Local Langlands Correspondences and Vanishing Cycles on Shimura Varieties

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Abstract. We report on the results and techniques of the author's recent joint work with Richard Taylor, which analyzes in detail the bad reduction of certain Shimura varieties in order to prove the compatibility of local and global Langlands correspondences, obtaining as a consequence the local Langlands conjecture for $GL(n)$ of a p-adic field. These Shimura varieties have natural models over p-adic integer rings, as moduli spaces for abelian varieties with additional structure. The starting point of the work with Taylor is the stratification of the special fiber of an integral model in minimal level, according to the isogeny type of the universal family of p-divisible groups attached to these abelian varieties. Similar stratifications can conjecturally be constructed for any Shimura variety, and indeed are known to exist for most PEL types. We discuss a series of conjectures regarding the behavior of vanishing cycles along these strata, with the aim of extending Kottwitz' conjectures on the cohomology of Shimura varieties to the case of bad reduction.

1. Introduction

Let F be a local field and n a positive integer. Let $\mathcal{A}(n,F)$ denote the set of equivalence classes of irreducible admissible representations of $GL(n,F)$, $\mathcal{A}_0(n,F)$ the subset of supercuspidal representations. Let $\mathcal{G}(n,F)$ denote the set of equivalence classes of n-dimensional complex representations of the Weil-Deligne group $WD(F)$ on which Frobenius acts semisimply, $\mathcal{G}_0(n,F)$ the subset of irreducible representations. A *local Langlands correspondence* (for general linear groups), a non-abelian generalization of local class field theory, is a family of bijections $\pi \mapsto \sigma(\pi)$ from $\mathcal{A}(n,F)$ to $\mathcal{G}(n,F)$, for all n, identifying $\mathcal{A}_0(n,F)$ with $\mathcal{G}_0(n,F)$, and satisfying a list of properties recalled below.

The existence of local Langlands correspondences, previously known in various special cases, has now been established in full generality. For F of positive characteristic, this was proved by Laumon, Rapoport and Stuhler [35], using a variant of Drinfeld's modular varieties; in particular, the techniques of [35] are

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global. For p-adic fields, the first proof was given in joint work with Richard Taylor [20], again using global methods, this time involving the geometry of certain Shimura varieties, together with cases of non-Galois automorphic induction proved in [19] (also using Shimura varieties). Shortly after distribution of the first version of [20], Henniart found a much simpler proof [25], obtaining the local Langlands correspondence directly from the results of [19]. All of these proofs rely crucially on a weaker version of the correspondence, the *numerical local Langlands correspondence*, proved by Henniart in [23].

The present article is a report on the results and techniques of [20]. Many of these techniques appear to apply to a more general class of Shimura varieties than those considered in [20]. Shimura varieties[1] are conjectured to be moduli spaces for certain kinds of motives. This was proved by Shimura for many Shimura varieties attached to classical groups, the motives in this case arising from abelian varieties with additional structure (PEL types). In this way many Shimura varieties, together with their Hecke correspondences, acquire natural models over p-adic integer rings. The varieties considered in [20] are of PEL type. The starting point of [20] is the stratification of the special fiber of the integral model, according to the isogeny type of the universal family of *p-divisible groups* (with additional structure) attached to the moduli problem. Such stratifications can be constructed for any Shimura variety realized as a moduli space for motives.

Let $Sh(G,X)$ be a Shimura variety, with G a connected reductive group over $\mathbb{Q}$, and X a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to G_{\mathbb{R}}$, satisfying a familiar list of axioms. Then $Sh(G,X)$ has a canonical model over the reflex field $E = E(G,X)$. Fix a rational prime p and a prime v of E above p with residue field $k(v)$, and assume $Sh(G,X)$ has a model over the v-adic integer ring $\mathcal{O}_v$. If π is a cohomological automorphic representation of (the adele group of) G, with finite part π_f, let $r_\ell(\pi)$ denote the virtual ℓ-adic representation of $\mathrm{Gal}(\bar{E}/E)$ on the π_f-isotypic component of $\sum_i (-1)^i H^i_c(Sh(G,X)_{\bar{E}}, \mathbb{Q}_\ell)$; more generally, $\mathbb{Q}_\ell$ can be replaced by an ℓ-adic local system. Following a technique introduced by Ihara and Langlands, and developed systematically by Kottwitz [29, 30], we study the local behavior at v of $r_\ell(\pi)$ by comparing the Grothendieck-Lefschetz trace formula on the cohomology of the special fiber at v of $Sh(G,X)$[2] with the Arthur-Selberg trace formula for G. Unlike Kottwitz, however, we do not assume $Sh(G,X)$ to have good reduction at v. Thus $r_\ell(\pi)$ is ramified in general, and the Lefschetz formula is applied to the cohomology with coefficients in the nearby cycle complex. In [20] it is proved that the sheaves of nearby cycles are locally constant on each stratum in the étale topology. This is unlikely to be true in general, but the stalks can be predicted in terms of the local uniformization theory of Rapoport and Zink [40]. In this way, we arrive at a conjectural expression for the contribution of each stratum to $r_\ell(\pi)$. Up to semisimplification, this contribution depends only on the local component of π at p.

[1] For experts: we always assume the weight morphism is rational over $\mathbb{Q}$.

[2] More precisely, the Shimura variety ${}_K Sh(G,X)$ at an appropriate finite level K.

Work on this report began in the summer of 1999, during a visit to the Sonderforschungsbereich in Münster. I thank Christopher Deninger for making that visit so enjoyable, and Matthias Strauch for encouraging me to lecture on [20].

2. The Local Langlands Correspondence

In what follows, p is a prime number. The local Langlands conjecture for $GL(n)$ is best stated for all positive integers n and all p-adic fields F simultaneously. Notation is as in the introduction.

Local Langlands Conjecture. *Let F be a finite extension of $\mathbb{Q}_p$.*

(i) *For every integer $n \geq 1$, there exists a bijection $\pi \to \sigma(\pi)$ between $\mathcal{A}(n,F)$ and $\mathcal{G}(n,F)$ that identifies $\mathcal{A}_0(n,F)$ with $\mathcal{G}_0(n,F)$.*

(ii) *Let χ be a character of $F^\times$, which we identify with a character of $WD(F)$ via the reciprocity isomorphism of local class field theory. Then $\sigma(\pi \otimes \chi \circ \det) = \sigma(\pi) \otimes \chi$. In particular, when $n = 1$, the bijection is given by local class field theory.*

(iii) *If $\pi \in \mathcal{A}(n,F)$ with central character $\xi_\pi \in \mathcal{A}(1,F)$, then $\xi_\pi = \det(\sigma(\pi))$.*

(iv) *$\sigma(\check{\pi}) = \check{\sigma}(\pi)$, where $\check{}$ denotes contragredient.*

(v) *Let $\alpha\colon F \to F_1$ be an isomorphism of local fields. Then α induces bijections $\mathcal{A}(n,F) \to \mathcal{A}(n,F_1)$ and $\mathcal{G}(n,F) \to \mathcal{G}(n,F_1)$ for all n, and we have $\sigma(\alpha(\pi)) = \alpha(\sigma(\pi))$. In particular, if F is a Galois extension of a subfield F_0, then the bijection σ respects the $\mathrm{Gal}(F/F_0)$-actions on both sides.*

(vi) *Let F'/F denote a cyclic extension of prime degree d. Let $BC\colon \mathcal{A}(n,F) \to \mathcal{A}(n,F')$ and $AI\colon \mathcal{A}(n,F') \to \mathcal{A}(nd,F')$ denote the local base change and automorphic induction maps* [1, 26]. *Let $\pi \in \mathcal{A}(n,F)$, $\pi' \in \mathcal{A}(n,F')$. Then $\sigma(BC(\pi)) = \sigma(\pi)|_{WD(F')}$, $\sigma(AI(\pi')) = \mathrm{Ind}_{F'/F}\,\sigma(\pi')$, where $\mathrm{Ind}_{F'/F}$ denotes induction from $WD(F')$ to $WD(F)$.*

Let n and m be positive integers, $\pi \in \mathcal{A}(n,F)$, $\pi' \in \mathcal{A}(m,F)$. Then

(vii) $L(s, \pi \otimes \pi') = L(s, \sigma(\pi) \otimes \sigma(\pi'))$.

(viii) *For any additive character ψ of F, $\epsilon(s, \pi \otimes \pi', \psi) = \epsilon(s, \sigma(\pi) \otimes \sigma(\pi'), \psi)$.*

Here the terms on the left of (vii) and (viii) are as in [27, 43] *and are compatible with the global functional equation for Rankin-Selberg L-functions. The right-hand terms are given by Artin and Weil (for (vii)) and Langlands and Deligne (for (viii)) and are compatible with the functional equation of L-functions of representations of the global Weil group. In particular both sides have Artin conductors and (viii) implies that $a(\sigma(\pi)) = a(\pi)$.*

I refer to Carayol's Bourbaki exposés [11, 12], and the introduction to [20], for more details on the history of this conjecture. A theorem of Henniart [24] implies that σ is uniquely determined by these properties. A version of the local Langlands conjecture for general connected p-adic reductive groups is recalled in §5 below, in connection with conjecture 5.3.

The proof of this conjecture in [20] is based on the following considerations. The logical first step is the

Theorem 2.1. *There is a family of maps* $\sigma_0^{\mathrm{van}}\colon \mathcal{A}_0(n,F) \to \mathcal{G}(n,F)$*, for all positive integers* n *and all* p*-adic fields* F*, satisfying (iii–vi).*

Theorem 2.1 summarizes the contents of Corollary 11.4 and Lemmas 12.1–12.3, 12.5, and 12.6 of [20]. The map, whose existence was conjectured by Carayol [10], is realized on a geometric model arising from the deformation theory of p-divisible groups. Its construction is global, however (see (17), below). A theorem of Henniart [23, 7] implies that any such family of maps is in fact a family of *bijections* $\mathcal{A}_0(n,F) \to \mathcal{G}_0(n,F)$ that preserves conductors and satisfies (i–vii).

It therefore remains to prove (viii). The proof is global in nature. We work over a CM field E, a quadratic extension of a totally real subfield E^+ of degree d. Let Π be a cuspidal automorphic representation of $GL(n)_E$, unramified outside a finite set S. For any finite set S' containing S, let $L^{S'}(s,\Pi)$ denote the partial standard L-function of π with the Euler factors at S' removed. Let L be a number field, $\{\lambda\}$ the set of finite places of L. Let $\sigma = \{\sigma_\lambda\}$ be a compatible family of n-dimensional λ-adic representations of $\mathrm{Gal}(\overline{E}/E)$. We say σ is *weakly associated* to Π if

$$L^{S'}(s,\sigma) = L^{S'}(s,\Pi) \tag{1}$$

as Euler products for some finite set S' containing S. Here $L^{S'}(s,\sigma)$ denotes the partial L-function of $\{\sigma\}$. Both sides of (1) are normalized so that the functional equation is symmetric around the line $\mathrm{Re}(s) = \frac{1}{2}$. Let $Reg(n,E)$ denote the set of Π for which Π_∞ is of cohomological type. We assume the coefficient system to be trivial, for simplicity, but this is not necessary. The following theorem is mainly due to Clozel, with improvements due to Taylor and Blasius, and depends crucially on Kottwitz' study of points on Shimura varieties over finite fields [13, 30, 31, 5]; cf. [20, Theorem 11.11]:

Theorem 2.2. **(**[13, Théorème 5.7]**,** [14]**)** *Let* c *denote complex conjugation on* E*. Let* $\Pi \in Reg(n,E)$*. Suppose (i) The local component* Π_v *at at least one finite place* v *is square-integrable (supercuspidal or generalized Steinberg); (ii)* Π *is dual to* Π^c*.*

Then there exists a compatible family $\sigma(\Pi) = \{\sigma_\lambda(\Pi)\}$ *of semi-simple* n*-dimensional* λ*-adic representations of* $\mathrm{Gal}(\overline{E}/E)$*, weakly associated to* Π*.*

To any $\pi \in \mathcal{A}(n,F)$ we associate its *supercuspidal support* $\mathrm{Supp}(\pi)$, consisting of a partition $n = n_1 + \cdots + n_r$ and, for each $i \in \{1,\ldots,r\}$, a $\pi_i \in \mathcal{A}_0(n_i,F)$, such that π is an irreducible constituent of the (normalized) induced representation $I_P^{GL(n,F)}(\pi_1 \otimes \cdots \otimes \pi_r)$, where $P \subset GL(n,F)$ is any parabolic subgroup with Levi factor $\prod_i (GL(n_i,F))$. Suppose we have bijections $\sigma_0\colon \mathcal{A}_0(m,F) \leftrightarrow \mathcal{G}_0(m,F)$ for all m and all F. We extend these bijections to all π in two steps. Let $\mathcal{G}_{ss}(n,F) \subset \mathcal{G}(n,F)$ denote the subset of representations of $WD(F)$ factoring through the Weil group $W(F)$. If $\pi \in \mathcal{A}(n,F)$, $\mathrm{Supp}(\pi) = \{(n_i,\pi_i)\}$, let

$$\sigma_{ss}(\pi) = \oplus_{i=1}^{r} \sigma(\pi_i) \in \mathcal{G}_{ss}(n,F)\,. \tag{2}$$

The set of $\pi \in \mathcal{A}(n,F)$ with support $\{(n_i,\pi_i)\}$ was classified by Zelevinski, who proved the existence of a natural extension of σ_{ss} to a set of bijections $\sigma\colon \mathcal{A}(m,F) \leftrightarrow \mathcal{G}(m,F)$ [45]. If σ_0 satisfies (iii), (iv), (vi), (vii), and (viii) for all m, then so does σ [24, 3.4]. The situation for (vi) is a bit more complicated, since BC and AI do not preserve supercuspidality in general, but allowing for this complication, it also suffices to verify (vi) for σ_0.

The main purpose of [20] is to remove the modifier "weakly" from theorem 2.2:

Theorem 2.3. ([20, Theorem 11.11]) *Let $\Pi \in Reg(n,E)$. Assume Π satisfies (i) and (ii) of theorem 2.2. Then for all primes v of E not dividing the characteristic of λ, the following relation holds:*

$$\sigma_\lambda(\Pi)_{v,ss} \xrightarrow{\sim} \sigma_{ss,\lambda}(\Pi_v)\,. \tag{3}$$

Here $\Pi_v \in \mathcal{A}(n,F)$ is the local component at v of Π, and $\sigma_\lambda(\Pi)_{v,ss}$ is the semi-simplification of the restriction of $\sigma_\lambda(\Pi)$ to $W(F)$.

Remarks 2.4. (i) *When Π_v is unramified, this comes down to the equality of local Euler factors asserted in theorem 2.2.*

(ii) *The article* [18] *uses rigid-analytic uniformization of slightly different Shimura varieties to obtain maps $\sigma_0^{\mathrm{rig}}\colon \mathcal{A}_0(n,F) \to \mathcal{G}(n,F)$, with the properties indicated in theorem 2.1. As above, Henniart's theorem implies that these maps define a family of bijections $\mathcal{A}_0(n,F) \to \mathcal{G}_0(n,F)$. Both σ_0^{rig} and σ_0^{van} satisfy theorem 2.3 when π_v is supercuspidal.* A posteriori, *it follows that the two correspondences coincide.*

(iii) *For $n = 2$, Carayol proved the stronger result* [9] *that $\sigma_\lambda(\Pi)_v \xrightarrow{\sim} \sigma_\lambda(\Pi_v)$ as representations of the full Weil-Deligne group; T. Saito has proved the analogous result without restriction on the residue characteristic of λ* [42][3]*. Removal of the subscript ss in theorem 2.3 seems to require proof of Deligne's conjecture on the purity of the monodromy weight filtration.*

The reduction of the local Langlands conjecture to theorem 2.3 is the subject of [19]. The point is to show that, as E varies over CM fields, the set of representations $Reg(n,E) \times Reg(m,E)$, for varying n and m, contains sufficiently many pairs (Π,Π') whose global L-functions are known *a priori* to satisfy two functional equations, one involving the automorphic local constants $\epsilon(s,\Pi\otimes\Pi',\psi)$, the other involving the Galois-theoretic local constants of Langlands and Deligne. A technique originating with Deligne then permits identification of the corresponding local constants at the prime v of interest.

[3] *Note added in proof.* Saito's work concerns the representation $D(\sigma_\lambda(\Pi)_v)$ of $WD(F)$ defined by Fontaine when v divides the characteristic of λ. Saito has recently announced joint work with K. Kato implying that, for general n, $D(\sigma_\lambda(\Pi)_v)$ and $\sigma_\lambda(\Pi_v)$ coincide after restriction to the wild ramification subgroup.

An argument involving Brauer's theorem shows that it suffices to construct $\Pi = \Pi(\chi) \in Reg(n, E)$ such that the associated $\sigma(\Pi)$ are induced from appropriate Hecke characters χ, of CM extensions E'/E, with fixed local behavior at primes dividing v, such that the Galois closure of E' over E is solvable. It follows that $\sigma(\Pi)$ is the compatible system associated to a complex representation of the global Weil group of E, hence that its L-function satisfies a functional equation of Artin-Weil type.

This version of non-Galois automorphic induction, carried out in §4 of [19], relies on Clozel's theorem 2.2, and thereby on Kottwitz' analysis of the good reduction of Shimura varieties. Henniart's simple proof of the local Langlands conjecture in [25] proceeds in the opposite direction. Starting with the $\pi(\chi)$ constructed in [19], he defines maps $\pi_n \colon \mathcal{G}_0(n, F) \to \oplus_{n\geq 1}\mathbb{Z} \cdot \mathcal{A}(n, F)$, where the target of π_n is a formal direct sum. The maps π_n are defined globally, but using properties of L-functions Henniart shows that they are well-defined, and that the image of π_n is contained in $\mathcal{A}_0(n, F)$. The results of [23, 24] then suffice to prove that π_n are independent of all choices and have properties (i)–(viii).

3. Shimura Varieties Attached to Twisted Unitary Groups

If G is a connected reductive group over a global or local field F, we denote by $\hat{G}$ its Langlands dual group, viewed as the points of a reductive group over an algebraically closed field $\mathbf{F}$. Denote by ${}^LG$ the L-group of G, a semi-direct product of $\hat{G}$ with either the Weil group $W(F)$ or the Galois group $\mathrm{Gal}(\bar{F}/F)$, depending on context.

Let (G, X) be a Shimura datum, as in the introduction. Let $\mu \colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the cocharacter attached to a point $h \in X$. The G-conjugacy class of μ is independent of h and its field of definition is the reflex field $E(G, X)$; we will write μ_X for any point in this conjugacy class. This is a cocharacter of some maximal torus of G, hence a character of a maximal torus $\hat{T} \subset \hat{G}$. We let r_μ be the irreducible representation of $\hat{G}$ with extreme weight μ. The weight μ is necessarily minuscule. Langlands has defined an extension of r_μ, also denoted r_μ, to a representation of the L-group ${}^LG$ of G over the base field $E(G, X)$. The Shimura variety $Sh(G, X)$ has a canonical model over the field $E(G, X)$.

The article [20] is concerned with a specific family of Shimura varieties. Let p be a rational prime, E an imaginary quadratic field in which p splits, F^+ a totally real field of degree d, and $F = F^+ \cdot E$; let $c \in \mathrm{Gal}(F/F^+)$ denote complex conjugation. Choose a prime u of E above p, and let $w = w_1, w_2, \ldots, w_r$ be the primes of F above u. Choose a distinguished embedding $\tau_0 \colon F \to \mathbb{C}$. Let σ_0 (resp. τ_E) denote the restriction of τ_0 to F^+ (resp. to E), and let Σ denote the set of complex embeddings of F restricting to τ_E on E.

Let B be a central division algebra of dimension n^2 above F, admitting an anti-automorphism # restricting to c on the center F. Assume B is split at w and at every prime that does not split over F^+, and that at every place B is either split or a division algebra.

Define a connected reductive $\mathbb{Q}$-algebraic group G by

$$G(R) = \{g \in (B^{\mathrm{op}} \otimes_{\mathbb{Q}} R)^\times \mid g \cdot g^\# = \nu(g) \in R^\times\} \tag{4}$$

for any $\mathbb{Q}$-algebra R. The kernel G_1 of the map $\nu\colon G \to \mathbb{G}_m$ is the restriction of scalars to $\mathbb{Q}$ of a group G^+ over F^+. Under a certain parity condition [13, §2]; [20, Lemma 1.1], which we assume, we can choose $\#$ so that G is quasi-split at all rational primes that do not split in $E/\mathbb{Q}$ and so that $G^+_{\sigma_0}$ is isomorphic to $U(1, n-1)$ but G^+_σ is a compact unitary group for all real places $\sigma \neq \sigma_0$.

Since p splits in E, we can identify

$$G(\mathbb{Q}_p) \xrightarrow{\sim} GL(n, F_w) \times \prod_{i>1} B^{\mathrm{op},\times}_{w_i} \times \mathbb{Q}_p^\times\,, \tag{5}$$

where the map $G(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is given by ν.

Choose an $\mathbb{R}$-algebra homomorphism $h_0\colon \mathbb{C} \to B^{\mathrm{op}} \otimes_{\mathbb{Q}} \mathbb{R}$ such that $h_0(z)^\# = h_0(\bar{z})$ for all $z \in \mathbb{C}$. The image is contained in G and we may assume it is centralized by a maximal compact subgroup of $G(\mathbb{R})$. Let (G, X) be the Shimura datum for which X is the $G(\mathbb{R})$-conjugacy class containing h_0. Then the reflex field $E(G, X)$ is isomorphic to F, identified with its image in $\mathbb{C}$ under τ_0.

3.1. The moduli problem

If A is an abelian scheme over a base scheme S, let $T_f(A)$ denote the direct product of the Tate modules $T_\ell(A)$ over all primes ℓ, $V_f(A) = \mathbb{Q} \otimes T_f(A)$. Let $K \subset G(\mathbf{A}_f)$ be a compact open subgroup. Consider the functor $\mathcal{A}_K(B, *)$ on schemes over F, which to S associates the set of equivalence classes, for the usual equivalence relation, of quadruples (A, λ, i, η), where A is an abelian scheme over S of dimension dn^2, $\lambda\colon A \to \hat{A}$ is a polarization, $i\colon B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ is an embedding, and $\eta\colon V \otimes_{\mathbb{Q}} \mathbf{A}_f \xrightarrow{\sim} V_f(A)$ an isomorphism of $B \otimes_{\mathbb{Q}} \mathbf{A}_f$-modules, modulo K [30, p. 390]. These data are assumed to satisfy the standard compatibilites. More importantly, i induces an action i_F of the center F of B on the $\mathcal{O}_S$-module $\mathrm{Lie}(A)$. For each embedding $\tau\colon F \to \mathbb{C}$, we let $\mathcal{O}_{S,\tau} = \mathcal{O}_S \otimes_{F,\tau} \mathbb{C}$, and let $\mathrm{Lie}(A)_\tau = \mathrm{Lie}(A) \otimes_{F,\tau} \mathbb{C}$. We then assume that

(i) $\mathrm{Lie}(A)_\tau = 0$, $\tau \in \Sigma$, $\tau \neq \tau_0$; module of rank n^2, $\tau \neq \tau_0$;
(ii) $\mathrm{Lie}(A)_{\tau_0}$ is a projective $\mathcal{O}_{S,\tau}$ module of rank n.

For K sufficiently small, $\mathcal{A}_K(B, *)$ is represented by a smooth projective scheme over F, also denoted $\mathcal{A}_K(B, *)$, isomorphic to $|\ker^1(\mathbb{Q}, G)|$ copies of the canonical model of ${}_K Sh(G, X)$, where $\ker^1(\mathbb{Q}, G)$ measures the defect of the Hasse principle for $H^1(\mathbb{Q}, G)$.

Assume K factors as $K_p \times K^p$, with K^p sufficiently small, $K_p = \prod_i K_{w_i} \times \mathbb{Z}_p^\times$, with respect to (5), and $K_w = K_{w_1} = GL(n, \mathcal{O}_w)$. Then $\mathcal{A}_K(B, *)$ has a smooth model over $\mathrm{Spec}(\mathcal{O}_w)$, also denoted $\mathcal{A}_K(B, *)$, that represents a slightly modified version of the functor considered above: in conditions (i)–(ii), the $\mathcal{O}_{S,\tau}$'s are replaced by $\mathcal{O}_S \otimes_{\mathcal{O}_w} \mathcal{O}_{\tilde{w}}$, where $\tilde{w}$ run through the primes of F above p. As above, $\mathcal{A}_K(B,*)$ is the union of $|\ker^1(\mathbb{Q},G)|$ copies of a smooth model $\mathcal{O}_w$-model $S_K(G,X)$ of ${}_KSh(G,X)$. We let $\bar{S}=\bar{S}_K(G,X)$ denote the special fiber of this model.

The moduli space $\mathcal{A}_K(B,*)$ admits a universal abelian scheme (with PEL structures) denoted $A=A_K(B,*)$. Let $A[w^\infty]$ denote the corresponding p-divisible $\mathcal{O}_w$-module. The action of a maximal order in $B_w \xrightarrow{\sim} M(n,F_w)$ breaks up $A[w^\infty]$ as a sum of n-copies of a p-divisible $\mathcal{O}_w$ module $\mathcal{G}$. Conditions (i)–(ii) imply that $\mathcal{G}$ is a one-dimensional height n divisible $\mathcal{O}_w$ module. The Serre-Tate theorem implies that the infinitesimal local structure of $\mathcal{A}_K(B,*)$ near a point s of the special fiber is controlled by the deformation theory of the fiber $\mathcal{G}_s$ at s. This is the basis of the stratification of the special fiber $\bar{S}$, discussed in the following section.

4. Stratifications of Shimura Varieties

We will work with a general Shimura variety $Sh(G,X)$, as in the introduction; to avoid complications, we assume the derived subgroup of G to be simply connected. Fix a prime p and a level subgroup $K=K_p\times K^p\subset G(\mathbf{A}_f)$. Assume G is quasi-split at p and K^p is sufficiently small, so that ${}_KSh(G,X)$ is smooth. Let v be a prime of the reflex field E dividing p, $F=E_v$. Let W denote the ring of Witt vectors of $k(v)$, $\mathcal{K}=\mathrm{Frac}(W)$, $\mathcal{L}$ the compositum of F and $\mathcal{K}$, σ the (arithmetic) Frobenius automorphism of $\mathcal{L}$ over F. We assume that ${}_KSh(G,X)$ has a smooth model S over $\mathrm{Spec}(\mathcal{O}_v)$, to which the Hecke correspondences extend; we let $\bar{S}$ denote the special fiber. The usual hypothesis is that p be unramified in E and that K_p be a hyperspecial maximal compact subgroup. This hypothesis is sufficient when $Sh(G,X)$ is of PEL type ([34, 30]), but is certainly stronger than necessary. Results of Labesse [33, Prop. 3.6.4] suggest it may suffice to take K_p to be a "very special" maximal compact subgroup, provided G splits over F; this is true in the cases considered in [20].

For any algebraic torus T, let $X^*(T)$ and $X_*(T)$ denote the group of its characters and cocharacters, respectively. Let $P_0\subset G$ be a $\mathbb{Q}_p$-rational minimal parabolic subgroup, with Levi factor T_0 and unipotent radical N_0. This determines an order on the root lattice of G and, dually, on that of $\hat{G}$. The prime p being fixed, we let $B(G)$ denote the set of σ-conjugacy classes in $G(\mathcal{L})$, and let $\kappa\colon B(G)\to X^*(Z((\hat{G}))^{\Gamma_p})$ be the invariant defined in [28], with $\Gamma_p=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $A\subset T_0$ be the maximal split torus, $\mathfrak{a}=X_*(A)\otimes_{\mathbb{Z}}\mathbb{R}$, $\mathfrak{a}_{\mathbb{Q}}=X_*(A)\otimes_{\mathbb{Z}}\mathbb{Q}$. Let $\bar{C}\subset\mathfrak{a}$ be the closed positive chamber corresponding to N_0, $\bar{C}_{\mathbb{Q}}=\bar{C}\cap\mathfrak{a}_{\mathbb{Q}}$. The *Newton map* $\bar{\nu}\colon B(G)\to\bar{C}_{\mathbb{Q}}$ is defined in [39, 32]; it is known that

$$\bar{\nu}\times\kappa\colon B(G)\to\bar{C}_{\mathbb{Q}}\times X^*(Z((\hat{G}))^{\Gamma_p})$$

is injective [32, 4.13]. The class b is *basic* if and only if $\bar{\nu}(b)$ is in the intersection of all root hyperplanes. In that case, b defines an inner twist of G [28, 4.4].

On the other hand, the cocharacter μ can be interpreted as a character of the dual torus $\hat{T}_0$; let $\mu^\#\in X^*(Z(\hat{G})^{\Gamma_p})$ denote the restriction of this character.

Following Kottwitz [32, §6], we let $B(G,\mu) = B(G_{\mathbb{Q}_p},\mu)$ be the set of $b \in B(G)$ satisfying $\kappa(b) = \mu^{\#}$ and such that $\bar{\nu}(b) \leq \mu_{\mathfrak{a}}$ with $\leq$ the usual lexicographic order.

Consider the Langlands representation r_μ of ${}^LG$. Let $P = LU \subset G$ be a standard parabolic. The representation r_μ decomposes, upon restriction to ${}^LL$, as a sum of irreducible components $\mathcal{C}_0(L,\mu)$, each intervening with multiplicity one. Indeed, μ is a minuscule weight, with stabilizer $W_\mu = W_{Q_\mu}$ for a certain parabolic subgroup $Q = Q_\mu \subset G$ defined over $\bar{\mathbb{Q}}$. Here W_{Q_μ} is the Weyl group of any Levi factor of Q_μ. The irreducible components of r_μ are indexed by $(W_P\backslash W_{\hat{G}}/W_{Q_\mu})$ where W_P is the Weyl group of the Langlands dual $\hat{L}$ of L and $W_{\hat{G}}$ is the absolute Weyl group of G. The highest weight of the component corresponding to w, relative to the standard ordering induced by P_0, is the one in the orbit containing $w\mu$. We identify two elements $\lambda, \lambda' \in \mathcal{C}_0(L,\mu)$ if they are associate; i.e., if there is an element of the (relative) Weyl group W_G that normalizes L that takes λ to λ'. Let $\mathcal{C}(L,\mu)$ be the set of equivalence classes for this relation, and let $\mathcal{C}(\mu) = \coprod_L \mathcal{C}(L,\mu)$, where L runs through the classes of standard Levi subgroups of G. The set $\mathcal{C}(\mu)$ of pairs $(L, w\mu)$ is partially ordered by inclusion on the first factor. If $L \subset G$ is an F-rational Levi factor, let $i_{LG}\colon B(L) \to B(G)$ denote the natural map.

Proposition 4.1. (i) *There is a canonical surjective map* $\mathrm{Strat}\colon \mathcal{C}(\mu) \to B(G,\mu)$ *such that, for any* $(L,w\mu) \in \mathcal{C}(\mu)$, $\mathrm{Strat}(L,w\mu) = i_{LG}(b_L(w\mu))$, *where* $b_L(w\mu) \in B(L)$ *is the unique basic class such that* $\kappa(b_L(w\mu)) = (w\mu)^{\#}$ *for* L*.*

(ii) *Let* $b \in B(G,\mu)$*, and let* $\mathrm{Rep}(b) = \mathrm{Strat}^{-1}(b) \subset \mathcal{C}(\mu)$*. The set* $\mathrm{Rep}(b)$ *contains a unique maximal element* $(M = M(b), w_b\mu)$*. Here* $M(b)$ *is the centralizer of the slope morphism attached to* b [39, 32] *and* b *is the image of a basic* σ*-conjugacy class* $b_M \in B(M)$ *under the natural map* $B(M) \to B(G)$*.*

(iii) *There is a bijection between* $\mathrm{Rep}(b)$ *and the set of* $\mathcal{P}(b)$ *of standard* $\mathbb{Q}_p$*-rational parabolics* $P \subset M = M(b)$ *that transfer to the inner form* $J(b)$ *of* M *defined by the basic* σ*-conjugacy class* b*. (We call such parabolics* b-relevant*.)*

The proof of this proposition makes use of simple properties of minuscule weights. It is an amusing exercise to work it out explicitly when $G = GL(n)$ and μ is any minuscule weight.

We assume $\bar{S}$ admits a stratification by locally closed reduced $k(v)$-rational subschemes

$$\bar{S} = \coprod_{b \in B(G,\mu)} S(b)\,, \tag{6}$$

with each $S(b)$ stable under Hecke correspondences. This is conjectured to be true in general. In the PEL case, the existence of such a stratification follows from Theorem 3.6 of [39]. This defines a map from $\bar{S}(\bar{k}(v))$ to $B(G)$, with image necessarily in $B(G,\mu)$ (by Mazur's theorem). We define $S(b)_{\mathrm{geom}}$ to be the inverse image of b with respect to this map. Theorem 3.6 of [39] asserts that $S(b)_{\mathrm{geom}}$ is the set of geometric points of a locally closed reduced subscheme $S(b)$. Stability under Hecke correspondences follows from invariance of isocrystals with respect to isogeny. The same theorem of [39] asserts moreover that $S(b') \subset \overline{S}(b)$ if and only if $\bar{\nu}(b') \leq \bar{\nu}(b)$. We assume our stratification to have this property as well.

4.1. Example (notation as in §3)

With respect to the factorization (5), we can write $\mu = (\mu_1, \ldots, \mu_r; \mu_0)$, where μ_0 is the $\mathbb{Q}_p^\times$-factor; likewise, we write $b = (b_1, \ldots, b_r; b_0)$. With our conventions, $\mu_i = 0$ for $i > 1$, and μ_1 is the minuscule coweight $(1, 0, \ldots, 0)$ in the usual coordinates; i.e., r_{μ_1} is the standard n-dimensional representation of $GL(n)$. Then $b \in B(G, \mu)$ if and only if $b_1 \in B(GL(n)_{F_w}, \mu_1)$, $b_i = 0$ for $i > 1$, and $\kappa(b_0) = \mu_0^\# \in X^*(\hat{\mathbb{G}}_m)$. Let $M_{\frac{r}{s}}$ denote a simple isocrystal with slope $\frac{r}{s}$. We can write $B(GL(n)_{F_w}, \mu_1) = \{b_1(h), 0 \leq h \leq n-1\}$, where $b_1(h)$ corresponds to the height n F_{w_1}-isocrystal M with slope decomposition $M_{\frac{1}{n-h}} \oplus M_0^h$. Let $b(h)$ be the element of $B(G, \mu)$ corresponding to $b_1(h)$. Then the group $J(b(h))$ is isomorphic to

$$D^\times_{\frac{1}{n-h}} \times GL(h, F_w) \times \prod_{i>1} B_{w_i}^{\mathrm{op},\times} \times \mathbb{Q}_p^\times , \tag{7}$$

where $D_{\frac{1}{n-h}}$ is the division algebra over F_w with invariant $\frac{1}{n-h}$.

Let $\bar{S}$ denote the special fiber of the smooth $\mathcal{O}_w$-model $S_K(G, X)$ of ${}_KSh(G, X)$. Then the stratification (6) holds with $S(b(h))$ defined as follows. As mentioned at the end of §3, $S_K(G, X)$ carries a natural family $\mathcal{G}$ of p-divisible $\mathcal{O}_w$-modules of height n and dimension 1. For each dimension $h = 0, 1, \ldots, n-1$, there is a unique stratum $S(h) = S(b(h)) \subset \bar{S}$ of dimension h, defined by the property that, over every geometric point $x \in S(h)$, the maximal étale quotient $\mathcal{H}_x^{et}$ of $\mathcal{H}_x$ is of height h (i.e., of height $h[F : \mathbb{Q}_p]$ as p-divisible group). It is proved in [20] that each stratum $S(h)$ is smooth. The proof makes use of Drinfeld's explicit deformation theory for one-dimensional formal $\mathcal{O}_w$-modules.

For general Shimura varieties, $S(b)$ is almost never expected to be smooth (see, e.g., [37]). Langlands and Rapoport have formulated a conjecture describing $\bar{S}(\bar{k}(v))$ as a disjoint union of subsets $S(\phi)$, each stable under Hecke correspondences. Here ϕ runs through the set $\Phi(G, X)$ of *Langlands-Rapoport parameters*: admissible homomorphisms $\phi\colon \mathfrak{P} \to \mathfrak{G}_G$, where $\mathfrak{P}$ is the pseudomotivic groupoid and $\mathfrak{G}_G$ is the neutral groupoid attached to G [34, 36].

There is a map $b\colon \Phi(G, X) \to B(G, \mu)$, defined by restricting ϕ to the Dieudonné groupoid $\mathcal{D}_p$ (cf. [36, p. 181]), so that $S(\phi) \subset S(b(\phi))(\bar{k}(v))$. Moreover, every point in $S(b)(\bar{k}(v))$ belongs to exactly one $S(\phi)$ with $b(\phi) = b$. We add the assumption that, to each admissible ϕ, we can associate a locally finite disjoint union $\tilde{S}(\phi)$ of closed reduced subschemes such that $S(\phi)$ is the set of $\bar{k}(v)$ points of $\tilde{S}(\phi)$. Under these conditions, the formal completion $S_{/\tilde{S}(\phi)}$ of the scheme S along the subset $\tilde{S}(\phi)$ of its special fiber can be defined as in [40, 6.22].

Let I_ϕ denote the automorphism group of the admissible homomorphism ϕ, in the sense of [34]. I_ϕ is a reductive algebraic group over $\mathbb{Q}$, an inner form of the centralizer in G of the torus $T(\phi) := \phi(\mathfrak{P}^+)$, where $\mathfrak{P}^+$ is the identity component of $\mathfrak{P}$. There is a natural homomorphism $I_\phi(\mathbb{Q}) \to J(b(\phi))$, well-defined up to conjugacy. At all finite primes $\ell \neq p$, the inner twist is trivial, hence there is a natural map $I_\phi(\mathbb{Q}) \to G(\mathbf{A}_f^p)$, well-defined up to conjugacy. For any $b \in B(G)$, we let $G^b(\mathbf{A}_f) = G(\mathbf{A}_f^p) \times J(b)$. The superscript $^{\mathrm{rig}}$ denotes the rigid-analytic generic fiber of a formal scheme.

Conjecture 4.2. (i) *For each $b \in B(G,\mu)$, there is a formal scheme $\breve{\mathcal{M}}(b,\mu)$ over $Spf(\mathcal{O}_{\mathcal{L}})$, with a Weil descent datum* [40, 3.45] *over $Spf(\mathcal{O}_v)$ and a compatible action of $J = J(b)$.*

(ii) *There is an étale $J(b)$-equivariant surjective rigid-analytic morphism*

$$\pi\colon \breve{\mathcal{M}}^{\mathrm{rig}}(b,\mu) \to \breve{\mathcal{F}}^{\mathrm{wa}}(b,\mu) \times \Delta',$$

where $\breve{\mathcal{F}}^{\mathrm{wa}}(b,\mu)$ is the rigid-analytic open subset of weakly admissible flags (relative to b) in the flag variety G/Q_μ [40, §1] *and Δ' is a discrete homogenous space for $G(\mathbb{Q}_p)$. The morphism π is compatible with Weil descent data on both sides.*

(iii) *For any open subgroup $K' \subset K_p$, there is a rigid-analytic covering $\pi_{K'}\colon \mathcal{M}_{K'}(b,\mu) \to \breve{\mathcal{M}}^{\mathrm{rig}}(b,\mu)$, and for any pair $K'' \subset K'$ of open subgroups of K_p, a morphism $\pi_{K'',K'}\colon \mathcal{M}_{K''}(b,\mu) \to \mathcal{M}_{K'}(b,\mu)$, such that $\pi_{K''} = \pi_{K'} \circ \pi_{K'',K'}$. The projective system $\mathcal{M}_{K'}(b,\mu)$ thus inherits a continuous action of $G(\mathbb{Q}_p)$, covering the natural action on the second factor of $\breve{\mathcal{F}}^{\mathrm{wa}}(b,\mu) \times \Delta'$.*

(iv) *Let ϕ be a Langlands-Rapoport parameter, and let $b = b(\phi)$. Let $x \in \tilde{S}(\phi)$ be a basepoint. There is an isomorphism of formal schemes (*local uniformization*):*

$$u_{\phi,x}\colon [\breve{\mathcal{M}}(b,\mu) \times (I_\phi(\mathbb{Q})\backslash G^{b(\phi)}(\mathbf{A}_f)/K^p)]/J(b(\phi)) \xrightarrow{\sim} S_{/\tilde{S}(\phi)}.$$

Here $J(b(\phi))$ acts diagonally, on the first factor as in (i) and on the second via the inclusion of $J(b(\phi))$ in $G^{b(\phi)}$. The Weil descent datum on $\breve{\mathcal{M}}(b,\mu)$ induces an effective descent datum on the left-hand side, and $u_{\phi,x}$ is an isomorphism of formal schemes over $Spf(\mathcal{O}_v)$. Moreover $u_{\phi,x}$ is equivariant with respect to the actions of the Hecke correspondences (relative to K) on both sides.

(v) *More generally, for any open subgroup $K' \subset K_p$, there is an isomorphism of rigid-analytic spaces*

$$u_{\phi,x,K'}\colon [\mathcal{M}_{K'}(b,\mu) \times (I_\phi(\mathbb{Q})\backslash G^{b(\phi)}(\mathbf{A}_f)/K^p)]/J(b(\phi)) \xrightarrow{\sim} (\Pi_{K'})^{-1}([S_{/\tilde{S}(\phi)}]^{\mathrm{rig}}.$$

Here $\Pi_{K'}\colon {}_{K'\times K^p}Sh(G,X) \to {}_K Sh(G,X)$ is the standard étale covering in characteristic zero. The isomorphism $u_{\phi,x,K'}$ is rational over E and equivariant with respect to Hecke correspondences (relative to $K' \times K^p$).

Sections 5 and 6 of [40] are largely devoted to proving a version of conjecture 4.2 for the PEL-type Shimura varieties considered there, in which Langlands-Rapoport parameters are replaced by isogeny classes of PEL abelian varieties[4]. In particular, the conjecture is true for the Shimura varieties considered in [20]. However, much more is true.

Theorem 4.3. *Let $Sh(G,X)$ be the Shimura variety considered in §3. Then*

(i) *Conjecture 4.2 is valid for $Sh(G,X)$, with Langlands-Rapoport parameters replaced by isogeny classes for the moduli problem 3.1.*

(ii) *For any isogeny class ϕ for $Sh(G,X)$, the connected components of $\tilde{S}(\phi)$ are closed points.*

(iii) *For any open subgroup $K' \subset K_p$, and any $b \in B(G,\mu)$, there is a formal scheme $\breve{\mathcal{M}}_{K'}(b,\mu)$ over $Spf(\mathcal{O}_{\mathcal{L}})$, with $J(b)$-action, a Weil descent datum over $Spf(\mathcal{O}_v)$, and a $J(b)$-equivariant isomorphism $\breve{\mathcal{M}}^{\mathrm{rig}}_{K'}(b,\mu) \xrightarrow{\sim} \mathcal{M}_{K'}(b,\mu)$. These isomorphisms are compatible with inclusions $K'' \subset K'$ of open subgroups, and the $G(\mathbb{Q}_p)$-action on the projective system $\{\breve{\mathcal{M}}^{\mathrm{rig}}_{K'}(b,\mu)\}$ induced by the isomorphism with $\{\mathcal{M}_{K'}(b,\mu)\}$ extends to a continuous $G(\mathbb{Q}_p)$-action on the projective system $\{\breve{\mathcal{M}}_{K'}(b,\mu)\}$ of formal schemes. Each $\breve{\mathcal{M}}_{K'}(b,\mu)$ is regular.*

(iv) *Similarly, there is a $G(\mathbf{A}_f)$-equivariant system of regular $\mathcal{O}_v$-schemes $S_{K'\times K^p}$ with $G(\mathbf{A}_f)$-equivariant isomorphisms*

$$S_{K'\times K^p} \otimes_{\mathrm{Spec}(\mathcal{O}_v)} \mathrm{Spec}(F) \xrightarrow{\sim} {}_{K'\times K^p}Sh(G,X)\,.$$

(v) *Let ϕ denote an isogeny class for the moduli problem (?), and let $b = b(\phi)$. Let $\tilde{S}_{K'\times K^p}(\phi)$ denote the scheme-theoretic inverse image of $\tilde{S}(\phi)$ in $S_{K'\times K^p}$. The rigid uniformization of 4.2(v) extends to a Hecke equivariant isomorphism of formal schemes over $Spf(\mathcal{O}_v)$:*

$$u\colon [\breve{\mathcal{M}}_{K'}(b,\mu) \times (I_\phi(\mathbb{Q})\backslash G^{b(\phi)}(\mathbf{A}_f)/K^p)]/J(b(\phi)) \xrightarrow{\sim} [S_{K'\times K^p}]_{/\tilde{S}_{K'\times K^p}(\phi)}\,.$$

As noted above, (i) is a special case of the results of [40]. Assertion (iii) is due to Drinfeld [16], and uses his theory of level structures for one-dimensional divisible $\mathcal{O}_v$-modules (Drinfeld bases); (ii) is a consequence of the explicit deformation theory of [16]. Assertion (iv) is proved using Drinfeld bases, and (v) is a formal consequence of the corresponding assertion when $K = K_p$.

5. Vanishing Cycles

Let E, v, and F be as above. Henceforward, we assume G to be anisotropic modulo its center; then $Sh(G,X)$ is a projective limit of projective varieties. For any open $K' \subset K_p$, the direct image $\pi_{K',*}\mathbb{Q}_\ell$ of the constant sheaf on ${}_{K'\times K^p}Sh(G,X)$ is a locally constant étale sheaf on ${}_KSh(G,X)$. We use the same notation for the corresponding étale sheaf on ${}_KSh(G,X)^{\mathrm{rig}}$. We let $R\Psi_{K'}$ denote the nearby cycles

[4] Assertion (ii) is proved in [40] assuming a conjecture of Fontaine, recently proved by Breuil.

complex $R\Psi(\pi_{K',*}\mathbb{Q}_\ell)$ on the special fiber $\bar{S}$. We view $R\Psi_{K'}$ as an object in the bounded derived category of constructible ℓ-adic complexes on $\bar{S}$. For $K'' \subset K'$, pullback via $\pi_{K'',K'}$ induces a canonical morphism $R\Psi_{K'} \to R\Psi_{K''}$. Define

$$R\Gamma(\bar{S}, R\Psi) := \varinjlim_{K',K^p} R\Gamma(\bar{S}, R\Psi_{K'})\,;$$

$$R\Gamma(Sh(G,X), \mathbb{Q}_\ell) = \varinjlim_{K',K^p} R\Gamma({}_{K'\times K^p}Sh(G,X), \mathbb{Q}_\ell)\,.$$

Here the subscript K^p is omitted from $\bar{S}$ for convenience. There is a canonical spectral sequence

$$E_2^{p,q} = H_c^p(\bar{S}, \Psi^q) \Rightarrow H_c^{p+q}(Sh(G,X), \mathbb{Q}_\ell) \tag{8}$$

of $G(\mathbf{A}_f) \times \mathrm{Gal}(\bar{F}/F)$-modules, with $\Psi^q = R^q\Psi$.

We work in a modified Grothendieck group $\mathrm{Groth}(G)$ of equivalence classes of admissible $G(\mathbf{A}_f) \times W(F)$-modules [6, 20]. If M is an admissible $G(\mathbf{A}_f) \times W(F)$-module, $[M]$ denotes the corresponding object of $\mathrm{Groth}(G)$. Let $[H(Sh(G,X), \mathbb{Q}_\ell)] = \sum_i (-1)^i H^i(Sh(G,X), \mathbb{Q}_\ell) \in \mathrm{Groth}(G)$. Then (8) corresponds to an equality in $\mathrm{Groth}(G)$:

$$[H(Sh(G,X), \mathbb{Q}_\ell)] = \sum_{p,q} (-1)^{p+q} [H_c^p(\bar{S}, \Psi^q)]\,. \tag{9}$$

Here and below, the constant sheaf $\mathbb{Q}_\ell$ can be replaced by the ℓ-adic local system $\mathcal{L}_\xi$ attached to a finite-dimensional absolutely irreducible representation ξ of G with coefficients in $\bar{\mathbb{Q}}_\ell$.

On the other hand, the stratification $\bar{S} = \coprod_{b \in B(G,\mu)} S(b)$ gives rise to a spectral sequence of dévissage, for each term on the right-hand side of (9). Let Ψ_b^q denote the pullback to $S(b)$ of Ψ^q. When Π_f is an irreducible admissible representation of $G(\mathbf{A}_f)$, let $[\Pi_f]$ denote the Π_f isotypic component, which makes sense in the Grothendieck group. Then

$$[H(Sh(G,X), \mathbb{Q}_\ell)][\Pi_f] = \sum_{p,q,b} (-1)^{p+q} [H_c^p(S(b), \Psi_b^q)][\Pi_f]\,, \tag{10}$$

for any Π_f.

Now let (b,μ) be a general weakly admissible pair for $G(\mathbb{Q}_p)$, in the sense of [40]. Consider the $G(\mathbb{Q}_p) \times J(b) \times W(F)$-equivariant system $\mathcal{M}_{K'}(b,\mu)$ of rigid-analytic converings of $\breve{\mathcal{M}}^{\mathrm{rig}}(b,\mu)$. Let $\mathrm{Groth}(G,b)$ be the Grothendieck group, in the above sense, now of smooth representations of $G(\mathbb{Q}_p) \times J(b) \times W(F)$. Define

$$[H_{(b,\mu)}] = \sum_i (-1)^i \varinjlim_{K'} H_c^i(\mathcal{M}_{K'}(b,\mu), \mathbb{Q}_\ell)]\,.$$

The cohomology is ℓ-adic étale cohomology, in the sense of Berkovich [2]. It is smooth as a representation of $G(\mathbb{Q}_p) \times J(b)$ [3]. It is reasonable to assume (cf. [18, 20]) it is a direct limit of finite-dimensional representations of $W(F)$. Let $R\Theta_{(b,\mu)} = \varinjlim_{K'} R\Theta(\pi_{K',*}\mathbb{Q}_\ell)$, where $\pi_{K',*}\mathbb{Q}_\ell$ is viewed as an étale sheaf on $\breve{\mathcal{M}}^{\mathrm{rig}}(b,\mu)$ and $R\Theta$ is Berkovich's nearby cycle functor for formal schemes [3, I.,

§4]: $R\Theta_{(b,\mu)}$ belongs to the derived category of constructible ℓ-adic sheaves on the special fiber $\bar{\mathcal{M}}(b,\mu)$ of the formal $Spf(\mathcal{O}_v)$-scheme $\breve{\mathcal{M}}(b,\mu)$. Writing $\Theta^j = R^j\Theta$, and letting $\check{\Theta}^j$ denote the $G(\mathbb{Q}_p)$-smooth contragredient, we then have

$$[H_{(b,\mu)}] = \sum_{i,j}(-1)^{i+j}[H^i_c(\bar{\mathcal{M}}(b,\mu), \check{\Theta}^j_{(b,\mu)}] \,. \tag{11}$$

Conjecture 5.1. *Let ϕ be a Langlands-Rapoport parameter for the special fiber $\bar{S}$ of $Sh(G,X)$ at v. Let $b = b(\phi)$, and let $\mu = \mu_X$. Let $\Psi^q(\phi)$ be the pullback of Ψ^q (or equivalently of Ψ^q_b) to the locally finite scheme $\tilde{S}(\phi)$. For every j, there is a $G(\mathbf{A}_f) \times W(F)$-equivariant isomorphism of ind-constructible sheaves:*

$$u^*_{\phi,x}\colon [\Theta^q_{(b,\mu)} \times (I_\phi(\mathbb{Q})\backslash G^{b(\phi)}(\mathbf{A}_f)/K^p)]/J(b(\phi)) \xrightarrow{\sim} \Psi^q(\phi)\,,$$

covering the uniformization map $u_{\phi,x}$.

When $Sh(G,X)$ is one of the PEL type Shimura varieties treated in [40], the analogue of this conjecture for torsion coefficients has been proved by L. Fargues.

5.1. Reduction to basic classes

Suppose $b \in B(G,\mu)$ is the image under the map Strat of a pair $(L, w\mu)$, as in proposition 4.1. Let $b_L = b_L(w\mu) \in B(L)$ be the corresponding basic class. Then there is (conjecturally!) a Rapoport-Zink space $\breve{\mathcal{M}}(b_L, w\mu)$, and thus a cohomology representation $[H_{(b_L,w\mu)}] \in \mathrm{Groth}(L, b_L)$. Let $P \subset G(\mathbb{Q}_p)$ be a parabolic subgroup with Levi factor L. Note that $J(b_L) = J(b)$. There is a thus a natural map (non-normalized parabolic induction)

$$\mathrm{Ind}^G_P\colon\ \mathrm{Groth}(L, b_L) \to \mathrm{Groth}(G, b)\,.$$

Suppose $(M(b), w\mu)$ is the maximal element in $\mathrm{Strat}^{-1}(b)$; we write b_M instead of $b_{M(b)}$ for the corresponding basic element of $B(M(b), w\mu)$. Then one parabolic P is better than the others. On any representation (ρ, V) of G, the class b defines the structure of an isocrystal. We may thus define the *slope filtration* on V to be the decreasing filtration F_a, indexed by $a \in \mathbb{Q}$, such that $F_a(V)$ is the sum of the isoclinic subspaces of slope $\geq a$. Let $P(b) \subset G$ be the parabolic stabilizing the slope filtration attached to b in any finite-dimensional representation of G. Since $M(b)$ is maximal, it is the centralizer of the slope morphism attached to b, hence $M(b)$ is a Levi factor of $P(b)$. On global grounds, it seems reasonable to propose the following

Conjecture 5.2. *There is an isomorphism*

$$[H_{(b,\mu)}] = \mathrm{Ind}^G_{P(b)}(\mathbb{Q}_p)[H_{(b_M,w\mu)}]\,.$$

There is apparently no geometric relation between the two sides, but Strauch's methods [44] may suffice to establish the corresponding identity of distribution characters[5].

[5] *Note added in proof.* In the PEL case, this conjecture should follow from recent work of Zink on the slope filtration of a p-divisible group.

Now assume b basic, so that $M(b) = G$. Write $J = J(b)$. Kottwitz has proposed a conjecture [38, §5] for the contribution to $[H_{(b,\mu)}]$ of discrete series representations of $G \times J(b)$, generalizing Carayol's conjectures in [10]. Kottwitz' conjecture is expressed in terms of the conjectural classification of L-packets by Langlands parameters, here denoted $\psi\colon W_{\mathbb{Q}_p} \to {}^LG$, and the characters of their centralizers S_ψ. Assume ψ is discrete; i.e., that the identity component S_ψ^0 of S_ψ is contained in $Z(\hat{G})^{\Gamma_p}$. Let $\hat{S}_\psi$ denote the set of equivalence classes of irreducible representations of S_ψ. We let

$$\Pi_\psi(G) = \{\tau \in \hat{S}_\psi | \tau|_{Z(\hat{G})^{\Gamma_p}} = 1\};\ \Pi_\psi(J) = \{\tau' \in \hat{S}_\psi | \tau'|_{Z(\hat{G})^{\Gamma_p}} = \kappa(b)\}\,.$$

The *local Langlands conjecture* for G identifies elements of $\Pi_\psi(G)$ (resp. $\Pi_\psi(J)$) with irreducible discrete series representations of $G(\mathbb{Q}_p)$ (resp. J), and conjectures that, as ψ varies over discrete Langlands parameters and τ varies over characters of $\hat{S}_\psi$ as above, this gives a complete parametrization of the discrete series.

For $\pi \in \Pi_\psi(J)$, let

$$[H_{(b,\mu)}][\pi'] = \sum_{i,j,k} (-1)^{i+j+k} [\mathrm{Ext}_J^k(H_c^i(\bar{\mathcal{M}}(b,\mu), \check{\Theta}^j_{(b,\mu)}), \pi')]\,. \tag{12}$$

The Ext groups are taken in the category of smooth J-modules. The result $[H_{(b,\mu)}][\pi]$ is a virtual representation of $G(\mathbb{Q}_p) \times W(F)$, and should belong to the Grothendieck group of admissible $G(\mathbb{Q}_p) \times W(F)$-modules in the naive sense (i.e., modules of finite length).

Recall that r_μ is taken to be a representation of $\hat{G} \rtimes W(F)$. In what follows, $\check{\pi}$ and $\check{\tau}$ denote the contragredients of π and τ, respectively.

Conjecture 5.3. (Kottwitz) *Let ψ be a discrete Langlands parameter for $G(\mathbb{Q}_p)$, and let $\pi' \in \Pi_\psi(J)$. Then*

$$[H_{(b,\mu)}][\pi'] = \sum_{\pi \in \Pi_\psi(G)} [\check{\pi} \otimes \mathrm{Hom}_{S_\psi}(\tau_\pi \otimes \check{\tau}_{\pi'}, r_\mu \circ \psi)]$$

as virtual representation of $G(\mathbb{Q}_p) \times W(F)$.

The global theory of Shimura varieties, in conjunction with proposition 4.1(iii), suggests a generalization of conjecture 5.3 to accomodate Langlands parameters induced from b-relevant parabolic subgroups of G. Let $P \subset G$ be a b-relevant parabolic, with Levi subgroup L, and let $P_J \subset J$ be a transfer of P, with Levi subgroup L_J. There is a basic $b_L \subset B(L)$ such that $i_{LG}(b_L) = b$. The bijection of proposition 4.1(iii) realizes b_L as $\mathrm{Strat}(L, \mu(b_L))$ for some minuscule character $\mu(b_L)$ of (a maximal torus of) $\hat{L}$, corresponding to a minuscule representation r_{μ,b_L}. We consider discrete Langlands parameters ψ for $L(\mathbb{Q}_p)$, and define the L-packets $\Pi_\psi(L)$ and $\Pi_\psi(J_L)$ as above. Let I_P^G, $I_{P_J}^J$ denote normalized induction. For $\pi \in \Pi_\psi(L)$, let

$$[H_{(b,\mu)}][\pi] = \sum_{i,j,k} (-1)^{i+j+k} [\mathrm{Ext}^k_{G(\mathbb{Q}_p)}(H_c^i(\bar{\mathcal{M}}(b,\mu), \check{\Theta}^j_{(b,\mu)}), I_P^G \pi)]\,. \tag{13}$$

We make the analogous definition for $\pi' \in \Pi_\psi(L_J)$.

Conjecture 5.4. *Let ψ be a discrete Langlands parameter for $L(\mathbb{Q}_p)$, and let $\pi' \in \Pi_\psi(L_J)$. Suppose I_{P_J} is irreducible. Then*

$$[H_{(b,\mu)}][\pi'] = \sum_{\pi \in \Pi_\psi(L)} [I_P^G(\check{\pi}) \otimes \mathrm{Hom}_{S_\psi}(\tau_\pi \otimes \check{\tau}_{\pi'}, r_{\mu,b_L} \circ \psi)]$$

as virtual representation of $L(\mathbb{Q}_p) \times W(F)$.

Remark 5.5. *Here and in conjecture 5.3, our sign conventions differ from those of* [38]*. There are analogous conjectures with the roles of G and J exchanged.*

It is not clear whether the formula in (5.4) should still hold when $I_P^G\pi$ is reducible. For example, is it compatible with conjecture 5.3 when $P = B$ is a Borel subgroup and $I_P^G\pi$ contains the Steinberg representation as subquotient?

5.2. The twisted unitary case

In the remainder of this section, we restrict attention to the Shimura varieties $Sh(G,X)$ of §3. In that case, conjecture 5.2 holds for straightforward geometric reasons, and nearly all the conjectures considered above are established in [20]. The starting point is theorem 4.3. Note that the conjectures simplify considerably. As in (4.1), we can factor

$$\bar{M}(b,\mu) \xrightarrow{\sim} \bar{M}(b_1,\mu) \times \prod_{i\neq 1} \bar{M}(b_i,\mu_i)\,. \tag{14}$$

For $i > 1$ the factors $\bar{M}(b_i,\mu_i)$ are discrete sets with trivial Galois action. Similarly, $\bar{M}(b_0,\mu_0) \xrightarrow{\sim} \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$; the Weil group action on this factor is non-trivial but elementary. Finally, let $L(h) \subset GL(n,F_w)$ be the Levi factor corresponding to $b_1(h) \in B(GL(n,F_w),\mu_1)$. Then

$$\bar{M}(b_1(h),\mu_1) \xrightarrow{\sim} GL(n,F_w) \times_{L(h)} [\bar{M}(h) \times GL(h,F_w)/GL(h,\mathcal{O}_w)]\,. \tag{15}$$

In the first place, the schemes $\bar{M}(b,\mu)$ are *zero-dimensional* (more precisely, are inductive limits of zero-dimensional reduced schemes) for all $b \in B(G,\mu)$. Thus $i = 0$ in (12) and $\check{\Theta}^j_{(b,\mu)}$ is just a vector space. Next, one sees easily from (14) and (15), and from compactness of $D^\times_{\frac{1}{n-h}}$ modulo its center, that $\check{\Theta}^j_{(b,\mu)}$ is a projective object in the category of smooth $J(b)$-modules. Thus (when G is replaced by J) the index k in (12) can also be taken to be zero. Finally, the L-packets Π_ψ are all singletons. Let $J = J(b) = J(b(h))$. There is a map $JL\colon \Pi_\psi(J) \to \Pi_\psi(G)$ defined as follows. We write $\Pi_\psi(J) = \pi' = [\pi'_1(n-h) \otimes \pi'_1(h)] \otimes \prod_{i>1} \pi'_i \otimes \pi'_0$, with respect to the factorization (7). Here $\pi'_1(n-h)$ (resp. $\pi'_1(h)$) is an irreducible representation of $D^\times_{\frac{1}{n-h}}$ (resp. $GL(h,F_w)$), and the other factors are clear. Let $P(h) = P(b(h))$. Define

$$JL(\pi') = \mathrm{Ind}_{P(h)}^{GL(n,F_w)}[JL(\pi'_1(n-h)) \otimes \pi'_1(h)] \otimes \prod_{i\neq 1} \pi'_i \otimes \pi'_0 \tag{16}$$

where the JL on the right-hand side is the Jacquet-Langlands correspondence [41, 15] between representations of $D^{\times}_{\frac{1}{n-h}}$ and discrete series representations of $GL(n-h, F_w)$. Thus conjecture 5.4 asserts

$$\sum_j (-1)^j [\mathrm{Hom}_{J(b)}(\check{\Theta}^j_{(b,\mu)}, \pi')] = JL(\check{\pi}') \otimes \sigma(JL(\check{\pi}')) \,. \tag{17}$$

This formula, a version of Carayol's conjecture [10], is the substance of Theorem 11.5 of [20], at least in the supercuspidal case.

6. Local Terms in the Lefschetz Formula

We return to the general setting of §§4 and 5, and assume the truth of conjectures 4.2 and 5.1. The identity (10) reduces study of the representation of $G(\mathbf{A}_f) \times W(F)$ on $[H(Sh(G,X), \mathbb{Q}_\ell)]$ to the determination of the representations on the individual strata. For fixed $b \in B(G, \mu)$, we write

$$[H(S(b), R\Psi)] = \sum_{p,q} (-1)^{p+q} [H^p_c(S(b), \Psi^q(b))] \,.$$

Fix an admissible irreducible representation Π_f of $G(\mathbf{A}_f)$, and suppose the p-adic component $\Pi_{f,p}$ is induced from a discrete representation π of the Levi component L of a parabolic subgroup $P \subset G(\mathbb{Q}_p)$. Let ψ be the corresponding discrete Langlands parameter for L. Roughly speaking, one expects that the map Strat of proposition 4.1 identifies the set of b for which $[H(S(b), R\Psi)][\Pi_f] \neq 0$ with the set $\mathcal{C}(L, \mu)$. Conjectures 5.3 and 5.4 suggest that the semisimplified local Galois representation on $\sum_b [H(S(b), R\Psi][\Pi_f]$ then should be something like

$$\sum_{w \in W_P \backslash W_G / W_{Q_\mu}} r_{w\mu} \circ \psi \,. \tag{18}$$

This natural generalization of (3) needs to be modified [29] when Π_f is attached to an endoscopic automorphic representation, but it provides a heuristic interpretation of proposition 4.1 as well as conjectures 5.3 and 5.4.

It is plausible that conjectures 5.3 and 5.4 can be established by global methods, using a trick due to Boyer [6]. Assuming conjecture 5.2 is true, as in [20] (and [6]), identity (10) shows that, if $\Pi_{f,p}$ is supercuspidal, then $[H(S(b), R\Psi)][\Pi_f] \neq 0$ only for b basic. If $b = b(\phi)$ is basic, I_ϕ is a twisted inner form of G. Conjecture 5.3 should then follow by comparing stable trace formulas for G and I_ϕ, at least when all $\pi \in \Pi_\psi$ are supercuspidal.

Remark 6.1. *Boyer's trick should also provide a means to apply the argument of Taylor-Wiles to study deformations of* $\pmod{\ell}$ *Galois representations in the part of the cohomology of* $Sh(G,X)$ *supercuspidal at* p*. See* [21] *for an application of this type in a case where the whole special fiber at* p *is basic.*

Assuming conjectures 5.2, 5.3, and 5.4, one obtains a formula similar to (18) for all strata, by purely local means, except that the terms $r_{w\mu}$ occur with undetermined multiplicities. It appears that the multiplicities can only be calculated by global means, generalizing Kottwitz' techniques for "counting points" in [29, 30].

Let $\mathcal{H}(G(\mathbf{A}_f))$ denote the big Hecke algebra of locally constant compactly supported functions on $G(\mathbf{A}_f)$, and define $\mathcal{H}(G(\mathbb{Q}_p))$, $\mathcal{H}(J(b))$, and $\mathcal{H}(G(\mathbf{A}_f^p))$ analogously. For $K' \subset G(\mathbf{A}_f)$ an open compact subgroup, we let $\mathcal{H}_{K'} \subset \mathcal{H}(G(\mathbf{A}_f))$ be the subalgebra of K'-biinvariant functions. These define correspondences on ${}_{K'}Sh(G,X)$, and we suppose these extend compatibly to cohomological correspondences on $[H(S(b),R\Psi)]^{K'} \subset [H(S(b),R\Psi)]$, covering an action by correspondences on $S(b)$. We will consider functions $f = f^p \otimes f_p \in \mathcal{H}_{K'}$, with $f^p \in \mathcal{H}(G(\mathbf{A}_f^p))$ and $f_p \in \mathcal{H}(G(\mathbb{Q}_p))$ such that the fixed point set $\mathrm{Fix}(f) \in S(b)(\bar{k}(v))$ is finite. As K' varies, such functions generate $\mathcal{H}(G(\mathbf{A}_f))$, hence their traces determine the admissible virtual $G(\mathbf{A}_f)$-module $[H(S(b),R\Psi)]$ up to isomorphism.

Since the actions of $G(\mathbf{A}_f)$ and $W(F)$ commute, the trace $\mathrm{Tr}(f|[H(S(b),R\Psi)])$ takes values in $\overline{\mathbb{Q}}_\ell \otimes \mathrm{Groth}(W(F))$, where $\mathrm{Groth}(W(F))$ is the Grothendieck group of virtual $W(F)$-modules. Indeed, if $f \in \mathcal{H}_{K'}$, for each p,q, and $a \in \overline{\mathbb{Q}}_\ell$, the generalized eigenspace $H_c^p(S(b),\Psi_b^q)^{K'}((f-a))$ of f with eigenvalue a is a finite-dimensional $W(F)$-module. We let

$$W\,\mathrm{Tr}(f|[H(S(b),R\Psi)]) = \sum_{p,q,a} (-1)^{p+q} a \otimes [H_c^p(S(b),\Psi_b^q)^{K'}((f-a))]\,, \qquad (19)$$

where [] denotes passage to $\mathrm{Groth}(W(F))$.

Similarly, if $f_p \in \mathcal{H}(G(\mathbb{Q}_p))$, and π' is any admissible representation of $J(b)$, we can define a virtual trace

$$\mathrm{Loc}(f_p,b,\pi') = W\,\mathrm{Tr}(f_p|[H_{(b,\mu)}][\pi']) = \sum_k (-1)^k W\,\mathrm{Tr}(f_p|\,\mathrm{Ext}^k_{J(b)}([H_{(b,\mu)}]),\pi')\,. \qquad (20)$$

Here $[H_{(b,\mu)}][\pi']$, defined as in (13), is an admissible virtual representation of $G(\mathbb{Q}_p) \times W(F)$, and $\mathrm{Loc}(f_p,b,\pi') \in \overline{\mathbb{Q}}_\ell \otimes \mathrm{Groth}(W(F))$. The following conjecture is presumably a straightforward application of the Paley-Wiener theorem [4].

Conjecture 6.2. *For any $f_p \in \mathcal{H}(G(\mathbb{Q}_p))$ there is a function $F_b(f_p) \in \mathcal{H}(J(b)) \otimes \mathrm{Groth}(W(F))$ such that*

$$\mathrm{Loc}(f_p,b,\pi') = \mathrm{Tr}(\pi')(F_b(f_p))\,.$$

For $\gamma \in I_\phi(\mathbb{Q})$, let $I_\gamma \subset I_\phi$ denote its centralizer, and let $v(\gamma) = \mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbf{A}_f))$. The choice of measure on $I_\gamma(\mathbf{A}_f)$ is irrelevant at this stage. The point-counting argument in [20] begins with a formula of the following type:

$$W\,\mathrm{Tr}(f|[H(S(b),R\Psi)]) = \sum_{\{\phi|b(\phi)=b\}} \sum_{\gamma \in I_\phi(\mathbb{Q})/\cong} c(\gamma)v(\gamma)O_\gamma(f^p \cdot F_b(f_p)). \qquad (21)$$

Here $I_\phi(\mathbb{Q})/\cong$ is the set of conjugacy classes in $I_\phi(\mathbb{Q})$, $O_\gamma(f^p \cdot f_{\pi'})$ is the orbital integral of f^p over the conjugacy class of γ in $G^b(\mathbf{A}_f)$, and $c(\gamma)$ is a "dimensionless constant" involving signs and the like. Moreover, it is assumed f belongs to a certain class of "acceptable" functions, sufficiently large to determine $[H(S(b), R\Psi)]$ up to isomorphism, for which Fujiwara's simple version of the Lefschetz formula [17] applies. Theorem 2.3 is obtained by relating this formula to the global trace formula on cohomology and combining the result with (10) and (17).

It is tempting to hope that a formula like (21) holds for general groups. However, it is not clear whether the coefficients $c(\gamma)$ can be defined to make this formula comparable to the global trace formula without assuming the fundamental lemma for endoscopy.

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