Functionals of Brownian Motion in Path-Dependent Option Valuation

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Abstract. Path-dependent options have become increasingly popular over the last few years, in particular in FX markets, because of the greater precision with which they allow investors to choose or avoid exposure to well-defined sources of risk. The goal of the paper is to exhibit the power of stochastic time changes and Laplace transform techniques in the evaluation and hedging of path-dependent options in the Black-Scholes-Merton setting. We illustrate these properties in the specific case of Asian options and continuously (de)activating double-barrier options and show that in both cases, the pricing, and as importantly, the hedging results are more accurate than the ones obtained through Monte Carlo simulations.

1. Introduction

Over the last ten years, the so-called "exotic" or path-dependent options have become increasingly popular in equity markets, and even more so in commodity and FX markets. As of today, ninety five per cent of options exchanged on oil and oil spreads are Asian. On the other hand, barrier options allow portfolio managers to hedge at a lower cost against extreme moves of stock or currency prices.

In order to overcome the technical difficulties associated with the valuation and hedging of path-dependent options even in the classical geometric Brownian motion setting of the Black-Scholes-Merton model, practitioners, taking advantage of the power of new computers and workstations, make with good reasons a great use of Monte Carlo simulations to price path-dependent options. However, our claim is that the results are not always extremely accurate: the most obvious example is the case of "continuously exploding" barrier options, heavily traded in the FX markets, and where the option is activated (or desactivated) at any point in the day where the underlying exchange rate hits a barrier. We recall that a barrier option provides the standard Black-Scholes payoff $\max(0, S_T - k)$ only if (or unless) the underlying asset S has reached a prespecified barrier L, smaller or greater than the strike price k, during the lifetime $[0, T]$ of the option. In the

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equity markets, the classical situation for barrier contracts is that S_t is compared with L only at the end of each day (daily fixings). In contrast, in the FX markets, the comparison is made quasi-continuously and (de)activation may occur at any point in time. Obviously, the valuation of the option by Monte Carlo simulations, built piecewise by definition, may lead to fatal inaccuracies, in particular when the value of the underlying instrument is near the barrier close to maturity (entailing at the same time hedging difficulties well-known by option traders).

Along the same lines, when computing the Value at Risk (VaR) of a complex position or of a portfolio (for a given horizon T and a confidence level p, the value at risk is the loss in market value over the time horizon T that is exceeded with probability $(1-p)$), Monte Carlo simulations allow one to represent different scenarios on the state variables. But if the price of every exotic security in each scenario is in turn computed through Monte Carlo simulations, one has to face "Monte Carlo of Monte Carlo" and it becomes impossible, even with powerful computers, to calculate the VaR of the portfolio overnight. In an analytical methodology, since we obtain quasi-explicit solutions, the new values of the options can immediately be computed by incorporating in the pricing formulas the parameters corresponding to the different scenarios; hence, the problems mentioned above in estimating VaR are dramatically reduced.

The remainder of the article is organized as follows. Section 2 recalls the definition of stochastic time changes and shows why they are very useful to price (and hedge), via Laplace transforms, path-dependent options. Section 3 examines the specific case of Asian options and offers comparisons with Monte Carlo simulation prices. Section 4 addresses the case of barrier and double-barrier options and illuminates the hedging difficulties near maturity when the underlying asset price is close to a barrier. Section 5 contains some concluding remarks.

2. Stochastic Time Changes and Laplace Transforms

Representing the randomness of the economy by the probability space (Ω, F, F_t, P) where F_t represents the filtration of information available at time t and P the objective probability measure, we assume, as in the classical Black-Scholes-Merton setting, that the dynamics of the underlying asset price process $(S(t))_{t\geq 0}$ are driven by the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, d\hat{W}_t \tag{1}$$

where μ is a real number, σ is a strictly positive number and $(\hat{W}_t)_{t\geq 0}$ is a P-Brownian motion. Introducing the assumption of no arbitrage, we know from the seminal papers by Harrison-Kreps [12] and Harrison-Pliska [13], that there exists a so-called risk adjusted probability measure Q under which the dynamics of $S(t)$ become

$$\frac{dS_t}{S_t} = (r-y)\, dt + \sigma\, dW_t \tag{1'}$$

where $(W_t)_{t\geq 0}$ is a Q-Brownian motion and y denotes the continuous dividend rate of the underlying stock, supposed to be constant over the lifetime $[0, T]$ of this option. Equation (1) expresses the key mathematical assumption in the Black-Scholes model, namely that the underlying asset price is a geometric Brownian motion under the true probability measure. Since there is in this representation *one* source of risk, namely the Brownian motion $(\hat{W}_t)_{t\geq 0}$, it follows from central results of finance such as the Capital Asset Pricing Model or the Arbitrage Pricing Theory (S. Ross, 1976) that the expected return on a risky security should out perform the risk-free rate r over the period by *one* risk premium, i.e.,

$$E_p\left(\frac{dS_t}{S_t}\right) = dt\ (r + \textit{risk premium})$$

where r is supposed to be constant in the Black-Scholes model. The risk premium can be written as the positive constant σ times a quantity λ called the market price of equity risk. It is then possible to rewrite equation (1) as

$$E_p\left(\frac{dS_t}{S_t}\right) = r\,dt + \sigma(\lambda\,dt + d\hat{W}_t)$$

and Girsanov's theorem allows to obtain equation (1') (it also provides the expression of the Radon-Nykodim derivative $\frac{dQ}{dP}$ in terms of λ —constant or not— and $(\hat{W}_t)$).

From now on, we will be working under the probability measure Q in order to be allowed to write the price of an option as the expectation of the discounted terminal pay-off.

As explained for instance by Kemna and Vorst [15] who studied the valuation of average-rate options when these instruments started becoming very popular in the financial markets, the pricing difficulties are fundamentally conveyed by the fact that the representation of $(S_t)_{t\geq 0}$ by a geometric Brownian motion as described in equation (1) (and which is crucial for the simple proofs-through a partial differential equation or a probabilistic approach-of the Black-Scholes formula) is not transmitted to the average of S; hence, the idea of searching for a class of stochastic processes stable under additivity and related to the geometric Brownian motion. The so-called squared-Bessel processes, denoted hereafter BESQ(t), have the remarkable property of being Markov processes (which is the assumption common to nearly all models of option pricing and yield curve deformations) and of being stable by additivity. Moreover, a remarkable theorem due to Williams (1974), establishes that

$$S(t) = \text{BESQ}[X(t)] \tag{2}$$

where the processes X and BESQ are completely defined in terms of the parameters of the geometric Brownian motion $(S(t))_{t\geq 0}$. Equation (2) simply states that the value of S at time t is equal to the value of the squared-Bessel process BESQ at time $X(t)$. X defines a *stochastic time change* and formula (2) expresses that a geometric Brownian motion is a time-changed squared-Bessel process. In full generality, the only condition a process X has to satisfy in order to define a time change is to be (almost surely) increasing since time cannot go backwards. Other properties such as independent or identical increments may or may not be satisfied; when both are satisfied, the time change is called subordination (see [5]). Stochastic time changes are very useful for the pricing of exotic options, as this chapter will try to show. They have also become extremely popular when studying asset price dynamics: $X(t)$ may represent random sampling times in a financial time series as a function of calendar time t. The time change $X(t)$ may also account for differences in market activity at different hours in the day or because of new information release. [7] and [2] show that normality of asset returns which does not hold in calendar time can be recovered through a stochastic time change defined by the number of transactions.

Coming back to exotic option pricing, and assuming that the underlying asset return is a Brownian motion with drift as stated in equation (1), a very useful time change is obtained by choosing for $X(t)$ an exponential variable independent of the Brownian motion $(W(t))$. Indeed, when pricing barrier, double-barrier or corridor options, the quantities whose expectations (under the right probability measure) provide the option prices involve functionals of Brownian motion such as the maximum M_T or the minimum I_T over the period $[0, T]$. The trivariate joint distribution of (M_T, I_T, S_T) has been known for some time (see [3]); however, its expression is complex. Pitman-Yor [19] show that this quantity becomes much simpler when the fixed time T (maturity of the option in our setting) is replaced by an exponential random variable τ independent of the Brownian motion contained in the dynamics of the process $(S_t)_{t>0}$. Remembering the expression of the density of an exponential variable, it is easy to see that in order to exploit the property above mentioned, we are naturally led to compute the Laplace transform of the option price with respect to its maturity since $\int_0^\infty C(T)e^{-\lambda T}\, dT$ can be interpreted (up to the factor λ) as the expectation of $C(\tau)$ where τ is an exponential variable. Lastly, let us observe that the integral $\int_0^\infty S(s)\, ds$ is distributed as an inverse gamma variable. This interesting result was first proved by Dufresne (1990) in the analysis of perpetuities; another proof was given by Yor (1992). It must be noted however that the integral only converges if $r - q < \frac{\sigma^2}{2}$ (hence may not exist for non-dividend paying assets since $\sigma = 0.02$ represents a fairly high volatility). Moreover, options traded in the financial markets have a finite maturity and the above described property cannot solve the Asian option valuation problem. The price $C(T)$ itself would have a much simpler expression for T infinite, which is obviously not the case for options traded on the markets.

3. The Case of Asian Options

As has been mentioned earlier and is substantiated by the continuously growing literature on the subject, Asian options have a number of attractive properties as financial instruments: for thinly traded assets and commodities (e.g., gold) or newly established exchanges, the averaging feature allows one to prevent possible manipulations on maturity day by investors or institutions holding large positions in the underlying asset. They are very popular among corporate treasurers who can hedge a series of cashflows denominated in a foreign currency by using an average-rate option as opposed to a portfolio of standard options; the hedge is obviously not identical but may be viewed as sufficient. Many domestic rates used in Europe and in the US as reference rates in floating-rate notes or interest rate swaps are defined as averages of spot rates; hence, caps and floors written on these rates are, by definition, Asian. To give an example very relevant in corporate finance, we can mention the so-called *contingent value rights*: a firm A wants to acquire a firm B. A is not willing to pay too high a price for the shares of company B but knows that this may lead to a failure of the takeover. Hence firm A will offer the shareholders of company B a share of the new firm AB accompanied by a contingent-value right on firm AB, maturing at time T (say two years later). This contingent-value right is nothing but an Asian put option. The put provides the classical protection of portfolio insurance; the Asian feature protects firm A for an exceptionally low market price of the share AB on day T, as well as the shareholders B in the case of a very high market price that day. These contingent-value rights were used when Dow Chemical acquired Marion Laboratory, when the French firm Rhône Poulenc acquired the American firm Rorer and more recently, when the insurance company Axa merged with Union des Assurances de Paris to form the second largest insurance company in the world (in the last case, the corresponding contingent value rights are still trading today). To give a last example of the usefulness of Asian options, we can mention that options written on oil or on oil spreads are mostly Asian since oil indices are generally defined as arithmetic averages. Many options written on gas have the same feature and the deregulation of the gas industry worldwide has entailed a significant growth of the gas derivatives market. Let us now turn to the valuation of these instruments.

3.1. The mathematical setting

We assume the asset price driven under the risk-adjusted probability measure Q by the dynamics described in (1')

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t)\,.$$

We also assume that the number of values whose average is computed is large enough to allow the representation of the average $A(T)$ over $[0, T]$ by the integral

$$A(T) = \frac{1}{T}\int_0^T S(u)\,du\,.$$

The value of an Asian call option at time t is expressed, by arbitrage arguments, as

$$C(t) = E_Q[e^{-r(T-t)} \max(A(T) - k, 0)/F_t] \tag{3}$$

where k is the strike price of the option and the discount factor may pulled out of the expectation since we assumed constant interest rates. We know that the option has a unique price: there is only one source of randomness represented by the Brownian motion and a money-market instrument traded together with the risky security, which implies market completeness.

As mentioned earlier, the (important) mathematical difficulty in formula (3) stems from the fact that, denoting $A(t) = \frac{1}{T}\int_0^T S(u)\,du$, the process $(A(t))_{t\geq 0}$ is *not* a geometric Brownian motion. Many practitioners (see for instance Levy 1992) make this simplifying assumption and can then recover a Black Scholes type pricing formula through the mere computation of the first two moments of $A(T)$. But to our knowledge, no upper bound of the error due to this approximation was ever provided.

Let us first observe that, when the option is traded at a date t posterior to date 0, the values of the underlying asset between 0 and t are fully known; the only randomness resides in the values to be taken by S between t and T. Hence, if the values observed between 0 and t are high enough, it may already be known at time t that the Asian call option will finish in the money and that we can make the simplification

$$\max(A(T) - k, 0) = A(T) - k$$

since

$$A(T) > \frac{1}{T}\int_0^t S(u)\,du > k$$

writing

$$A(T) = \frac{1}{T}\int_0^t S(s)\,ds + \frac{1}{T}\int_t^T S(s)\,ds$$

and observing that the first term is fully known at date t, we obtain

$$E_Q[A(T) - k/F_t] = \frac{1}{T}\int_0^t S(s)\,ds - k + E_Q\left[\frac{1}{T}\int_0^T S(s)\,ds/F_t\right].$$

The linearity of the operators integral and expectation and the martingale property satisfied by the discounted price of S_t under Q allow one to compute explicitly the last term, namely

$$\begin{aligned} E_Q\left[\frac{1}{T}\int_t^T S(s)\,ds/F_t\right] &= \frac{1}{T}\int_t^T E_Q[S(s)/F_t]\,ds \\ &= \frac{1}{T}\int_t^T S(t)e^{r(s-T)}\,ds \\ &= S(t)\frac{e^{r(t-T)-1}}{r(T-t)}. \end{aligned}$$

We then obtain the Asian call price (when it is known at date T that the call is in the money) as

$$C(t) = S(t)\frac{1 - e^{-r(T-t)}}{r(T-t)} + e^{-r(T-t)}\left[\frac{1}{T}\int_0^t S(s)\,ds - k\right]. \tag{4}$$

It is worth noting that the same type of considerations (Fubini theorem) allows one to compute fairly easily the exact moments of all orders of the arithmetic average, in contrast with the unnecessary approximations which are often offered in the literature.

Formula (4) has some striking resemblances with the Black-Scholes-Merton formula, the sign plus in the second term translating the moneyness of the Asian call in this situation.

Obviously, in most cases, this formula does not hold since at date t the quantity $\frac{1}{T}\int_0^t S(s)\,ds - k$ is likely to be non positive. To address this difficult situation in an exact approach, a solution consists (see [9]) in

a) writing $S(t)$ as a time-changed squared Bessel process
b) choosing not to compute the option price itself but rather its Laplace transform with respect to maturity, namely the function $\varphi(\lambda) = \int_0^{+\infty} C(T)e^{-\lambda T}\,dT$.

Geman-Yor [9] give the details of the different mathematical steps which lead to the following expression for the call price

$$C(t) = \frac{4S(t)}{\sigma^2 T} e^{-r(T-t)} C^{(\nu)}(h,q) \tag{5}$$

where

$$\nu = \frac{2r}{\sigma^2} - 1; \quad h = \frac{\sigma^2}{4}(T-t); \quad q = \frac{\sigma^2}{4S(t)}\left\{kT - \int_0^t S(u)\,du\right\}$$

and the Laplace transform of the quantity $C^{(\nu)}$ with respect to h is given by

$$\int_0^\infty E^{-\lambda h} C^{(\nu)}(h,q)\,dh = \frac{\int_0^{1/2q} dx e^{-x} x^{\frac{\mu-\nu}{2}} (1-2qx)^{\frac{\mu-\nu}{2}+1}}{\lambda(\lambda - 2 - 2\nu)\Gamma\left(\frac{\mu-\nu}{2} - 1\right)} \tag{6}$$

where Γ denotes the gamma function and $\mu = \sqrt{2\lambda + \nu^2}$.

We can observe that when the underlying asset is a stock paying a continuous dividend y (y may also be the convenience yield of a commodity or the foreign interest rate in the case of a currency), the above results prevail exactly by replacing r by $r - y$ and ν by

$$\frac{2(r-y)}{\sigma^2} - 1\,.$$

The inversion of the Laplace transform in (6) provides not only the call price but also its delta through in the same methodology. Indeed, the differentiation of formula (5) with respect to S gives

$$\Delta = \frac{\partial C_t}{\partial S_t} = \frac{C_t}{S_t} - \frac{e^{-r(T-t)}}{T} \frac{1}{S(t)} \left\{ kT - \int_0^t S(u)\, du \right\} \frac{\partial C^\nu(h,q)}{\partial q} \tag{7}$$

and we face an analogous problem of inversion of the Laplace transform.

Geman-Eydeland [8] on one hand, Fu-Madan-Wang [6] on the other hand apply different algorithms to invert the Geman-Yor formula but come up with results remarkably close (Fu-Madan-Wang use an algorithm developed by Abate and Whitt [1]; Geman-Eydeland use a method based on contour integration in the complex plane). The latter authors also provide comparisons with Monte Carlo simulations since this mathematically simple approach is very popular among practitioners and does not raise particular problems in the case of the smooth payoff of the average-rate option (in contrast to barrier options).

The following table gives some numerical results (the stock is assumed to pay no dividend over the period and the date of analysis t to be 0).

Interest rate r	Volatility σ	Maturity T	Strike price k	Initial value $S(0)$	$G-Y$	Monte Carlo
0.05	0.5	1	2	1.9	0.195	0.191
0.05	0.5	1	2	2	0.248	0.248
0.05	0.5	1	2	2.1	0.308	0.306
0.02	0.1	1	2	2	0.058	0.056
0.0125	0.25	2	2	2	0.1772	0.1771
0.05	0.5	2	2	2	0.351	0.347

The Monte Carlo values are obtained through a sample of 50 evaluations, each evaluation being performed on 500 Monte Carlo paths.

Turning to the computation of the delta of the option, for instance for $S(0) = 2$, we know that many practitioners use an elementary finite difference method with Monte Carlo values, which means in our example a delta equal to $\frac{0.306-0.191}{0.2} = 0.575$; by doing so, a much higher error appears in the delta than in the option price itself.

On the contrary, in the Laplace transform approach and thanks to the linearity of integration and derivation, the error does not deteriorate and the delta obtained in the above example is 0.56, a number significantly different than 0.575.

To end this section, let us observe that we have addressed the so-called fixed strike Asian option. A less popular type of Asian option has a floating strike, meaning that the pay-off at maturity is expressed as $\max(A_T - S_T, 0)$. Ingersoll in his book [14] conjectured that this case would be much simpler than the fixed-strike case. Indeed, taking the stock price as the numéraire (see Geman-El Karoui-Rochet 1995), one obtains a fairly simple partial differential equation satisfied by the Asian call option. The powerful change of numéraire technique, though still feasible, does not provide as simple a result for the fixed-strike Asian call options.

4. Barrier and Double-Barrier Options

Barrier options to which a vast body of literature is currently dedicated, represent the most common type of exotic options: they were traded in over-the counter markets in the United States many years before plain vanilla options were listed (see Snyder, 1969). The pricing of "single barrier" options is not very difficult in the standard Black-Scholes-Merton framework and closed-form solutions have been available for some time. The price of a down-and-out option was already in Merton [18] seminal paper and in 1979, Goldman-Sosin-Gatto offered explicit solutions for all types of single barrier options.

We focus in this paragraph on double-barrier options which have become very popular recently. Not only, as mentioned earlier, do they provide a less expensive hedge which may be good enough in a number of situations. But they also allow investors with a specific view on the range of a stock price without any specific anticipation on the terminal value to take a position accordingly. We will be addressing the so-called "continuously desactivating" double-barrier options (meaning that the option vanishes at any time where the underlying asset price hits the upper barrier U or the lower barrier L), as opposed to comparing the daily fixings of a stock with the numbers U and L. This is the situation which prevails in the FX markets, where double-barrier options represent a significant fraction of options written every day. The methodology described below allows to price, as a simpler case, the so-called corridor options which pay one at maturity if the underlying asset price has remained in the corridor during the lifetime of the option.

4.1. The mathematical setting

Assuming that the dynamics of the underlying asset are driven under Q by the same equation as before

$$\frac{dS_t}{S_t} = (r - y)\,dt + \sigma\,dW_t$$

and denoting by L the lower barrier and by U the upper barrier, we consider an option which vanishes as soon as either the upper or lower barrier is hit. The price of the call at time t is equal to

$$C(t) = E_Q\left[e^{-r(T-t)}\max(0, S(T) - k)1_{(\Sigma > T)}/F_t\right] \tag{8}$$

where $\Sigma = \{\inf t/S(t) \geq U \text{ or } S(t) \leq L\}$ is the first exit time of the process $(S(t))$ out of the interval $[L, U]$ and interest rates are supposed constant (as well as the possible dividend payment y). It is slightly easier to compute the quantity

$$D(t) = E_Q\left[e^{-r(T-t)}\max(0, S_t - k)1_{(\Sigma \leq T)}/F_t\right]\,.$$

Obviously, the knowledge of $D(t)$ would give $C(t)$ since the two quantities add up to the Black-Scholes price.

Again, the expression whose expectation is computed (which is a functional of the brownian motion W_t through S_T and Σ) would be simpler if the fixed maturity date was replaced by an exponential time τ independent of (W_t). This leads us to compute the Laplace transform $\Psi(\lambda)$ of $D(t)$ with respect to maturity T. Geman-Yor [10] show that

$$\Psi(\lambda) = \frac{1}{\sigma^2}\zeta(\lambda/\sigma^2)$$

where

$$\zeta(\lambda) = \frac{sh(\mu b)}{sh[\mu(a+b)]}g_1(e^{-a}) + \frac{sh(\mu a)}{sh[\mu(a+b)]}g_2(e^b)$$

with

$$\alpha = \frac{1}{\sigma^2}\left(r - y - \frac{\sigma^2}{2}\right);\ L/S(t) = e^{-a};\ U/S(t) = e^b;\ h = k/S(t);\ \mu = \sqrt{2\lambda + \alpha^2}$$

$$g_1(e^{-a}) = \frac{h^{\alpha+1-\mu}e^{-\mu a}}{\mu(\mu-\alpha)(\mu-\alpha-1)}$$

$$g_2(e^b) = 2\left[\frac{e^{b(\alpha+1)}}{\mu^2 - (\alpha+1)^2}\right] + \frac{e^{-\mu b}h^{\alpha+1+\mu}}{\mu(\mu+\alpha)(\mu+\alpha+1)}.$$

Again, the numerical results obtained through the inversion of the Laplace transformed are compared with Monte Carlo simulations. A first set of tests is performed with $t = 0$, $T = 1$ year

Parameters	$S(0) = 2$ $\sigma = 0.2$ $r = 0.02$ $k = 2$ $L = 1.5$ $u = 2.5$	$S(0) = 2$ $\sigma = 0.5$ $r = 0.05$ $k = 2$ $L = 1.5$ $U = 3$	$S(0) = 2$ $\sigma = 0.5$ $r = 0.05$ $k = 1.75$ $L = 1$ $U = 3$
$G - Y$ price	0.0411	0.0178	0.07615
Monte Carlo price (st. dev = 0.003)	0.0425	0.0191	0.0722

where the standard deviation is computed on a sample of 200 evaluations, each evaluation being performed on 5000 Monte Carlo paths with a step size of 1/365 year.

In order to show the nonrobustness of Monte Carlo methods when we approach maturity while the price of the underlying asset is close to one of the barriers, we take the same parameters as in the first column of the above table except that $S(0)$ is supposed to be 2.4 and the time to maturity one month. The $G - Y$ method gives a call price equal to 0.17321 and there is no change in the accuracy nor in the computing time since the Laplace transform method is insensitive to the position of the underlying asset price with respect to the barrier. On the contrary, keeping the same step size of 1/365 year gives a Monte Carlo standard deviation equal to 0.073 (which is clearly too high for practical purposes) and a Monte Carlo value for the call of 0.1930. By making the step four times smaller, the standard deviation is reduced to 0.008 and the price becomes 0.1739, which happens to be much closer to the $G-Y$ price and to be lower than 0.1930 (since in the first simulations, the option may have been overpriced through some trajectories "missing" the barrier while, in reality, the underlying asset had hit it, entailing the desactivation of the option).

5. Conclusion

The methodology involving stochastic time changes and Laplace transforms has been proved to be very efficient in the valuation and hedging of the most notoriously difficult European path-dependent options, namely the Asian and double-barrier options. The results have been obtained in the classical Black–Scholes-Merton of a constant volatility. We can observe, however, that the introduction of a stochastic volatility in the underlying asset dynamics necessitates the use of a tree or of some numerical procedure (Monte Carlo or other). In all cases, the quasi-exact values obtained in the constant volatility case could be used as control variates to improve the accuracy of the numerical procedure.

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