Reassigned Scalograms and Singularities

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Abstract. Reassignment is a general nonlinear technique aimed at increasing the localization of time-frecuency and time-scale distributions. Its principle consists in supplementing an energy distribution with a suitable vector field, thanks to which energy contributions are moved on the plane so as to sharpen the initial distribution. Reassignment can be performed in an efficient way and, in the case of scalograms (i.e., wavelet-based energy densities), it can be equipped with fast algorithms too. When applied to isolated Hölder singularities, scalogram reassignment acts as a squeezing operator upon the influence cone of the underlying wavelet transform, thus permitting a sharper localization and a higher contrast as compared to conventional scalograms. Closed form expressions can be obtained for the specific family of Klauder wavelets, with the Morlet wavelet as a limiting case. When considered as a function of scale at the time instant of the singularity, reassigned scalograms are shown to undergo a power-law evolution from which the Hölder exponent can be estimated.

1. Introduction

Time-frecuency analysis of nonstationary signals can be performed in many different ways, with techniques ranging from short-time Fourier or wavelet transforms to Wigner-type methods [7, 14]. Whereas the former approaches exhibit poor localization properties, the latter ones are impaired by interference phenomena which limit the effectiveness of their sharper localization in multicomponent situations [10]. A nonlinear technique, referred to as *reassignment*, has been proposed to overcome both limitations. In a nutshell, reassignment is a two-step process which consists in smoothing out oscillating interference terms while squeezing the localized terms which have been spread over the plane by the smoothing operation. Originally proposed for the only spectrograms (short-time Fourier energy densities) [12, 13], the method has since been generalized far beyond [1], with a possible application to time-scale techniques such as scalograms (wavelet energy densities). The purpose of this paper is to address some specific issues related to scalogram reassignment, and especially to investigate how reassigned scalograms may be used for characterizing isolated Hölder singularities.

2. Reassigning Scalograms

Given an admissible wavelet $\psi(t) \in L^2(\mathbb{R})$ and a scale factor $a > 0$, the continuous wavelet transform (CWT) of a signal $x(t) \in L^2(\mathbb{R})$ is classically defined by [14]

$$T_x^{(\psi)}(t,a) := \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} x(s)\, \overline{\psi\left(\frac{s-t}{a}\right)}\, ds\,. \tag{1}$$

As written in (1), the CWT is a function of time and scale but one can remark that, under mild conditions on the analyzing wavelet $\psi(t)$ (namely that its spectrum[1] energy density $|\Psi(\omega)|^2$ is unimodal and characterized by some reference (angular) frequency $\omega_0 > 0$), it can also be expressed as a function of time and frequency $\omega > 0$, with the identification $\omega := \omega_0/a$.

As it is well-known, a by-product of the admissibility condition imposed to $\psi(t)$ (essentially that it is zero-mean, together with a proper normalization) is that an energetic interpretation can be attached to a CWT, thanks to the relation

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty} |T_x^{(\psi)}(t,a)|^2\, \frac{dt\, da}{a^2} = \|x\|_2^2\,. \tag{2}$$

The energy distribution $S_x^{(\psi)}(t,a) := |T_x^{(\psi)}(t,a)|^2$ associated to a CWT is referred to as a *scalogram* and, whereas it is usually defined as the squared modulus of a CWT, it turns out [16] that it can be expressed as well as

$$S_x^{(\psi)}(t,a) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} W_x(s,\xi)\, W_\psi\left(\frac{s-t}{a}, a\xi\right) \frac{ds\, d\xi}{2\pi}, \tag{3}$$

by introducing the *Wigner-Ville distribution* (WVD) of $x(t)$ [7] :

$$W_x(t,\omega) := \int_{-\infty}^{+\infty} x(t+s/2)\, \overline{x(t-s/2)}\, e^{-i\omega s}\, ds\,. \tag{4}$$

The above relation (3) makes explicit the fact that a scalogram results from an affine smoothing of the WVD, with the consequence that the scalogram value at a given point (t,a) of the plane cannot be considered as pointwise. In fact, thanks to the smoothing operation (3), it rather results from the summation of all WVD contributions within some time-frecuency domain defined as the essential time-frecuency support of the wavelet, properly shifted by t and scaled by a. A whole distribution of values is therefore summarized by a single number, and this number is assigned to the geometrical center of the domain over which the distribution is considered. Reasoning with a mechanical analogy, the situation is as if the total mass of an object was assigned to its geometrical center, an arbitrary point which —except in the very specific case of an homogeneous distribution over the domain— has no reason to suit the actual distribution. A much more meaningful choice is to assign the total mass to the *center of gravity* of the distribution within

[1]Throughout this paper, we will use the convention of using lower case symbols for representing signals in the time domain, while labelling by the corresponding upper case symbols their spectra in the frequency domain.

the domain, and this is precisely what reassignment does: at each point where a scalogram value is computed, we also compute the local centroïd $(\hat{t}_x(t,a), \hat{\omega}_x(t,a))$ of the WVD W_x, as seen through the time-frecuency "window" W_ψ centered in $(t, \omega = \omega_0/a)$, and the scalogram value is moved from the point where it has been computed to a new time-scale point deduced from this centroïd. Operating directly on the time-scale plane, this leads to define the reassigned scalogram as [1]:

$$\check{S}_x^{(\psi)}(t,a) := \int_{-\infty}^{+\infty}\int_0^{+\infty} S_x^{(\psi)}(s,b)\, \delta\left(t - \hat{t}_x(s,b), a - \hat{a}_x(s,b)\right) \frac{b^2}{\hat{a}_x^2(s,b)}\, ds\, db\,, \quad (5)$$

with the centroïd in time given by

$$\hat{t}_x(t,a) = \frac{1}{S_x^{(\psi)}(t,a)} \int_{-\infty}^{+\infty}\int_0^{+\infty} s\, W_x(s,\xi)\, W_\psi\left(\frac{s-t}{a}, a\xi\right) \frac{ds\, d\xi}{2\pi}\,, \quad (6)$$

whereas the associated centroïd in scale $\hat{a}_x(t,a)$ needs some intermediate step in frequency. Precisely, it is necessary to first compute the quantity:

$$\hat{\omega}_x(t,a) = \frac{1}{S_x^{(\psi)}(t,a)} \int_{-\infty}^{+\infty}\int_0^{+\infty} \xi\, W_x(s,\xi)\, W_\psi\left(\frac{s-b}{a}, a\xi\right) \frac{ds\, d\xi}{2\pi}\,, \quad (7)$$

from which a frequency-to-scale conversion is then achieved according to:

$$\hat{a}_x(t,a) = \frac{\omega_0}{\hat{\omega}_x(t,a)}\,, \quad (8)$$

where ω_0 is the mean frequency of the wavelet spectrum energy density $|\Psi(\omega)|^2$, assumed to be of unit integral over $\mathbb{R}_+$. Whereas the proposed displacement rule is natural from the viewpoint of the mechanical analogy mentioned above, it is worth noting that other prescription rules can be derived from group theory arguments [5].

From a practical point of view, the centroïds (6) and (8) can be evaluated in a much more efficient way. Given a wavelet $\psi(t)$, this requires the introduction of the two additional wavelets $(\mathbf{T}\psi)(t) := t\,\psi(t)$ and $(\mathbf{D}\psi)(t) := (d\psi/dt)(t)$, thanks to which we have [1]:

$$\hat{t}_x(t,a) = t + a\ \mathrm{Re}\left\{T_x^{(>\mathbf{T}\psi)}(t,a)/T_x^{(\psi)}(t,a)\right\}\,; \quad (9)$$

$$\hat{a}_x(t,a) = -\frac{a\,\omega_0}{\mathrm{Im}\left\{T_x^{(>\mathbf{D}\psi)}(t,a)/T_x^{(\psi)}(t,a)\right\}}\,. \quad (10)$$

As compared to conventional scalograms based on one single CWT, reassigned scalograms involve three different CWT's (and even only two when using a Morlet wavelet $g(t)$ for which $(\mathbf{T}g)(t)$ and $(\mathbf{D}g)(t)$ are proportional). The computational burden remains therefore of the same order, with moreover the possibility of an efficient implementation based on fast algorithms [6].

3. Reassignment as Squeezing

Whereas there is no time-scale curve onto which conventional scalograms can perfectly localize, reassigned scalograms inherit automatically of the localization properties of the WVD on straight lines of the time-frecuency plane. More precisely, it is well-known that, for unimodular signals of the form $x(t) = \exp\{i(\omega_0 t + \alpha t^2/2)\}$, we have $W_x(t,\omega) = \delta\left(\omega - (\omega_0 + \alpha t)\right)$ for any α (linear "chirp"), and not only for $\alpha = 0$ (pure tone) as is the case for the Fourier transform [7]. Letting α go to $\pm\infty$ leads formally to idealized impulses for which we also have:

$$x(t) = \delta(t - t_0) \Rightarrow W_x(t,\omega) = \delta\left(t - t_0\right) . \tag{11}$$

In this case, it immediately follows from (9) that $\hat{t}_x(t,a) = t_0$ for any a and any $\psi(t)$, thus guaranteeing that $\check{S}_x^{(\psi)}(t,a) \propto \delta(t - t_0)$, as does the WVD in (11). This situation contrasts with that of the ordinary scalogram for which

$$x(t) = \delta(t - t_0) \Rightarrow S_x^{(\psi)}(t,a) = \frac{1}{a}\left|\psi\left(\frac{t_0 - t}{a}\right)\right|^2 . \tag{12}$$

In this latter case, the essential support of the scalogram corresponds to a time-scale domain (referred to as its *influence cone*) that is limited by the two lines of equations $t = t_0 \pm a\,\Delta t_\psi/2$, where Δt_ψ stands for a measure of effective width of the wavelet $\psi(t)$. Reassignment acts therefore as a squeezing operator that permits to end up with a perfectly localized distribution in the case of a perfectly localized impulse[2]. For the picture to be complete, one has to also consider the reassignment operator acting on scale and it turns out, from (8), that

$$\hat{a}_x(t,a) = \frac{\omega_0}{\omega_{\psi_a}(t)}, \tag{13}$$

where $\omega_{\psi_a}(t)$ refers to the instantaneous frequency of $\psi_a(t) := \psi(t/a)/\sqrt{a}$, i.e., of the wavelet at scale a. Reassignment in scale depends therefore on the chosen wavelet and, except in Morlet-type cases for which the instantaneous frequency $\omega_g(t)$ is constant, reassignment trajectories will in general be curves that are not parallel to the time axis.

4. Singularity Characterization from Reassigned Scalograms

Time-scale techniques —including scalograms— are known to provide a tool simultaneously adapted for the detection and the characterization of singularities [14, 15]. Inspired by the impulse example sketched above, it is therefore natural to consider reassigned scalograms of singularities, the underlying motivation being that reassignment methods may improve the contrast in the representation of singularities and therefore their detection.

[2] As such, reassignment shares much with related techniques such as those discussed in [4, 8, 9].

We will, here, limit ourselves to isolated Hölder singularities. Any isolated Hölder singularity can be written as (details about the construction of this family can be found in [3]):

$$X(\omega) = A_\nu |\omega|^{-\nu-1}\,, \tag{14}$$

where the amplitude function A_ν is parameterized by the Hölder exponent ν according to:

$$A_\nu = \begin{cases} 2\Gamma(\nu+1)\,(-\sin(\nu\pi/2))\,, & \text{if } \nu \in \mathbb{R}-\mathbb{Z}\,, \\ 2(\nu\,!)\,(-1)^{(\nu+1)/2}\,, & \text{if } \nu \in \mathbb{N}\,, \\ \pi(-1)^{\nu/2}/|\nu+1|\,, & \text{if } \nu \in \mathbb{Z}_-^*\,. \end{cases} \tag{15}$$

These functions are obviously a particular case of singularities. Nevertheless, they are good local approximations for a wide variety of observable singular behaviours.

Defining the fractional derivative of order α of a signal $x(t)$ as

$$x^{(\alpha)}(t) := \int_0^{+\infty} (i\omega)^\alpha\, X(\omega)\, e^{it\omega}\, \frac{d\omega}{2\pi}\,, \tag{16}$$

it turns out [9] that isolated Hölder singularities of the type (14) have the property that their CWT is equal to a rescaled version of their fractional derivative of order $\alpha = -\nu - 1$:

$$T_x^{(\psi)}(t,a) = A_{-\alpha-1}\, a^{-(\alpha+1/2)}\, i^\alpha\, \overline{\psi^{(\alpha)}(-t/a)}\,. \tag{17}$$

One can see in (17) two important characteristics of the scalogram structure of Hölder singularities. First, the energy is almost entirely concentrated in influence cone defined by the support of $|\psi^{(\alpha)}(-t/a)|$. Second, from the restriction of (17) at time $t = 0$, namely

$$\log|T^{(\psi)}(0,a)|^2 = \log|A_\nu \psi^{(-\nu-1)}(0)|^2 + (2\nu+1)\log a\,, \tag{18}$$

one can get a simple estimate of ν by measuring the scalogram slope along the scale axis in a log-log diagram [15].

In order to go further, the wavelet $\psi(t)$ has to be specified, and it proves convenient to make use of the so-called *Klauder* [11] (or Cauchy) wavelet, defined in the time domain as

$$\kappa_{\beta,\gamma}(t) = \frac{C_{\beta,\gamma}}{(\gamma - it)^{\beta+1}}\,, \tag{19}$$

where the constant $C_{\beta,\gamma} = (2\gamma)^{\beta+1/2}\Gamma(\beta+1)/\sqrt{2\pi\Gamma(2\beta+1)}$ ensures a unit energy normalization. From the Fourier transform of $\kappa_{\beta,\gamma}(t)$,

$$K_{\beta,\gamma}(\omega) = C_{\beta,\gamma}\,\frac{2\pi}{\Gamma(\beta+1)}\,\omega^\beta\, e^{-\gamma\omega}\, U(\omega)\,, \tag{20}$$

with the convergence conditions $\beta > -1/2$ and $\gamma > 0$, and where $U(\cdot)$ is the Heaviside step function, one can see that the Klauder wavelet family is covariant to fractional differentiation:

$$\kappa_{\beta,\gamma}^{(\alpha)}(t) = \left(\frac{i}{2\gamma}\right)^{\alpha} \sqrt{\frac{\Gamma(2(\alpha+\beta)+1)}{\Gamma(2\beta+1)}}\, \kappa_{\alpha+\beta,\gamma}(t)\,. \tag{21}$$

This last equation gives us the possibility of obtaining the following closed form expression for the (Klauder) CWT of a Hölder singularity:

$$T_x^{(\kappa_{\beta,\gamma})}(t,a) = A_{-\alpha-1}\,(2\gamma a)^{\beta+1/2}\,\frac{\Gamma(\alpha+\beta+1)}{\sqrt{\Gamma(2\beta+1)}}\,|\gamma a - it|^{-(\alpha+\beta+1)}\,, \tag{22}$$

whence that of the corresponding scalogram, by taking the squared modulus.

Concerning the computation of the centroïds (9) and (10), the two wavelet transforms $T_x^{(\mathbf{T}\psi)}$ and $T_x^{(\mathbf{D}\psi)}$ need to be expressed. For this, we can use the property that the Klauder wavelet is stable by multiplication by t and by differentiation (the second property resulting from (21) with $\alpha = 1$):

$$\begin{aligned} (d\kappa_{\beta,\gamma}/dt)(t) &= (i/2\gamma)\,\sqrt{(2\beta+3)(2\beta+2)}\,\kappa_{\beta+1,\gamma}(t), \\ t\,\kappa_{\beta,\gamma}(t) &= i\gamma\sqrt{(2\beta)/(2\beta-1)}\,\kappa_{\beta-1,\gamma}(t) - i\gamma\kappa_{\beta,\gamma}(t)\,. \end{aligned} \tag{23}$$

Combining (17), (21) and (23), we can get the algebraic form of the three wavelet transforms involved in (9) and (10), leading finally to (see [3] for details):

$$\hat{t}(t,a) = \frac{\alpha}{\alpha+\beta}\,t\;; \tag{24}$$

$$\hat{a}(t,a) = \frac{\omega_0}{\alpha+\beta+1}\left(\gamma a + \frac{t^2}{\gamma a}\right). \tag{25}$$

with the reference frequency of the Klauder wavelet equal to $\omega_0 = (\beta+1/2)/\gamma$.

A remarkable feature of the above equation is that, although the Klauder wavelet is of infinite support in time (with an effective width $\Delta t_{\kappa_{\beta,\gamma}} \propto \gamma/\sqrt{2\beta-1}$), it leads to a reassigned scalogram whose time-scale support is *strictly* limited to a cone centered at the time of occurrence $t = 0$ of the singularity. Border lines of this cone are given by

$$\hat{t} = \pm\,\frac{\alpha\gamma\,(\alpha+\beta+1)}{(\alpha+\beta)(2\beta+1)}\,\hat{a}\,, \tag{26}$$

thus showing that the sharpness of the cone is controlled by both the singularity strength (through α) and the chosen wavelet (through β and γ). More precisely, it is easy to check that, for any fixed β and γ (i.e., for any fixed Klauder wavelet), the angle θ of the cone goes to zero as $\theta \sim [(\gamma/\beta)(\beta+1)/(\beta+1/2)] \times \alpha$ when $\alpha \to 0$, as expected from the result we obtained in the impulse case. Conversely, we also get that θ goes to zero as $\theta \sim \gamma\alpha/\beta$ when α is fixed and $\beta \to \infty$. In such a limiting case, the Klauder wavelet tends to become a Morlet wavelet, thus evidencing that a Morlet scalogram perfectly localizes Hölder singularities, even in cases where $\alpha \neq 0$.

The explicit evaluation of the reassigned scalogram can finally be achieved by inverting the reassignment operators (24) and (25), so as to identify which quantities sum up at a given reassignment point $(\hat{t}, \hat{a})$, leading to

$$\check{S}_x^{(\kappa)}(\hat{t}, \hat{a}) = \frac{\hat{C}\,\hat{a}^{-(\alpha+\beta-1)}}{(2C)^{\alpha+\beta+1}} \sum_{\epsilon=\pm 1} \left(C\hat{a} + \epsilon\sqrt{(C\hat{a})^2 - (1+\beta/\alpha)^2\hat{t}^2} \right)^{-\alpha+\beta-2} \quad (27)$$

where the constants are given by $\hat{C} = A^2_{-\alpha-1} 2^{2\beta+1} \gamma^{\alpha+\beta+3} \Gamma^2(\alpha+\beta+1)/\Gamma(2\beta+1)$ and $C = \gamma(\alpha+\beta+1)/(2\beta+1)$.

At the time of occurrence $t = 0$ of the singularity, (27) simplifies to

$$\check{S}_x^{(\kappa)}(0, \hat{a}) = \frac{\hat{C}}{(2C)^{2\alpha-3}}\, \hat{a}^{-(2\alpha+1)}\,, \quad (28)$$

equation from which we conclude that (i), as for the scalogram, the reassigned scalogram undergoes a power-law behaviour with respect to scales, and (ii) the exponent $-(2\alpha+1) = 2\nu+1$ of this power-law is the same as in the scalogram case (see (18)). This means that the measurement of the Hölder exponent ν can be possibly done with a reassigned scalogram.

5. Conclusion

Reassigned scalograms have been shown to be candidates for the detection and characterization of isolated Hölder singularities, and by extension for the measurement of the local regularity index of an arbitrary signal. Some technical details (such as existence conditions for the Klauder wavelet, or the consideration of special cases where the reassignment operators cannot be inverted) and questions concerning practical implementation (influence of the time-frecuency grid resolution) have been omitted here but can be found in [3].

Software — Matlab codes for computing reassigned time-frecuency and time-scale distributions are available as part of a Toolbox [2], freely distributed on the Internet.

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