Contact Topology in Dimension Greater than Three

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Abstract. The aim of this talk is to give a survey of the known methods for constructing contact structures on manifolds of dimension greater than three. We give an extensive list of contact manifolds that can be constructed via these methods, including some recent examples of contact structures on 5-dimensional manifolds found by a combination of contact surgery and cobordism theoretic techniques.

1. Introduction

Let M be a differentiable manifold of odd dimension $2n-1$. A codimension one distribution ξ on M (i.e. a smooth tangent hyperplane field) is called a **contact structure** if any differential 1-form α that locally defines ξ as $\xi = \ker \alpha$ satisfies the condition that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere zero. This condition is independent of the choice of α: Replacing α by $f\alpha$ with f some nowhere zero function, we have $f\alpha \wedge (d(f\alpha))^{n-1} = f^n \alpha \wedge (d\alpha)^{n-1}$.

If ξ is coorientable, such an α exists globally (by a partition of unity argument) and is then called a **contact form**. For convenience we restrict attention to orientable manifolds and coorientable contact structures.

In the present survey we concentrate on the question as to what can be said about the existence of contact structures, in particular on manifolds of dimension greater than three. For a discussion of contact topology in a wider context of its recent and not so recent history see [8, 14].

2. Some Classical Constructions

In this section we briefly recall the constructions of contact manifolds known before the early 1990s. According to the usage of C. T. C. Wall, as related to me by C. B. Thomas, the attribute ‘classical’ may be employed with reference to any result prior to one’s thesis.

2.1. Hypersurfaces in symplectic manifolds

Let (V, ω) be a symplectic manifold of dimension $2n$. This means that ω is a closed 2-form ($d\omega = 0$) of maximal rank ($\omega^n \neq 0$). A vector field X on V is called a **Liouville vector field** if $L_X\omega = \omega$. With the help of Cartan's formula for the Lie derivative $L_X = d\circ i_X + i_X \circ d$ this may be rewritten as $d(i_X\omega) = \omega$. Then the 1-form $\alpha = i_X\omega$ defines a contact form on any hypersurface M in V transverse to X. Indeed,

$$\alpha\wedge(d\alpha)^{n-1} = (i_X\omega)\wedge(d(i_X\omega))^{n-1} = i_X\omega\wedge\omega^{n-1} = \frac{1}{n} i_X(\omega^n)\,,$$

which restricts to a volume form on $M\subset V$ provided M is transverse to X. This is how contact structures arise on suitable energy hypersurfaces in Hamiltonian systems.

Example 2.1. *$M = S^{2n-1}\subset\mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^n dx_i\wedge dy_i$ and Liouville vector field $X = \frac{1}{2}\sum_{i=1}^n (x_i\partial_{x_i} + y_i\partial_{y_i})$, whence $\alpha = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$. Notice that if we regard S^{2n-1} as the unit sphere in $\mathbb{C}^n$ with complex structure J, then $\xi = \ker\alpha$ defines at each point $x\in S^{2n-1}$ the $(n-1)$-dimensional complex subspace of T_xS^{2n-1}, since $\alpha = -\frac{1}{2} r\, dr\circ J$ in terms of the radial coordinate r. The hermitian form $d\alpha(\,\cdot\,, J\,\cdot\,)$ on ξ is called the Levi form of the hypersurface $S^{2n-1}\subset\mathbb{C}^n$, and the contact condition for α corresponds to the positive definiteness of that Levi form, or the strict pseudoconvexity of the hypersurface.*

Example 2.2. *The cotangent bundle T^*N of an arbitrary manifold N of dimension n carries a canonical symplectic form ω which in local coordinates q_i on N and dual coordinates p_i can be written as $\sum_{i=1}^n dp_i\wedge dq_i$. The radial vector field in fibre direction, $X = \sum_{i=1}^n p_i\partial_{p_i}$, is a Liouville vector field on (T^*N,ω). So the unit cotangent bundle of N (with respect to any Riemannian metric on N) inherits a contact form, given in local coordinates by $\alpha = \sum_{i=1}^n p_i dq_i$.*

2.2. S^1-bundles

Let (B,ω) be a symplectic manifold such that ω defines an integral de Rham cohomology class, i.e. $[\omega]$ lies in the image of the natural homomorphism $H^2(B;\mathbb{Z})\hookrightarrow H^2(B;\mathbb{R})\cong H^2_{dR}(B)$. Let M be the total space of the S^1-bundle over B with Euler class $e = [\omega]$. Then, as observed by Boothby and Wang [4], one can find a connection 1-form α for this bundle such that the curvature equation reads $d\alpha = \pi^*\omega$, where $\pi\colon\ M\to B$ is the bundle projection. This easily implies that α is a contact form on M.

Example 2.3. *We can recover example 2.1 by considering the generalised Hopf fibration $\pi\colon\ S^{2n-1}\to\mathbb{C}P^{n-1}$, where $\mathbb{C}P^{n-1}$ is equipped with its standard Fubini-Study symplectic form. By computing the Betti numbers of S^1-bundles over certain Kähler manifolds with integral Kähler class, Boothby and Wang produced the first systematic examples of contact manifolds that are not spheres or cotangent bundles.*

This construction was extended by C. B. Thomas [31] to certain Seifert fibrations. As an application, he could show that most simply connected, indecomposable 5-dimensional spin manifolds carry a contact structure, since they can be realised as Seifert S^1-bundles over $\mathbb{C}P^2$.

2.3. Brieskorn manifolds

In the mid 1970s several teams of authors (see [26, 30, 1]) observed that Brieskorn manifolds

$$\Sigma(a_0,\dots,a_n)=\left\{(z_0,\dots,z_n)\in\mathbb{C}^{n+1}\,|\,z_0^{a_0}+\dots+z_n^{a_n}=0\right\}\cap S^{2n+1}$$

(here the a_j are natural numbers ≥ 2) admit a contact structure. One of the several equivalent ways of writing this contact structure is as the kernel of the contact form

$$\alpha=\frac{i}{4}\sum_{j=0}^{n}(z_j d\overline{z}_j-\overline{z}_j dz_j)\,.$$

Example 2.4. *Every odd dimensional homotopy sphere that bounds a parallelisable manifold can be realised as a Brieskorn manifold, see* [22]*, and hence admits a contact structure. This includes all 28 seven-dimensional and all 992 eleven-dimensional homotopy spheres.*

Example 2.5. *Using the Brieskorn examples, and the connected sum construction for contact manifolds due to Meckert* [28] *(this will be discussed below in the context of contact surgery), C. B. Thomas* [32] *proved the existence of contact structures on a range of* $(n-1)$*-connected* $(2n+1)$*-manifolds.*

2.4. 3-dimensional constructions

Our main emphasis is on higher dimensions, but for completeness we mention some 3-dimensional constructions, so that below we can see to what extent they have higher-dimensional analogues. For more detailed surveys see [8, 17, 33].

The existence of contact structures on every closed, orientable 3-manifold was first shown by Martinet [27], using the surgery presentation for 3-manifolds due to Lickorish and Wallace. In dimension 3, surgery preserving a contact structure can be performed on any S^1 embedded *transversely* to a given contact structure (and with any framing), and the standard contact structure on S^3 provides the starting point.

Thurston and Winkelnkemper [34] gave a short alternative proof using the Alexander open book decomposition for 3-manifolds, and Gonzalo [20] obtained the same result from a branched covering description of 3-manifolds due to Hilden, Montesinos, and Thickstun.

A further proof of the existence of contact structures on every closed, orientable 3-manifold M was given by Altschuler [2]. His proof is based on a parabolic deformation of 1-forms. Starting from a nowhere zero 1-form α which satisfies $\alpha\wedge d\alpha\geq 0$ but not $\alpha\wedge d\alpha\equiv 0$ (plus some additional restrictions), a heat equation is used to diffuse the 'positivity' of the form over the whole 3-manifold. Eliashberg and Thurston [10] considerably extended these ideas, using more geometric arguments.

3. Contact Surgery and Cobordism Theory

In 1990 Eliashberg [6] proved a remarkable theorem about the topology of Stein manifolds (affine complex analytic manifolds). As was known from the work of Lefshetz, Serre, Frankel-Andreotti and Milnor, a Stein manifold of real dimension $2n$ admits a proper Morse function with all critical points of index $\leq n$, so in particular it has the homotopy type of an n-dimensional CW complex. Eliashberg's theorem says that the converse is true provided $n \geq 3$.

Theorem 3.1. (Eliashberg) *Let (W, J) be a $2n$-dimensional compact almost complex manifold with boundary ∂W. If $n \geq 3$ and W admits a Morse function constant on the boundary and with all critical points of index $\leq n$, then J is homotopic to an integrable almost complex structure J' such that ∂W is strictly J'-convex and the interior of W is Stein.*

The situation for $n = 2$ is more complicated, see [19].

3.1. Contact surgery

As mentioned in example 2.5, in 1982 Meckert had given a proof that the connected sum of two contact manifolds also admits a contact structure, but her construction did not seem to allow any simple generalisation to surgeries along higher-dimensional spheres (observe that the connected sum of two manifolds may be thought of as a surgery along a zero-dimensional sphere S^0, one each of the two points constituting S^0 being embedded in the two manifolds).

Recall from example 2.1 that the strict J-convexity of a hypersurface M in an almost complex manifold (V, J) is equivalent to saying that the J-invariant hyperplanes in the tangent spaces of M define a contact structure on M. So theorem 3.1 may be interpreted as a statement about contact surgery. Weinstein [36] gave an alternative description of this aspect of Eliashberg's theorem, using hypersurfaces transverse to a Liouville vector field. The basic idea in Weinstein's construction is strikingly simple.

We illustrate this idea for connected sums of 3-manifolds, see figure 1. Consider $\mathbb{R}^4$ with coordinates x, y, z, t and standard symplectic form $\omega = dx \wedge dy + dz \wedge dt$. The vector field $X = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + 2z\partial_z - t\partial_t$ is a Liouville vector field for ω. We can find a smooth handle, with boundary transverse to X, joining the hyperplanes $\{t = 1\}$ and $\{t = -1\}$. Thus $\alpha = \omega(X, \cdot\,)$ induces a contact form on these hyperplanes and the (boundary of) the smooth handle joining them.

On $\{t = \pm 1\}$, this 1-form α restricts to $\frac{1}{2}x\,dy - \frac{1}{2}y\,dx \pm dz$. By the Darboux theorem for contact forms, any contact form can in suitable local coordinates be written exactly like that. Thus, by choosing the handle in figure 1 thin enough, we can glue its ends into two given contact 3-manifolds to perform a connected sum (or a 0-surgery on one component).

By replacing the coordinate axes in this picture by higher-dimensional euclidean spaces, one obtains the corresponding handle for surgeries along higher-dimensional **isotropic** spheres (i.e. spheres tangent to a given contact structure). Notice that an isotropic submanifold in (M^{2n-1}, ξ) has dimension at most $n - 1$.

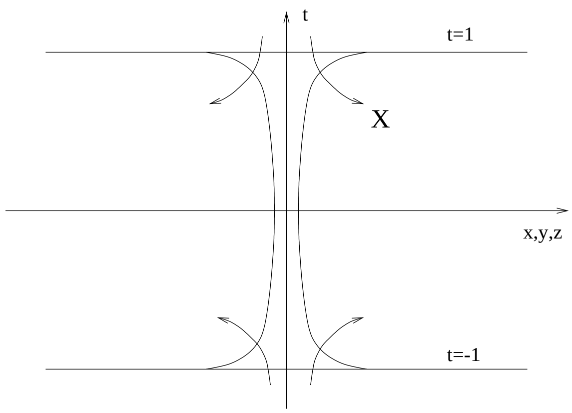

FIGURE 1. Handle used in contact surgery.

Thanks to an h-principle for isotropic embeddings (cf. [21, 6, 11]), and a neighbourhood theorem for isotropic submanifolds in place of the Darboux theorem, the question which kinds of surgeries can be performed as contact surgeries now essentially becomes a problem in algebraic topology.

It is important to bear in mind that there are two questions that need to be adressed when trying to perform a contact surgery:

(1) Can a given embedding $S^k \hookrightarrow (M, \xi)$ be approximated by an isotropic embedding (i.e. can the h-principle be applied)?
(2) Which framings (i.e. trivialisations of the normal bundle of $S^k \hookrightarrow M$) allow contact surgery?

These issues are discussed at length in [12, 15] for various applications. The introductory sections in those papers might provide a useful guide.

Example 3.2. *Using contact surgery, an essentially complete solution to the existence problem for contact structures on* $(n-1)$*-connected* $(2n+1)$*-manifolds was given in* [11, 12]. *The essential point is that these manifolds can be obtained from a sphere by doing surgery below the middle dimension. The precise existence statement is complicated by the presence of exotic spheres and certain exceptional invariants in the classification of these highly connected manifolds. Here are some examples that are easy to state:*

(a) *Any simply-connected* 5*-manifold* M *whose second Stiefel-Whitney class* $w_2(M) \in H^2(M; \mathbb{Z}_2)$ *admits an integral lift* $c \in H^2(M; \mathbb{Z})$ *carries a contact structure* ξ *such that the conformally symplectic (and hence complex) bundle* $(\xi, d\alpha)$ *has first Chern class* $c_1 = c$*.*
(b) *Every* 2*-connected* 7*-manifold and every* 4*-connected* 11*-manifold admits a contact structure (cf. example 2.4).*

A coorientable codimension one distribution η on M with a complex bundle structure or, equivalently, a reduction of the structure group of TM^{2n+1} to $\mathrm{U}(n)\times 1$, is called an **almost contact structure**. Clearly, the existence of an almost contact structure is a necessary condition for the existence of a coorientable contact structure. In dimension 5 (without any assumption on the fundamental group of M), almost contact structures are classified by the integral lifts of $w_2(M)$.

3.2. Plumbing

By arguments related to those in the proof of theorem 3.1, Eliashberg [7] showed the following: Given a symplectic manifold (W,ω) and $L \looparrowright W$ an immersed Lagrangian submanifold (i.e. $\omega|TL \equiv 0$ and $\dim L = (\dim W)/2$) with only transverse double points, one can find a small neighbourhood of L whose boundary inherits a contact structure. In particular —recall the remarks in example 2.2— this allows the plumbing of cotangent disc bundles (for a discussion of plumbing see for instance [22]).

Example 3.3. *In* [7] *this was used by Eliashberg to construct exotic contact structures on S^{2n-1}, i.e. contact structures that are not diffeomorphic to the standard structure of example 2.1. The exoticity of the contact structure is detected by a symplectic filling not containing any symplectic spheres, but different from a disc D^{2n}.*

Two contact structures on a manifold are called **homotopically equivalent** if they induce homotopic almost contact structures. A contact structure ξ on $S^{2n-1} = \partial D^{2n}$ is called **homotopically standard** if it is homotopically equivalent to the standard contact structure, or equivalently, if the stable almost complex structure on the stable tangent bundle $TS^{2n-1} \oplus \varepsilon^1 \cong TD^{2n}|S^{2n-1}$ defined by ξ extends as an almost complex structure over D^{2n}.

Example 3.4. *For n odd, in which case the obstruction group $\pi_{2n-1}(\mathrm{SO}_{2n}/\mathrm{U}_n)$ detecting homotopical standardness is finite, Eliashberg obtained exotic but homotopically standard contact structures on S^{2n-1} by taking the connected sum of spheres as in example 3.3. By refining the homotopical considerations and making direct use of contact surgery, this argument was extended in* [12] *to prove the existence of exotic but homotopically standard contact structures also on S^7 and S^{8k+3}. The case S^{8k+7} with $k \geq 1$ still seems to be open (in spite of the result attributed to me in* [9]*).*

A more effective way of detecting homotopically equivalent but nondiffeomorphic contact structures is provided by the newly developed contact homology, see [9]. For instance, Ustilovsky [35] has used this to show that on S^{4n+1} there exist in fact infinitely many nondiffeomorphic, but homotopically standard contact structures (constructed from different descriptions of S^{4n+1} as Brieskorn manifold).

3.3. Spin bordism

One strategy to produce examples of contact manifolds is to find classes $\mathcal{C}$ of manifolds with the following properties:

(i) $\mathcal{C}$ contains a subclass $\mathcal{C}_0$ of manifolds which carry a contact structure by some explicit construction (e.g. Brieskorn manifolds).

(ii) Any manifold in $\mathcal{C}$ is obtained from one in $\mathcal{C}_0$ by surgery below the middle dimension, and all the necesary surgeries can be performed as contact surgeries.

For instance, in the results from [11, 12] illustrated in example 3.2, this strategy is applied to the class $\mathcal{C}$ of all highly connected manifolds carrying an almost contact structure, starting from the class $\mathcal{C}_0$ of homotopy spheres bounding parallelisable manifolds (cf. example 2.4). The main difficulty lies in distilling from Wall's classification of highly connected manifolds the information that is necessary to answer questions (1) and (2) from section 3.1.

As explained in [15, Lemma 3], these questions are easy to answer in dimension 5. Here we only have to deal with questions (1) and (2) in the case of 1- and 2-spheres. There are no obstructions to approximating an embedding $S^1 \hookrightarrow (M, \xi)$ by an isotropic one, and for an embedding $i\colon\ S^2 \hookrightarrow (M, \xi)$ one only has to require $i^* c_1(\xi) = 0$ to be able to find an isotropic approximation. In the case of 1-spheres all framings can be realised by contact surgeries (this is false for 3-manifolds). In the case of 2-spheres there is no problem with framings because they are controlled by the group $\pi_2(\mathrm{SO}_3) = 0$.

In [15, 16] this has been used to prove the existence of contact structures on classes $\mathcal{C}_\pi$ of 5-dimensional spin manifolds M with fundamental group π_1 isomorphic to some given group π. The key steps in the argument are as follows, for details see [16].

1. Bordism theorem: Let $f\colon\ M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$ and σ a spin structure on M. If the bordism class $[f\colon\ M \to B\pi, \sigma] \in \Omega_5^{\mathrm{spin}}(B\pi)$ contains a contact manifold (M_0, ξ_0) with $c_1(\xi_0) = 0$, then M admits a contact structure.

This is analogous to results about metrics of positive scalar curvature, cf. [29]. What the assumptions of the theorem really imply (in any dimension) is that M can be obtained from M_0 by surgery in codimension ≥ 3. This information is sufficient for constructing a positive scalar curvature metric on M, given one on M_0, but for contact structures this does not allow going beyond dimension 5.

2. Reduction theorem: To solve the problem for the class $\mathcal{C}_\pi$, it suffices to solve it for all the classes $\mathcal{C}_{\pi_p}$, where π_p denotes a p-Sylow subgroup of π. This is formally analogous to the result of Kwasik and Schultz [23] for positive scalar curvature metrics.

3. It remains to find a class $\mathcal{C}_{\pi_p,0}$ of contact manifolds (with $c_1 = 0$) generating $\Omega_5^{\mathrm{spin}}(B\pi_p)$. In [16], step 3 is completed for certain groups π having periodic cohomology (periodic groups, for short). The result of [15] for $\pi = \mathbb{Z}_2$ can also be interpreted in this framework, but by *ad hoc* arguments a somewhat stronger result was achieved there (both concerning the existence of contact structures and the topological structure of 5-manifolds with fundamental group of order 2). We summarise the main existence results of these two papers in the following example.

Example 3.5. *Let M be a closed oriented 5-manifold with universal cover $\widetilde{M}$. The manifold M admits a contact structure in either of the following cases:*

(a) $\pi = \pi_1(M)$ *is periodic,* $|\pi|$ *is odd,* $9 \not| \, |\pi|$*, and* M *is spin, i.e.* $w_2(M) = 0$.

(b) $\pi_1(M) = \mathbb{Z}_2$ *and* $\widetilde{M}$ *is spin.*

By the reduction theorem (and the structure of periodic groups of odd order), case (a) reduces to the study of cyclic groups. This causes the problems at the prime 3, since the spin bordism group $\Omega_5^{\mathrm{spin}}(B\pi_{p^k})$ has a different structure for $p = 3$ or $p \geq 5$. The class $\mathcal{C}_{Z_{p^k},0}$ may be taken to consist of lens spaces, i.e. quotients of S^5 under some fixed point free representation $\rho\colon \mathbb{Z}_{p^k} \to \mathrm{U}(3)$. The standard contact structure on S^5 descends to these quotients.

In (b), for the class $\mathcal{C}_{\mathbb{Z}_2,0}$ one needs ten manifolds, constructed explicitly as $\mathbb{Z}_2$-quotients of Brieskorn manifolds.

An interesting illustrative example is the following:

Example 3.6. *Let $D_{s,3}$ denote the metacyclic group*

$$\left\{x, y \,|\, x^s = y^3 = 1, yxy^{-1} = x^r\right\},$$

where s is an odd integer, r is a primitive third root of $1 \bmod s$*, and furthermore* $\gcd(3(r-1), s) = 1$*. This group acts freely and smoothly, but not freely and linearly, on S^5. The quotient $S^5/D_{s,3}$ admits a contact structure by the results above, see also* [12] *for a more direct argument. The contact structure given by this construction lifts to an exotic contact structure on S^5. Indeed, there is evidence that $D_{s,3}$ cannot act compatibly with the standard contact structure.*

4. Other Constructions and Examples

We conclude this survey with a summary of other methods for constructing contact manifolds. For a more detailed discussion of the first three of the following constructions see [13].

4.1. Branched covers

As mentioned above, Gonzalo had introduced a branched cover construction for 3-dimensional contact manifolds. Gromov [21, p. 343] observed that contact structures lift to branched covers in arbitrary dimension, provided the branching locus is a codimension two contact submanifold. For a proof see [13].

Example 4.1. *Let M^3 be a closed, orientable* 3*-manifold and Σ_g a closed, orientable surface of genus g. Then $M^3 \times \Sigma_g$ admits a contact structure. Indeed, M^3 is parallelisable, so by example 2.2 we have a contact form on $M^3 \times S^2$, thought of as the unit cotangent bundle of M^3. A contact structure on $M^3 \times \Sigma_g$ can now be obtained by writing this manifold as a branched cover of $M^3 \times S^2$, branched along* $2 + 2g$ *suitable sections of the unit cotangent bundle.*

Contact structures on more general *torus* bundles over 3-manifolds had been found earlier by Lutz [25].

4.2. Fibre connected sum

Again this is a construction that was first observed by Gromov [21, p. 343]. In the symplectic case this construction was worked out by Gompf [18] and used to remarkable effect. For the contact case a proof can again be found in [13].

Let $j_i\colon (N,\alpha_N) \hookrightarrow (M,\alpha)$, $i = 1,2$, be two disjoint contact embeddings ($j_i^*\alpha = \alpha_N$). Let $\psi\colon V_1 - j_1(N) \to V_2 - j_2(N)$ be an orientation preserving diffeomorphism of punctured tubular neighbourhoods (reversing the radial coordinate), induced by an orientation reversing isomorphism of the two normal bundles. Then the fibre connected sum of M along the $j_i(N)$, obtained from $M-(j_1(N)\cup j_2(N))$ by gluing with the help of ψ, admits a contact structure.

Example 4.2. *The manifold $M^3\times\Sigma_g$ can also be obtained by starting with $M^3\times S^2$ and performing g fibre connected sums along suitable sections of the unit cotangent bundle of M^3.*

4.3. Contact reduction

This construction is the contact analogue of symplectic reduction, which has been studied much more intensively. A concise description is given in [13], but the possibility of contact reduction has been observed independently by several authors. This construction has recently attracted a great deal of attention, see for instance [5, 24]. It has already led to some interesting results, but its potential for the construction of higher-dimensional contact manifolds remains to be explored.

4.4. Heat flow

Recently, Altschuler, in collaboration with L. F. Wu [3], has extended his construction (alluded to in section 2.4) from dimension 3 to higher dimensions. Independently, R. Gulliver, M. Schwarz and the author have obtained results in a similar direction. The set-up, however, becomes more restrictive, and so far the applications of this method remain scarce. One of the concrete examples of Altschuler and Wu is yet another construction of contact structures on trivial surface bundles over 3-manifolds.

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