The Mathematics of M-Theory

Robbert Dijkgraaf

Abstract. String theory, ot its modern incarnation M-theory, gives a huge generalization of classical geometry. I indicate how it can be considered as a two-parameter deformation, where one parameter controls the generalization from points to loops, and the other parameter controls the sum over topologies of Riemann surfaces. The final mathematical formulation of M-theory will have to make contact with the theory of vector bundles, K-theory and non-commutative geometry.

1. Introduction

Over the years there have been many fruitful interactions between string theory [14] and various fields of mathematics. Subjects like algebraic geometry and representation theory have been stimulated by new concepts such as mirror symmetry [3], quantum cohomology [12] and conformal field theory [4]. But most of these developments have been based on the perturbative formulation of string theory, either in the Lagrangian formalism in terms of maps of Riemann surfaces into manifolds and the quantization of loop spaces. This perturbative approach is however only an approximate description that appears for small values of the quantization parameter.

Recently there has been much progress in understanding a more fundamental description of the theory that has become known as M-theory. M-theory seem to be the most complex and richest mathematical object so far in physics. It seems to unify three great ideas of twentieth century theoretical physics:

(1) General relativity – the idea that gravity can be described by the Riemannian geometry of space-time.

(2) Gauge theory – the description of forces between elementary particles using connections on vector bundles. In mathematics this involves K-theory and index theorems.

(3) Strings, or more generally extended objects, as a natural generalization of point particles. Mathematically this means that we study spaces primarily through their (quantized) loop spaces.

At present it seems that these three independent ideas are closely related, and perhaps essentially equivalent. To some extend physics is trying to build a dictionary between geometry, gauge theory and strings.

It must be said that in all developments there have been two further ingredients that are absolutely crucial. The first is quantum mechanics —the description of physical reality in terms of operator algebras acting on Hilbert spaces. In most attempts to understand string theory quantum mechanics has been the foundation, and there is little indication that this is going to change.

The second ingredient is supersymmetry —the unification of matter and forces. In mathematical terms supersymmetry is closely related to de Rham complexes and algebraic topology. In some way much of the miraculous interconnections in string theory only work if supersymmetry is present. Since we are essentially working with a complex, it should not come to a surprise to mathematicians that there are various 'topological' indices that are stable under perturbation and can be computed exactly in an appropriate limit. From a physical perspective supersymmetry is perhaps the most robust prediction of string theory.

1.1. A two-parameter deformation of classical geometry

For pedagogical purposes in this lecture M-theory will be considered as a two parameter family of deformations of "classical" Riemannian geometry. Let us introduce these two parameters heuristically. (We will give a more precise definition later.)

First, in perturbative string theory we study the loops in a space-time manifold. These loops can be thought to have an intrinsic length ℓ_s, the *string length.* At least at an heuristic level it is clear that in the limit $\ell_s \to 0$ the string degenerates to a point, a constant loop. The parameter ℓ_s controls the "stringyness" of the model. We will see how the quantity $\ell_s^2 = \alpha'$ plays the role of Planck's constant on the worldsheet of the string. That is, it controls the quantum correction of the two-dimensional field theory on the world-sheet of the string. An important example of a stringy deformation is quantum cohomology [12].

Secondly, strings can split and join, sweeping out a surface Σ of general topology in space-time. According to the general rules of quantum mechanics we have to include a sum over all topologies. Such a sum over topologies can be regulated if We can introduce a formal parameter $\lambda \in \mathbb{R}_+$, the *string coupling*, such that a surface of genus g gets weighted by a factor λ^{2g-2}. Higher genus topologies can be interpreted as virtual processes wherein strings split and join —a typical quantum phenomenon. Therefore the parameter λ controls the quantum corrections. In fact we can equate λ^2 with Planck's constant in space-time. Only for small values of λ can string theory be described in terms of loop spaces and sums over surfaces.

In fact, in the case of particles we know that for large values of λ it is better to think in terms of waves, or more precisely quantum fields. So we expect that for large λ and α' the right framework is string field theory [21]. This is partly true, but it is in general difficult to analyze this string field theory directly. In particular the occurrence of branes, higher-dimensional extended objects that will play an important role in the subsequent, is often obscure. (See however the recent work [17].)

Summarizing we can distinguish two kinds of deformations: stringy effects parametrized by α', and quantum effects parametrized by λ. This situation can be described with the following table

$\alpha' > 0$	conformal field theory *strings*	M-theory *string fields, branes*
$\alpha' = 0$	quantum mechanics *particles*	quantum field theory *fields*
	$\lambda = 0$	$\lambda > 0$

It is perhaps worthwhile to put some related mathematical fields in a similar table

$\alpha' > 0$	quantum cohomology (*Gromov, Witten*)	non-commutative geometry (*Connes*)
$\alpha' = 0$	combinatorical knot invariants (*Vassiliev, Kontsevich*)	4-manifold, 3-manifolds, knots (*Donaldson, Witten, Jones*)
	$\lambda = 0$	$\lambda > 0$

We will now briefly review these various generalizations. More background material can be found in [5].

2. Quantum Mechanics and Particles

In classical mechanics we describe point particles on a Riemannian manifold X that we think of as a (Euclidean) space-time. Pedantically speaking we look at X through maps

$$x\colon pt \to X$$

of an abstract point into X. Quantum mechanics associates to the classical configuration space X the Hilbert space $\mathcal{H} = L^2(X)$ of square-integrable wavefunctions. We want to think of this Hilbert space as associated to a point

$$\mathcal{H} = \mathcal{H}_{pt}\,.$$

For a supersymmetric point particle we instead work with the space of de Rham differential forms $\mathcal{H} = \Omega^*(X)$.

Classically a particle can go in a time t from point x to point y along some preferred path, typically a geodesic. Quantum mechanically we instead have a linear evolution operator

$$\Phi_t\colon \mathcal{H} \to \mathcal{H}\,,$$

that describes the time evolution. Through the Feynman path-integral this operator is associated to maps of the line interval of length t into X. More precisely, the kernel $\Phi_t(x,y)$ of the operator Φ_t is given by the path-integral

$$\Phi_t(x,y) = \int_{x(\tau)} [dx] e^{-\int_0^t d\tau |\dot{x}|^2}$$

over all paths $x(\tau)$ with $x(0) = x$ and $x(t) = y$. Φ_t is the kernel of the heat equation

$$\frac{d}{dt}\Phi_t = \Delta\Phi_t, \qquad \Phi_0 = \delta(x-y)\,.$$

These path-integrals have a natural gluing property: if we first evolve over a time t_1 and then over a time t_2 this should be equivalent to evolving over time $t_1 + t_2$.

$$\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}\,. \tag{1}$$

This allows us to write

$$\Phi_t = e^{-tH}$$

with H the Hamiltonian. In the case of a particle on X the Hamiltonian is of course simply given by the Laplacian $H = -\Delta$.

The composition property (1) is a general property of quantum field theories. It leads us to Segal's functorial view of quantum field theory, as a functor between the categories of manifolds (with bordisms) to vector spaces (with linear maps) [15].

The Hamiltonian can be written as

$$H = -\Delta = -(dd^* + d^*d)\,.$$

Here the differentials d, d^* play the role of the supercharges. Ground states satisfy $H\psi = 0$ and are therefore harmonic forms and in 1-to-1 correspondence with the de Rham cohomology group

$$\psi \in \mathrm{Harm}^*(X) \cong H^*(X)\,.$$

We want to make two additional remarks. First we can consider also a closed 1-manifold, namely a circle S^1 of length t. Since a circle is obtained by identifying two ends of an interval we can write

$$\Phi_{S^1} = \mathrm{Tr}_{\mathcal{H}}\, \Phi_t\,.$$

Here the partition function Φ_{S^1} is a number associated to the circle S^1 that encodes the spectrum of Δ. We can also compute the supersymmetric partition function by using the fermion number F (defined as the degree of the corresponding differential form). It computes the Euler number

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F \Phi_t = \chi(X)\,.$$

Secondly, to make the step from the quantum mechanics to the propagation of a particle in quantum field theory we have to integrate over the metric on the

1-manifold. In case of an interval we so obtain the usual propagator, the Greens' function of the Laplacian

$$\int_0^\infty dt\, e^{t\Delta} = \frac{1}{\Delta}\,.$$

3. Conformal Field Theory and Strings

We will now introduce our first deformation parameter α' and generalize from point particles and quantum mechanics to strings and conformal field theory.

3.1. Sigma models

A string can be considered as a parametrized loop. So, in this case we study the manifold X through maps

$$x\colon S^1 \to X$$

that is, through the free loop space $\mathcal{L}X$.

Quantization will associate a Hilbert space to this loop space. Roughly one can think of this Hilbert space as $L^2(\mathcal{L}X)$, but it is better to think of it as a quantization of an infinitesimal thickening of the locus of constant loops $X \subset \mathcal{L}X$. These constant loops are the fixed points under the obvious S^1 action on the loop space. The normal bundle to X in $\mathcal{L}X$ decomposes into eigenspaces under this S^1 action, and this gives a description (valid for large volume of X) of the Hilbert space $\mathcal{H}_{S^1}$ associated to the circle as the normalizable sections of an infinite Fock space bundle over X.

$$\mathcal{H}_{S^1} = L^2(X, \mathcal{F}_+ \otimes \mathcal{F}_-)$$

where the Fock bundle is defined as

$$\mathcal{F} = \bigotimes_{n\geq 1} S_{q^n}(TX) = \mathbb{C} \oplus qTX \oplus \cdots$$

Here we use the formal variable q to indicate the $\mathbb{Z}$-grading of $\mathcal{F}$ and we use the standard notation

$$S_q V = \bigoplus_{N\geq 0} q^N\, S^N V$$

for the generating function of symmetric products of a vector space V.

When a string moves in time it sweeps out a surface Σ. For a free string Σ has the topology of $S^1 \times I$, but we can also consider at no extra cost interacting strings that join and split. In that case Σ will be a oriented surface of arbitrary topology. So in the Lagrangian formalism one is let to consider maps

$$x\colon \Sigma \to X\,.$$

There is a natural action for such a sigma model if we pick a Hogde star or conformal structure on Σ (together with of course a Riemannian metric g on X)

$$S(x) = \int_\Sigma g_{\mu\nu} dx^\mu \wedge * dx^\nu \,.$$

The critical points of $S(x)$ are the harmonic maps.

In the Lagrangian quantization formalism one considers the formal path-integral over all maps $x\colon \Sigma \to X$

$$\Phi_\Sigma = \int_{x\colon \Sigma \to X} e^{-S/\alpha'} \,.$$

Here the constant α' plays the role of Planck's constant on the string worldsheet Σ. It can be absorbed in the volume of the target X by rescaling the metric as $g \to \alpha' \cdot g$. The semi-classical limit $\alpha' \to 0$ is therefore equivalent to the limit $\mathrm{vol}(X) \to \infty$.

3.2. Functorial description

In the functorial description of conformal field theory the maps Φ_Σ are abstracted away from the sigma model definition.

Starting point is now an arbitrary (closed, oriented) Riemann surface Σ with boundary. This boundary consists of a collections of oriented circles. One declares these circles in-coming or out-going depending on whether there orientation matches that of the Σ. To a surface Σ with m in-coming and n out-going boundaries one associates a linear map

$$\Phi_\Sigma \colon \mathcal{H}_{S^1}^{\otimes n} \to \mathcal{H}_{S^1}^{\otimes n} \,.$$

These maps are not independent but satisfy gluing axioms that generalize the simple composition law (1)

$$\Phi_{\Sigma_1} \circ \Phi_{\Sigma_2} = \Phi_\Sigma$$

where Σ is obtained by gluing Σ_1 and Σ_2 on their out-going and incoming boundaries respectively.

In this way we obtain what is known as a modular functor. It has a rich algebraic structure. For instance, the sphere with three holes gives rise to a product

$$\Phi\colon \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1} \,.$$

Using the fact that a sphere with four holes can be glued together from two copies of the three-holed sphere one shows that this product is essentially commutative and associative. For more details see e.g. [12, 5].

3.3. *B*-fields and gerbes

There is a straightforward generalization of the sigma model action (3.1) that includes a 2-form B on X. This so-called B-field adds an extra phase

$$\exp i \int_\Sigma x^* B$$

in the path-integral. The two-form B should be considered as a two-form analogue of a connection. Its curvature H which is locally given by $H = dB$ actually represents a class $[H/2\pi] \in H^3(X, \mathbb{Z})$. So more generally, if Σ is considered as the boundary of a 3-manifold W and the map x is extended over W, then the phase factor is defined as

$$\exp i \int_W x^* H \, .$$

By pulling back B to the loop space we see that it acts as a connection on a line bundle over $\mathcal{L}X$.

Such a B-field can be considered as a connection on a gerbe [9] over the manifold X. Gerbes are generalizations of line bundles that naturally support p-form connections. Extended objects such as strings and branes are closely connected to these p-form theories.

3.4. Strings and gravity

The way in which general relativity emerges from string theory is deep and I want to use some time to explain this here. We have already seen that at the classical level every two-dimensional sigma model is conformal invariant. In order to write down the classical action and the resulting equation of motion (the harmonic maps condition) for maps $\Sigma \to X$ we only need the Hodge star or a conformal structure on Σ. This is no longer true at the quantum level. Heuristically, we need a full metric γ on Σ to define the measure in the path-integral over all maps $\Sigma \to X$, and this metric dependence of the path-integral distroys the conformal invariance of the quantum theory.

So the partition function is actually a function of both the metric g on X and γ on Σ. We can now consider a (constant) rescaling

$$\gamma \to \mu \cdot \gamma$$

of the worldsheet metric and compute the μ-dependence of the partition function $\Phi(g, \gamma)$. In general one finds a non-zero result.

The amazing result of standard renormalization theory in quantum field theory is that a rescaling in γ can be absorbed into a redefinition of the metric g on X. This leads to the famous renormalization group flow equation [14]

$$\left[\mu \frac{\partial}{\partial \mu} + \int_X \beta_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}} \right] Z = 0 \, .$$

Here the beta-function $\beta_{\mu\nu}$ is a vector field on the space $MET(X)$ of Riemannian metrics on X.

This effect can be observed most easily in a semi-classical expansion of the sigma model in α'. That is, one consider maps that are approximately constant

$$x = x_0 + \xi$$

with $\xi\colon \Sigma \to T_{x_0}X$. In this approximation the action can be written to subleading order as (schematically)

$$\int g_{\mu\nu}(x_0) d\xi^\mu \wedge *d\xi^\nu + R_{\mu\nu\lambda\rho}(x_0)\xi^\mu \xi^\nu d\xi^\lambda \wedge *d\xi^\rho \,.$$

Here the Riemann curvature $R_{\mu\nu\lambda\rho}$ can be considered as a small perturbation in the large volume limit. To leading order one then finds that

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \mathcal{O}({\alpha'}^2) \,.$$

So to this level of approximations the Ricci flat metrics with

$$R_{\mu\nu} = 0$$

correspond to the quantum CFTs.

An important property of the beta-function β is that it is a gradient vector field on $MET(X)$

$$\beta = \nabla \mathcal{S}, \qquad \mathcal{S} = \int \sqrt{g} R(g)$$

with $\mathcal{S}$ the Einstein-Hilbert action for the metric g on X. Conformal field theories are given by the zeroes of β and thus correspond to the critical points of $\mathcal{S}$. In this sense a quantum CFT corresponds to a semi-classical solution of gravity.

3.5. 'Stringy' geometry and T-duality

Two-dimensional sigma models give a natural one-parameter deformation of classical geometry. The deformation parameter is Planck's constant α'. In the limit $\alpha' \to 0$ we localize on constant loops and recover quantum mechanics or point particle theory. For non-zero α' the non-constant loops contribute.

This structure is most familiar now in the form of quantum cohomology for Kähler manifolds. Under suitable circumstances the path-integral localizes to holomorphic maps that get weighted by a phase factor q^d where d is the degree and $q = e^t$ with t the pull-back of the complexified Kähler form $\frac{\omega}{\alpha'} + iB$. So the CFT maps Φ_Σ have a typical expansion [3]

$$\Phi_\Sigma = \sum_d q^d N_\Sigma(d)$$

where $N_\Sigma(d)$ counts the number of holomorphic maps $\Sigma \to X$. Since essentially $q \sim e^{-1/\alpha'}$ we see that these corrections are non-perturbative from a world-sheet point of view. There are invisible in a Taylor expansion in α'.

In fact we can picture the moduli space of CFT's roughly as follows. It will have components that can be described in terms of a target spaces X. For these models the moduli parametrize Ricci-flat metrics plus a choice of B-field. These components have a boundary 'at infinity' which describe the large volume manifolds. We can use the parameter α' as local transverse coordinate on the collar around this boundary. If we move away from this boundary stringy corrections set in.

In the middle of the moduli space exotic phenomena can take place. For example, the automorphism group of the CFT can jump, which gives rise to orbifold singularities at enhanced symmetry points.

The most striking phenomena that the moduli space can have another boundary that allows again for a semi-classical interpretation in terms of a second classical geometry $\hat{X}$. These points look like quantum or small volume in terms of the original variables on X but can also be interpreted as large volume in terms of a set of dual variables on a dual or mirror manifold $\hat{X}$. In this case we speak of a T-duality. In this way two manifold X and $\hat{X}$ are related since they give rise to the same CFT.

The most simple example of such a T-duality occurs for toroidal compactification. If $X = T$ is an torus, the CFT's on T and its dual T^* are isomorphic. We will explain this in more detail in §5.

4. M-Theory and Branes

We have seen how CFT gives rise to a rich structure in terms of the modular geometry as formulated in terms of the maps Φ_Σ. To go from CFT to string theory we have to make two more steps.

4.1. Summing over string topologies

First, we want to generalize to the situation where the maps Φ_Σ are not just functions on the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces but more general differential forms. In fact, we are particular interested in the case where they are volume forms since then we can define the so-called string amplitudes as

$$A_g = \int_{\mathcal{M}_g} \Phi_\Sigma \,.$$

This is also the general definition of Gromov-Witten invariants [12].

Although we suppress the depends on the CFT moduli, we should realize that the amplitudes A_g (now associated to a *topological* surface of genus g) still have (among others) α' dependence.

Secondly, it is not enough to consider a string amplitude of a given topology. Just as in field theory one sums over all possible Feynman graphs, in string theory we have to sum over all topologies of the string world-sheet. In fact, we have to ensemble these amplitudes into a generating function.

$$A(\lambda) \approx \sum_{g \geq 0} \lambda^{2g-2} A_g \,.$$

Here we introduce the so-called string coupling constant λ. Unfortunately this generating function can be at best an asymptotic series expansion of an analytical function $A(\lambda)$. A rough estimate of the volume of $\mathcal{M}_g$ shows that typically

$$A_g \sim 2g!$$

Indeed, general physics arguments tell us that the *non-perturbative* amplitudes $A(\lambda)$ have corrections of the form

$$A(\lambda) = \sum_{g\geq 0} \lambda^{2g-2} A_g + \mathcal{O}(e^{-1/\lambda})\,.$$

Clearly to approach the proper definition of the string amplitudes these non-perturbative corrections have to be understood.

4.2. M-theory

The last five years have seen remarkable progress in this direction. It involves two remarkable new ideas.

1. String theory is not a theory of strings. It is not enough to consider loop spaces. We should also include other extended objects, collectively known as branes. One can try to think of these objects as associated to more general maps $Y \to X$ where Y is a higher-dimensional space. But the problem is that there is not a consistent quantization starting from 'small' branes along the lines of string theory, that is, an expansion where we control the size of Y (through α') and the topology (through λ). However, through the formalism of D-branes [13] these can be analyzed exactly in string perturbation theory.
2. As we stressed, the amplitudes A depend on many parameters or moduli. Apart from the string coupling λ all other moduli have a geometric interpretation, in terms of the metric and B-field on X. The second new ingredient is the insight that string theory on X with string coupling λ can be given a fully geometric realization in terms of a new theory (called M-theory) on the manifold $X \times S^1$, where the length of the circle S^1 is λ [18].

5. Torus Compactifications

So we see that there is a natural hierarchy of generalized geometries, roughly associated to particles, strings and branes. According this point of view a 'classical' manifold can be considered as an element of three different categories. Viewed as an object in such a category it can inherit different symmetries.

Although we are not yet in a position to give a completely rigorous definition of what M-theory is, we do know what kind of data we want to associate to a space X. These data contain at least the following

1. A moduli space $\mathcal{M}$ of geometric structures on X. This can be a Ricci-flat metric, but also B-fields or their generalizations.
2. A charge lattice Γ that labels the various sectors of the theory.
3. A discrete symmetry group G (the duality group) that acts on the lattice Γ, which will typically form an irreducible representation.

4. A Hilbert space bundle

$$\mathcal{H} = \bigoplus_{\gamma\in\Gamma} \mathcal{H}(\gamma)$$

over the moduli space $\mathcal{M}$ that is graded by the charge lattice Γ.

This hierarchy of structures can be nicely illustrated with a very simple class of manifolds: n-tori, written as quotients

$$T = \mathbb{R}^n/L$$

with L a rank n lattice.

5.1. Particles on a torus

States of a quantum mechanical point particle on T are conveniently labeled by their momentum

$$p \in L^* .$$

The wavefunctions e^{ipx} form a basis of $\mathcal{H} = L^2(T)$ that diagonalizes the Hamiltonian $H = -\Delta = p^2$. So we can write

$$\mathcal{H} = \bigoplus_{p\in L^*} \mathcal{H}(p) ,$$

where the graded pieces $\mathcal{H}(p)$ are one-dimensional. There is a natural action of the symmetry group

$$G = SL(n,\mathbb{Z}) = \mathrm{Aut}\, L$$

on the lattice $\Gamma = L$ and the Hilbert space $\mathcal{H}$. (These transformations will in general not leave the metric invariant, but instead give by pull-back another flat metric on T.)

5.2. Strings on a torus

In the case of a string moving on the torus T states are also labeled by their winding number

$$w \in L .$$

The winding number simply distinguishes the various connected components of the loop space $\mathcal{L}T$, since

$$\pi_0\mathcal{L}T = \pi_1 T \cong L .$$

We therefore see a natural occurrence of the so-called Narain lattice

$$\Gamma^{n,n} = L \oplus L^* .$$

This is an even self-dual lattice of signature (n,n) with inner product

$$p = (w,k), \qquad q^2 = 2w\cdot k .$$

It turns out that the symmetries of the lattice $\Gamma^{n,n}$ lift to symmetries of the full conformal field theory. The symmetry group

$$SO(n,n,\mathbb{Z}) = \mathrm{Aut}\,\Gamma^{n,n}$$

are called T-dualities. A particular example is the interchange $T \leftrightarrow T^*$. From this perspective the string can be considered to be moving on the space $T \times T^*$. If we consider chiral or BPS states in the superstring, then the graded Hilbert space $\mathcal{H}(q)$ is given by

$$\mathcal{H}(p) = \mathcal{F}(\tfrac{1}{2}p^2)\,,$$

in terms of the Fock space

$$\mathcal{F}_q = \bigotimes_{n=1}^{\infty} S_{q^n}(\mathbb{R}^8) \otimes \textstyle\bigwedge_{q^n}(\mathbb{R}^8) = \bigoplus_{N\geq 0} q^N \mathcal{F}(N)\,.$$

Note that the Dynkin diagram of the corresponding lie algebra $D_n \cong so(n,n)$ is obtained from $A_{n-1} \cong sl(n)$ by adding an extra root. Reflections in this root represent the T-duality that maps T to T^*.

5.3. Branes on a torus

If we move to the full M-theory the charge lattice becomes more complicated. For small values of n ($n \geq 4$) it can be written as

$$\Gamma^{n,n} \oplus H^{\mathrm{even/odd}}(T)\,.$$

Here we note that the lattice of branes (which are even or odd depending on the type of string theory that we consider)

$$H^{\mathrm{even/odd}}(T) \cong \textstyle\bigwedge^{\mathrm{even/odd}} L^*$$

transform as half-spinor representations under the T-duality group $SO(n,n,\mathbb{Z})$. The full duality group turns out to be the exceptional group over the integers

$$E_{n+1}(\mathbb{Z})\,.$$

So we see that our hierarchy is reflected in the symmetry groups

$$SL(n,\mathbb{Z}) \subset SO(n,n\mathbb{Z}) \subset E_{n+1}(\mathbb{Z})$$

of rank $n-1$, n and $n+1$ respectively. It is already a very deep (and generally unanswered) question what the 'right' mathematical structure is associated to a n-torus that gives rise to the group $E_{n+1}(\mathbb{Z})$.

In this case we also have indirect evidence how the graded Hilbert space $\mathcal{H}$ should behave. If one considers so-called BPS states the graded pieces $\mathcal{H}(\gamma)$ should be finite dimensional, and for large γ we can estimate their growth

$$\dim \mathcal{H}(\gamma) \sim \exp S(\gamma)$$

with $S(\gamma)$ the entropy. Arguments from black hole physics tell us that

$$S(\gamma) = \sqrt{Q(\gamma)}\,,$$

with $Q(\gamma)$ an algebraic invariant of the duality group G. For example, for $n = 5$ with $G = E_6$ the lattice Γ has rank 26 and Q is the famous cubic invariant. Similarly for $n = 6$ and $G = E_7$ we obtain the quartic invariant of the 56-dimensional representation.

6. D-Branes

The crucial ingredient to extend string theory beyond perturbation theory are D-branes [13]. From a mathematical point of view D-branes can be considered as a relative version of Gromov-Witten theory. The starting point is now a pair of relative manifolds (X, Y) with X a d-dimensional manifold and $Y \subset X$ closed. The string worldsheets are defined to be Riemann surfaces Σ with boundary $\partial\Sigma$, and the class of maps $x \colon \Sigma \to X$ should satisfy

$$x(\partial\Sigma) \subset Y\,.$$

That is, the boundary of the Riemann surfaces should be mapped to the subspace Y.

Note that in a functorial description there are now two kinds of boundaries to the surface. First there are the time-like boundaries that we just described. Here we choose a definite boundary condition, namely that the string lies on the D-brane Y. Second there are the space-like boundaries that we considered before. These are an essential ingredient in any Hamiltonian description. On these boundaries we choose initial value conditions that than propagate in time. In closed string theory these boundaries are closed and therefore a sums of circles. With D-branes there is a second kind of boundary: the open string with interval $I = [0, 1]$.

The occurrence of two kinds of space-like boundaries can be understood because there are various ways to choose a 'time' coordinate on a Riemann surface with boundary. Locally such a surface always looks like $S^1 \times \mathbb{R}$ or $I \times \mathbb{R}$. This ambiguity how to slice up the surface is a powerful new ingredient in open string theory.

To the CFT described by the pair (X, Y) we will associate an extended modular category. It has two kinds of objects or 1-manifolds: the circle S^1 (the closed string) and the interval $I = [0, 1]$ (the open string). The morphisms between two 1-manifolds are again bordisms or Riemann surfaces Σ now with a possible boundaries. We now have to kinds of Hilbert spaces: closed strings $\mathcal{H}_{S^1}$ and open strings $\mathcal{H}_I$.

Semi-classically, the open string Hilbert space is given by

$$\mathcal{H}_I = L^2(Y, \mathcal{F})$$

with Fock space bundle

$$\mathcal{F} = \bigotimes_{n \geq 1} S_{q^n}(TX)\,.$$

Note that we have only a single copy of the Fock space $\mathcal{F}$, the boundary conditions at the end of the interval relate the left-movers and the right-movers. Also the fields are sections of the Fock space bundle over the D-brane Y, not over the full space-time manifold X. In this sense the open string states are localized on the D-brane.

We have seen that an important new ingredient in the step from classical geometry to 'stringy' geometry was the 2-form$B$ field, technically a connection on a gerbe. It coupled to the string worldsheet via

$$\int_\Sigma B\,.$$

The phase factor

$$\exp i \int_\Sigma B$$

that appears in the path-integral should be considered as the generalization of the holonomy of a connection on a line bundle associated to a loop. It satisfies a gauge invariance

$$B \to B + d\Lambda\,.$$

If we now work in the category of Riemann surfaces with boundary, we see that such maps are not gauge invariant but pick up a phase factor

$$\exp i \int_{\partial\Sigma} \Lambda\,.$$

However, on surfaces with boundary we can weight the path-integral with an extra phase factor. Let A be a connection (on a trivial line bundle) on Y. Then we can add the holonomy phase factor

$$\exp i \int_{\partial\Sigma} A\,.$$

Now we see that the system has a generalized gauge invariance where apart from the transformation of B we have

$$A \to A - \Lambda\,.$$

This leads to a generalized notion of gauge invariant curvature

$$F = dA + B\,.$$

This equation immediately implies that *when restricted to the D-brane* Y the curvature $H = dB$ should vanish.

6.1. Branes and matrices

One of the most remarkable facts is that D-branes can be given a multiplicity N which naturally leads to a non-abelian structure [19].

Given a modular category as described above there is a simple way in which this can be tensored over the $N \times N$ hermitean matrices. We simply replace the Hilbert space $\mathcal{H}_I$ associated to the interval I by

$$\mathcal{H}_I \otimes Mat_{N\times N}$$

with the hermiticity condition

$$(\psi \otimes M_{IJ})^* = \psi^* \otimes M_{JI}\,.$$

The maps Φ_Σ are generalized as follows. Consider for simplicity first a surface Σ with a single boundary C. Let C contain n 'incoming' open string Hilbert spaces with states $\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n$. These states are now matrix valued. Then the new morphism is defined as

$$\Phi_\Sigma(\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n) = \Phi_\Sigma(\psi_1, \ldots, \psi_n)\,\mathrm{Tr}(M_1 \cdots M_n)\,.$$

In case of more than one boundary component, we simply have an additional trace for every component.

In particular we can consider the disk diagram with three open string insertions. By considering this as a map

$$\Phi_\Sigma \colon \mathcal{H}_I \otimes \mathcal{H}_I \to \mathcal{H}_I$$

we see that this open string interaction vertex is now given by

$$\Phi_\Sigma(\psi_1 \otimes M_1, \psi_2 \otimes M_2) = (\psi_1 * \psi_2) \otimes (M_1 M_2)\,.$$

So we have tensored the associate string product with matrix multiplication.

If we consider the geometric limit where the CFT is thought of as the semi-classical sigma model on X, the string fields that correspond to the states in the open string Hilbert space $\mathcal{H}_I$ will become matrix valued fields on the D-brane Y, *i.e.* they can be considered as sections of $\mathrm{End}(E)$ with E a (trivial) vector bundle over Y.

This matrix structure naturally appears if we consider N different D-branes $Y_1, \ldots, Y_N$. In that case we have a matrix of open strings that stretch from brane Y_I to Y_J. In this case there is no obvious vector bundle description. But if all the D-branes coincide $Y_1 = \cdots = Y_N$ a $U(N)$ symmetry appears.

6.2. D-branes and K-theory

The relation with vector bundles has proven to be extremely powerful. The next step is to consider D-branes with *non-trivial* vector bundles. It turns out that these configurations can be considered as a composite of branes of various dimensions [6]. There is a precise formula that relates the topology of the vector bundle E to the brane charge $\mu(E)$ that can be considered as a class in $H^*(X)$. (For convenience we consider first maximal branes $Y = X$.) It reads [8]

$$\mu(E) = ch(E)\widehat{A}^{1/2} \in H^*(X)\,. \tag{2}$$

Here $ch(E)$ is the (generalized) Chern character $ch(E) = \mathrm{Tr}\exp(F/2\pi i)$ and $\widehat{A}$ is the genus that appears in the Atiyah-Singer index theorem. Note that the D-bane charge can be fractional.

Branes of lower dimension can be described by starting with two branes of top dimension, with vector bundles E_1 and E_2, of opposite charge. Physically two such branes will annihilate leaving behind a lower-dimensional collection of branes. Mathematically the resulting object should be considered as a virtual bundle $E_1 \ominus E_2$ that represents a class in the K-theory group $K^0(X)$ of X [20]. In fact the map μ in (2) is a well-known correspondence

$$\mu\colon K^0(X) \to H^{\mathrm{even}}(X)$$

which is an isomorphism when tensored with the reals. In this sense there is a one-to-one map between D-branes and K-theory classes [20]. This relation with K-theory has proven to be very useful.

6.3. Example: the index theorem

A good example of the power of translating between open and closed strings is the natural emergence of the index theorem. Consider the cylinder $\Sigma = S^1 \times I$ between two D-branes described by (virtual) vector bundles E_1 and E_2. This can be seen as closed string diagram with in-state $|E_1\rangle$ and out-state $|E_2\rangle$

$$\Phi_\Sigma = \langle E_2, E_1\rangle\,.$$

Translating the D-brane boundary state into closed string ground states (given by cohomology classes) we have

$$|E\rangle = \mu(E) \in H^*(X)$$

so that

$$\Phi_\Sigma = \int_X ch(E_1)ch(E_2^*)\widehat{A}\,.$$

On the other hand we can see the cylinder also as a trace over the open string states, with boundary conditions labeled by E_1 and E_2. The ground states in $\mathcal{H}_I$ are sections of the Dirac spinor bundle twisted by $E_1 \otimes E_2^*$. This gives

$$\Phi_\Sigma = \mathrm{Tr}_{\mathcal{H}_I}(-1)^F = \ \mathrm{index}(D_{E_1\otimes E_2^*})\,.$$

So the index theorem follows rather elementary.

7. U-Duality

We indicated that in M-theory we do not want to include only strings but also D-branes (and even further objects that I will suppress in this discussion such as NS 5-branes and Kaluza-Klein monopoles). So in the limit of small string coupling λ the full (second quantized) string Hilbert space would look something like

$$\mathcal{H} = S^*(\mathcal{H}_{\mathrm{string}}) \otimes S^*(\mathcal{H}_{\mathrm{brane}}) .$$

Of course our discussion up to now has been very skew. In the full theory there will be symmetries, called U-dualities, that will exchange strings and branes.

There are at present very few formulations of M-theory that present such a manifest duality-invariant approach. Only for very special compactifications (such as low-dimensional tori) matrix theory [1] or the famous AdS-CFT correspondence [11] gives a precise non-perturbative definition.

We will give a rather simple example of such a symmetry that appears when we compactify the (Type IIA) superstring on a four-torus $T^4 = \mathbb{R}^4/L$. In this case the charge lattice has rank 16 and can be written as

$$\Gamma^{4,4} \oplus K^0(T^4) .$$

It forms an irreducible spinor representation under the U-duality group

$$G = SO(5,5,\mathbb{Z}) .$$

Notice that the T-duality subgroup $SO(4,4,\mathbb{Z})$ has three inequivalent 8-dimensional representations (related by triality). The strings with Narain lattice $\Gamma^{4,4}$ transform in the vector representation while the even-dimensional branes labeled by the K-group $K^0(T^4) \cong \wedge^{\mathrm{even}} L^*$ transform in the spinor representation. (The odd-dimensional D-branes that are labeled by $K^1(T)$ and that appear in the Type IIB theory transform in the conjugate spinor representation.)

To compute the spectrum of superstrings we have to introduce the corresponding Fock space. It is given by

$$\mathcal{F}_q = \bigotimes_{n=1}^{\infty} S_{q^n}(\mathbb{R}^8) \otimes \textstyle\bigwedge_{q^n}(\mathbb{R}^8) = \displaystyle\bigoplus_{N \geq 0} q^N \mathcal{F}(N) .$$

The Hilbert space of BPS strings with momenta $p \in \Gamma^{4,4}$ is then given by

$$\mathcal{H}_{\mathrm{string}}(p) = \mathcal{F}(\tfrac{1}{2}p^2) .$$

For the D-branes we take a completely different approach. Since we only understand the system for small string coupling we have to use semi-classical methods. Consider a D-brane that corresponds to a K-theory class E with charge vector $\mu = ch(E) \in H^*(T)$. To such a vector bundle we can associate a moduli space $\mathcal{M}_\mu$ of self-dual connections. (If we work in the holomorphic context we could equally well consider the moduli space of holomorphic sheaves of this topological class.) Now luckily a lot is know about these moduli spaces. They are hyper-Kähler and (for primitive μ) smooth. In fact, they are topologically Hilbert schemes [10] which are deformations of symmetric products

$$\mathcal{M}_\mu \cong \mathrm{Hilb}^{\mu^2/2}(T^4) \sim S^{\mu^2/2}T^4 \,.$$

Computing the BPS states through geometric quantization we find that

$$\mathcal{H}_{\mathrm{brane}}(\mu) = H^*(\mathcal{M}_\mu) \,.$$

The cohomology of these moduli spaces have been computed [7] with the result that

$$\bigoplus_{N\geq 0} q^N H^*(\mathrm{Hilb}^N(T^4)) = \mathcal{F}_q \,.$$

This gives the final result

$$\mathcal{H}_{\mathrm{brane}}(\mu) = \mathcal{F}(\mu^2/2) \cong \mathcal{H}_{\mathrm{string}}(p)$$

where μ and p are related by an $SO(5,5,\mathbb{Z})$ U-duality transformation.

8. Non-Commutative Geometry

From the present point of view, D-branes and the corresponding open strings seem to indicate that in a more final formulation of M-theory a fundamental role is played by non-commutative geometry [2, 16]. One of the indications is the occurrence of the B-field in string theory. Roughly in the presence of such a (constant) B-field the space-time coordinates do not longer commute, but instead satisfy

$$[x_\mu, x_\nu] = B_{\mu\nu} \,.$$

One of the most striking results is that the D-brane gauge theory, in the presence of such a B-field indeed is invariant under the T-duality group through the concept of Morita equivalence. Further explorations of the links between string theory and noncommutative geometry can well give the key to a final understanding of M-theory.

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Korteweg-de Vries Institute for Mathematics,
University of Amsterdam,
Plantage Muidergracht 24,
1018 TV Amsterdam, The Netherlands
E-mail address: rhd@science.uva.nl