_0^- = \text{constant} > 0$, $\nu^- = \mu^-/\rho_0^-$, $\Delta' = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, $x' = (x_1, x_2)$.

We take the Fourier transform on the tangent space variables $(x_1, x_2) = x'$ and the Laplace transform on t. We denote the dual variables by $\xi = (\xi_1, \xi_2)$ and s, respectively. The problem (16) then goes over into the system of ordinary differential equations with unknown functions $\widetilde{\boldsymbol{v}}$, $\widetilde{p}$. We solve this system and we write a solution in the form convenient for the following estimates:

$$\begin{aligned} \widetilde{\boldsymbol{v}} &= \boldsymbol{W} e_0^{\pm} + \boldsymbol{V}^{\pm} e_1^{\pm}, \quad \pm x_3 > 0\,, \\ \widetilde{p} &= -C_3^{+}\rho_0^{+} s e^{-|\xi| x_3} = -\mu^{+} C_3^{+}(r^{+} - |\xi|)(r^{+} + |\xi|)e^{-|\xi|x_3}, \quad x_3 > 0\,, \end{aligned} \tag{17}$$

where $r^{\pm} = \sqrt{\frac{s}{\nu^{\pm}} + \xi^2}$, $r_1^{-} = \sqrt{\frac{s}{(2+\beta^{-})\nu^{-}} + \xi^2}$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $|\arg \sqrt{z}| < \pi/2$ for $\forall z$, $\beta^{-} = \kappa^{-}/\nu^{-}$; $e_0^{\pm} = e^{\mp r^{\pm} x_3}$, $\quad e_1^{+} = \frac{e^{-r^{+}x_3} - e^{-|\xi|x_3}}{r^{+} - |\xi|}$, $\quad e_1^{-} = \frac{e^{r^{-}x_3} - e^{r_1^{-}x_3}}{r^{-} - r_1^{-}}$,

$$\boldsymbol{W} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \ \boldsymbol{V}^{+} = -C_3^{+}(r^{+} - |\xi|) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -|\xi| \end{pmatrix}, \ \boldsymbol{V}^{-} = -C_3^{-}(r^{-} - r_1^{-}) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ r_1^{-} \end{pmatrix},$$

$$\begin{aligned} C_3^{+} &= -\frac{A\{[2\mu^{+} r^{+} + \frac{\sigma}{s}\xi^2](r^{-} r_1^{-} - \xi^2) + \mu^{-} r^{+}(r^{-2} - 2r^{-} r_1^{-} + \xi^2) + \rho_0^{-} s r^{-}\}}{(r^{+} - |\xi|)P} \\ &\quad - \frac{\widetilde{b}_3'\{\mu^{+}(r^{-} r_1^{-} - \xi^2)(r^{+2} + \xi^2) + \mu^{-}\xi^2(r^{-2} + \xi^2 - 2r^{-} r_1^{-}) + \rho_0^{-} s r^{+} r_1^{-}\}}{(r^{+} - |\xi|)P}, \\ C_3^{-} &= -\frac{A}{P}\Big\{\mu^{+}\left[r^{+}(r^{+} + |\xi|) + r^{-}(r^{+} - |\xi|)\right] + 2\mu^{-} r^{-}|\xi| + \frac{\sigma}{s}|\xi|^3\Big\} + \\ &\quad + \frac{\widetilde{b}_3'}{P}\Big\{\mu^{+}\left[r^{+}|\xi|(r^{-} + |\xi|) + (r^{-} - |\xi|)\xi^2\right] + \mu^{-}(r^{-2} + \xi^2)|\xi|\Big\}, \\ \omega_\alpha &= \frac{\{\widetilde{b}_\alpha + i\xi_\alpha[\mu^{+}(r^{+} - |\xi|)C_3^{+} + \mu^{-}(r^{-} - r_1^{-})C_3^{-} + (\mu^{+} - \mu^{-})\omega_3]\}}{\mu^{+} r^{+} + \mu^{-} r^{-}}, \\ &\qquad \alpha = 1, 2, \\ \omega_3 &= \frac{-(r^{+} - |\xi|)|\xi| C_3^{+} + (r^{-} r_1^{-} - \xi^2)C_3^{-}}{r^{+} + r^{-}}. \end{aligned} \tag{18}$$

In formulas (18) we have used the notation: $A = i\xi_1\widetilde{b}_1 + i\xi_2\widetilde{b}_2$, $\quad \widetilde{b}_3' = \widetilde{b}_3 + \frac{\sigma}{s}\widetilde{B}$,

$$\begin{aligned} P &= \mu^{+} s(r^{+} + |\xi|)\{\rho_0^{+}(r^{-} r_1^{-} - \xi^2) + \rho_0^{-}(r^{-}|\xi| + r^{+} r_1^{-} + 2\xi^2)\} + \rho_0^{-2} s^2|\xi| + \\ &\quad + 4(\mu^{+} - \mu^{-})\xi^2\{\mu^{+} r^{+}(r^{-} r_1^{-} - \xi^2) - \mu^{-} r^{-}(r^{-} - r_1^{-})|\xi|\} + \\ &\quad + \frac{\sigma|\xi|^3}{s}\{\mu^{+}(r^{+} + |\xi|)(r^{-} r_1^{-} - \xi^2) + \rho_0^{-} r_1^{-} s\}\,. \end{aligned}$$

We observe that solution (17), (18) may be obtained by the passage to the limit $r_1^{+} \to |\xi|$ from a solution of the model problem with a plane interface between two compressible fluids (see [3]). On the other hand, it goes over as $r_1^{-} \to |\xi|$ into a solution of the corresponding problem for two incompressible liquids [1]. This passage may also be demonstrated in the equations. Let us turn r_1^{-} to $|\xi|$ that corresponds to $\beta^{-} \to \infty$. In this case, the Navier-Stokes equations for a compressible fluid in the domain $\{x_3 < 0\}$ become the equations for the incompressible one.

The estimates of solution (17), (18) has been considered in detail in [2]. We remark only that C_3^+ is contained in all the expressions with multiplier $(r^+ - |\xi|)$, which is cancelled with the denominator of C_3^+. Hence, the evaluation of solution (17), (18) depends only on the lower bound of $|P|$.

It should be noted also that the uniqueness of solution (17), (18) for $\gamma \geqslant \gamma_0 > 0$ is guaranteed by the fact that in this case $|P|$ is separated from zero.

Lemma 3.1. *Assume that for the viscosities of the fluids the inequalities*

$$\nu^+ < \nu^-, \quad \mu^+ > \mu^- \tag{19}$$

hold and that $\sigma \geqslant 0$. *Then for* $\forall \xi \in \mathbb{R}^2$, $\forall s \in \mathbb{C}$, $\operatorname{Re} s = \gamma > 0$,

$$|P| \geqslant c\left(|s|^2 + |s|^{3/2}|\xi| + |s|\xi^2 + \sigma|\xi|^3\right)\left(|s|^{1/2} + |\xi|\right).$$

The main difficulty in the evaluation of $|P|$ has been to find an universal multiplier q for P in the following sense: the quotient of the division of P by q has a positive real part. There are many ways to write down P. We has needed to find such expression for $P = \sum_{j=1}^{m} I_j$ that $\operatorname{Re}\frac{I_j}{q} \geqslant 0$, $j = 1, \ldots, m$, and we could apply the inequality

$$|P| \geq |q|\left|\operatorname{Re}\left(\sum_{j=1}^{m}\frac{I_j}{q}\right)\right| \geqslant |q|\sum_{j=1}^{m}\operatorname{Re}\frac{I_j}{q} > 0\,.$$

We observe that it is the polynomial multiplier of the term with $\sigma|\xi|^3/s$ what plays the role of such multiplier q in all three cases. For example, in the case of two incompressible fluids [1] $q = \mu^+(r^+ + |\xi|) + \mu^-(r^- + |\xi|)$.

Remark 3.1. *We note that inequalities* (13) *follow from* (19) *so as for the case of two compressible fluids: restrictions* (15) *appear in the estimate process of the corresponding denominator P (see* [3]*).*

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