Sobolev Spaces and Quasiconformal Mappings on Metric Spaces

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Abstract. Heinonen and I have recently established a theory of quasiconformal mappings on Ahlfors regular Loewner spaces. These spaces are metric spaces that have sufficiently many rectifiable curves in a sense of good estimates on moduli of curve families. The Loewner condition can be conveniently described in terms of Poincaré inequalities for pairs of functions and upper gradients. Here an upper gradient plays the role that the length of the gradient of a smooth function has in the Euclidean setting. For example, the Euclidean spaces and Heisenberg groups and the more general Carnot groups admit the type of a Poincaré inequality we need. We describe the basics and discuss the associated Sobolev spaces that, for example, allow for a very abstract setting for variational integrals. We also discuss the concept of a Sobolev mapping between two metric spaces.

1. Introduction

Let X and Y be metric spaces and $f : X \to Y$ a homeomorphism. Then the distortion of f at a point $x \in X$ is

$$H(x) := \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)}\,, \tag{1}$$

where

$$\begin{aligned} L(x,r) &:= \sup\{d_Y(f(x),f(y)) : d_X(x,y) \le r\}\,,\\ l(x,r) &:= \inf\{d_Y(f(x),f(y)) : d_X(x,y) \ge r\}\,. \end{aligned}$$

We say that f is quasiconformal if there is a constant H so that $H(x) \le H$ for every $x \in X$. This infinitesimal condition is easy to state but not easy to use. For instance, it is not clear from the definition if the inverse mapping is quasiconformal as well. It is thus desirable to find conditions on X and Y that would guarantee the stronger, global requirement:

$$H(x,r) := \frac{L(x,r)}{l(x,r)} \le H' < \infty$$

for all $x \in X$ and all $r > 0$ whenever f is a quasiconformal mapping of X onto Y. We call this global condition quasisymmetry and a homeomorphism satisfying it a quasisymmetric mapping. Our notion of quasisymmetry is equivalent to the concept of a quasisymmetric function introduced by Beurling and Ahlfors in [2] when $X = Y$ is the real line and to the concept of quasisymmetry defined by Tukia and Väisälä in [28] for many metric spaces X and Y, especially for all metric spaces discussed below in theorem 1.1. It is a fundamental fact that quasiconformal mappings between Euclidean spaces of dimension at least two are quasisymmetric. This fails in dimension one; consider, for example, $f(x) = x + e^x$. It is immediate that the inverse mapping of a quasisymmetric mapping is also quasisymmetric and many other important properties of quasiconformal mappings follow as well easily from quasisymmetry.

This infinitesimal-to-global principle was shown to hold on the Heisenberg group by Koranyi and Reimann, [17], and it holds for mappings between spaces that occur as conformal boundaries of rank-one symmetric spaces by results of Mostow and Pansu, [23, 24]. This extends to the case of Carnot groups by results of Heinonen and Koskela in [11] and to the more general case of Carnot-Caratheodory spaces by the work of Margulis and Mostow, [22].

A natural metric setting that covers all the above cases turns out to be that of an Ahlfors regular metric space that supports a suitable Poincaré inequality. Here X is Ahlfors Q-regular if X is equipped with a Borel measure μ and there is a constant $C_\mu \geq 1$ such that

$$C_\mu^{-1} r^Q \leq \mu(B(x,r)) \leq C_\mu r^Q \,,$$

for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$. We also assume that X is proper: each closed ball in X is compact.

Theorem 1.1. *Let $Q > 1$. If X is a proper, Q-regular metric space that supports a Q-Poincaré inequality, then each quasiconformal self-homeomorphism of X is quasisymmetric.*

For the assumption concerning Poincaré inequalities see section 2. This result of Heinonen and Koskela is from [13] where versions with more general target can be found as well. We arrived at theorem 1.1 from results more intrinsic to quasiconformal mappings. This includes moduli of curve families and so-called Loewner spaces. The quasisymmetry gives Hölder continuity and other pleasant properties. If one assumes a p-Poincaré inequality for some $p < Q$, then the volume derivative of a quasiconformal mapping is a Muckenhoupt weight. For all this see [13]. We will now leave quasiconformal mappings with the following philosophical point of view: for quasiconformal mappings to be quasisymmetric, the infinitesimal distortion of shapes needs to integrate to global control and a Poincaré inequality tells us that the infinitesimal control of a function by its "gradient" results in global estimates.

In this paper we concentrate on metric spaces that support a Poincaré inequality. Section 2 deals with the basics of such spaces. One of the crucial concepts here is the notion of an upper gradient which will be our substitute for the length of the gradient of a smooth function. In Section 3 we discuss Sobolev spaces and notice that one obtains a rich theory as soon as a Poincaré inequality is available. The issue of a Sobolev mapping between metric spaces gets also briefly addressed. In the final short section, section 4, we comment on the the consequences related to calculus of variations in the metric setting and point out some other applications.

2. Spaces that Support Poincaré Inequalities

To simplify the things we will assume that the spaces we consider are relatively nice. Our standing assumption will be that X is a proper (recall the definition from Introduction), *doubling* metric space: X is equipped with a Borel measure μ so that

$$\mu(2B) \leq C_d \mu(B) \tag{2}$$

whenever B is a ball and $2B$ is the ball with the same center as B and with radius twice that of B. Notice that each Ahlfors regular metric space is automatically doubling. Moreover, a simple iteration argument shows that the doubling condition (2) is equivalent to the existence of a constant $C > 0$ and an exponent $s > 0$ such that

$$\frac{\mu(B)}{\mu(B_0)} \geq C \left(\frac{r}{r_0}\right)^s \tag{3}$$

whenever B_0 is an arbitrary ball of radius r_0 and $B = B(x, r)$, $x \in B_0$, $r \leq r_0$.

2.1. Upper gradients

In the abstract setting of a metric space we cannot talk about partial derivatives. However, the length of a gradient of a smooth function has a rather natural generalization.

Definition 2.1. *Let $u\colon A \to \overline{\mathbf{R}}$, $A \subset X$. An* upper gradient *of u on A is a Borel function $g\colon A \to [0, \infty]$ such that for each rectifiable path $\gamma\colon [0, l] \to A$*

$$|u(\gamma(l)) - u(\gamma(0))| \leq \int_\gamma g\, ds\,. \tag{4}$$

This definition is from [12, 13] except that there functions g as in (4) where not called upper gradients. The current terminology comes from [20].

If u is Lipschitz:

$$|u(x) - u(y)| \leq Ld(x, y)$$

for all x, y, then trivially $g \equiv L$ is an upper gradient of u, but the local Lipschitz constant provides us with a smaller upper gradient than the global Lipschitz constant: Given u, define

$$\operatorname{Lip} u(x) = \liminf_{r \to 0^+} \sup_{\{y : d(x,y) \leq r\}} \frac{|u(x) - u(y)|}{r}\,.$$

See e.g. [6].

2.2. Further Examples

1. $g \equiv \infty$ is always an upper gradient.
2. If there are no rectifiable, non-constant curves in A, then $g \equiv 0$ is an upper gradient.
3. Each function u in the usual Sobolev class $W^{1,p}(\mathbf{R^n})$ has a representative that has a p-integrable upper gradient. See e.g. [19, 26].

Notice that the existence of an upper gradient that is integrable on many curves gives good control on u: If $\int_\gamma g\, ds < \infty$, then u is continuous and real-valued on γ.

To save space we only mention here that upper gradients behave quite the way one expects, see [6, 19, 25, 26].

2.3. Poincaré inequalities

Definition 2.2. *Let (X, d, μ) be as before. X is said to support a p-Poincaré inequality, $p \geq 1$, if there exist $C > 0$ and $\lambda \geq 1$ such that*

$$\fint_B |u - u_B|\, d\mu \leq C\, diam\,(B) \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}, \tag{5}$$

for all balls B, for all continuous functions u on λB and for every upper gradient g of u on λB. Here u_B is the average of u over the ball B and the barred integrals mean averaged integrals.

When compared with the usual Poincaré inequalities in the Euclidean setting, two differences appear: instead of the (p,p)-inequality we ask for a $(1,p)$-inequality and the ball g gets integrated over is larger (when $\lambda > 1$) than the corresponding ball for u. It follows from theorem 2.4 that we could as well assume a (p,p)-inequality. It is however immediate from the $(1,p)$-formulation (using Hölder's inequality) that (5) becomes weaker as p increases whereas this is substantially harder to see from the (p,p)-formulation. In fact, (5) becomes strictly weaker when p dicreases [10, 13]. Regarding the size of balls, we can omit λ if the geometry of balls is sufficiently nice, e.g. if the metric is a length metric, but not in general, see [10].

In the Euclidean setting the Poincaré inequality for Sobolev functions follows from that inequality for smooth functions. In the abstract setting we cannot speak about smooth functions but Lipschitz functions make sense. It is then natural to ask if a Poincaré inequality for Lipschitz functions guarantees that the space supports a Poincaré inequality. This turns out to be true in great generality.

Theorem 2.3. *Assume that X is path-wise connected. Then X supports a p-Poincaré inequality if and only if (5) holds for each ball B and for all Lipschitz functions u and their upper gradients. In fact, (5) will then hold for all measurable functions u.*

Here our standing assumption that X be proper is essential (see [10]). For a proof of theorem 2.3 see [14, 19].

Recall that we required inequality (5) for every continuous function u and each upper gradient of u. It is then natural to inquire how good the functions u are for which (5) holds for some L^p-function g. According to the following result, such functions share many of the good properties of Euclidean Sobolev functions. The various constants C below are not necessarily the same as the constant in (5) but they depend only on the given data, of which u, g are not part of.

Theorem 2.4. *Let X be a doubling space and s be an exponent as in (3). Assume that $g \in L^p(X)$, $p \geq 1$, and a locally integrable u satisfy inequality (5) for all balls B.*

1. *If $p < s$, and $q < \frac{ps}{s-p}$, then*
$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leq C \, diam\,(B) \left(\fint_{5\lambda B} g^p \, d\mu\right)^{1/p} .$$
for all balls B.
2. *If X is connected and $p = s > 1$, then*
$$\fint_B \exp\left((t|u - u_B|)^{\frac{s}{s-1}}\right) \, d\mu \leq C_2 \, . \tag{6}$$
for each ball B, where
$$t = \frac{C_1 \mu(B)^{1/s}}{diam\,(B) \|g\|_{L^s(5\lambda B)}} \, .$$
3. *If $p > s$, then u (after redefinition in a set of measure zero) is locally Hölder continuous:*
$$|u(x) - u(y)| \leq C r_0^{s/p} d(x,y)^{1-s/p} \left(\fint_{5\lambda B_0} g^p \, d\mu\right)^{1/p} \tag{7}$$
for all $x, y \in B_0$, where B_0 is an arbitrary ball of radius r_0.
4. *Let $x \in X$. If $p > s-1$, then u (after redefinition in a set of measure zero) is uniformly Hölder continuous with exponent $1 - (s-1)/p$ for almost every $r > 0$ on the set $\{y : d(y,x) = r\}$.*
5. *Suppose that a sequence of pairs $(u_i, g_i)_{i \in \mathbb{N}}$ satisfies inequality (5) with uniform constants. If $||u_i||_{L^1(B)} + ||g_i||_{L^p(5\lambda B)} \leq C$ for each i for a ball B, then there exists a subsequence $(u_{i_k})_{k \in \mathbb{N}}$ that converges in $L^q(B)$ to a function u. Here one can take any $1 \leq q < ps/(s-p)$ when $p < s$ and any finite $q \geq 1$ when $p \geq s$. Moreover, when $p > 1$, there is a function g in $L^p(5\lambda B)$ so that (5) holds for the pair (u, g) (with a possibly different constant C).*

For a proof of these results see [10] (part 1 can be found already in [9]). The last conclusion in part 5 is however not covered by [10] but it easily follows by selecting a weakly convergent subsequence of the subsequence g_{i_k} and using Mazur's lemma to get a strongly convergent sequence of convex combinations of these functions.

By theorem 2.4, the Poincaré inequality (5) implies versions of Sobolev-Poincaré and Trudinger inequalities, a version of the Sobolev embedding theorem on spheres and of the Rellich-Kondrachov theorem. Besides the Sobolev-Poincaré inequality and the Sobolev embedding on spheres, we obtain as good results as in the Euclidean setting. If we assume that X supports a p-Poincaré inequality, then we have the full analog of the Sobolev-Poincaré inequality in (1) with $q = \frac{ps}{s-p}$ for function - upper gradient pairs (see [10]).

When does X then support a Poincaré inequality? A look at (4) should soon convince the reader that this is the case whenever the concept of a smooth function makes sense and one has a "usual" Poincaré inequality. This covers the case of Carnot groups and the vector field setting (c.f. [10]). Regarding necessary conditions, a Poincaré inequality implies the existence of "many" short curves, in particular, that any pair of points in X can be joined with a curve whose length is no more than a fixed constant times the distance between the points. Spaces with this property are called *quasiconvex*. For sufficient conditions and "exotic" examples see [4, 10, 13, 21, 25, 27]. We wish to stress that the spaces we consider need not be integer dimensional. Indeed, Laakso [21] constructs for each $Q > 1$ Ahlfors Q-regular spaces that support even a 1-Poincaré inequality. The first non-integer-dimensional examples were constructed by Bourdon and Pajot ([4]) as the Gromov boundaries of certain hyperbolic buildings.

2.4. Stability

One good property of the concept of a metric space supporting a p-Poincaré inequality is that it is stable both under bi-Lipschitz changes of the metric and under pointed, measured Gromov-Hausdorff convergence. The first stability is immediate; recall that $f\colon X \to Y$ is bi-Lipschitz if there is a constant L so that

$$d_X(x,y)/L \leq d_Y(f(x), f(y)) \leq L d_X(x,y)$$

for all $x, y \in X$. The second result is more complicated and essentially due to Cheeger [6]. Recall that all the spaces we consider are assumed to be proper and doubling.

Theorem 2.5. *Suppose that (X_i, d_i, μ_i) is a sequence of spaces that all support a p-Poincaré inequality with fixed C, λ and are all doubling with a fixed doubling constant C_d. If this sequence converges (as a subsequence always does) in the pointed, measured Gromov-Hausdorff sense to a space (X, d, μ), then X also supports a p-Poincaré inequality.*

Cheeger only obtains a q-Poincaré inequality for X for all $q > p$ but relying on his work one can verify even a p-Poincaré inequality, as was done in [19].

Above the Poincaré inequality, doubling and the properness of the spaces guarantee that each of the spaces X_i is quasiconvex with a fixed constant. Thus, using approriate bi-Lipschitz changes of metric, each X_i can be assumed to be a length space, and it follows that also X carries a length space metric. The requirement that the convergence be measured then means that, whenever $x_i \in X_i$

converge to $x \in X$, the measures of the balls $B(x_i, r)$, when normalized by the measure of $B(x_i, 1)$, converge to the measure of $B(x, r)$ for $r > 0$; here the balls are given in the length space metrics.

3. Sobolev Spaces

3.1. The real-valued case

Following Shanmugalingam [26] we define the Newtonian space $N^{1,p}(X)$ by

$$N^{1,p}(X) = \{u \in L^p(X) : \exists \text{ an upper gradient } g_u \in L^p(X)\}$$

and equip it with the norm

$$\|u\|_{1,p} = \|u\|_p + \inf_{g_u} \|g_u\|_p \,.$$

Naturally, we need to identify u and v if $\|u - v\|_{1,p} = 0$. Motivated by the Euclidean setting we further define

$$H^{1,p}(X) = \overline{\mathrm{Lip}\,(X)}^{\|\cdot\|_{1,p}} \,.$$

Finally we let

$$P^{1,p}(X) = \{u \in L^p(X) : \exists g \in L^p(X) \text{ such that (5) holds for the pair } u,\, g\} \,.$$

If X supports a p-Poincaré inequality, then all these spaces coincide, and the Newtonian space has many of the properties of the classical Sobolev spaces.

Theorem 3.1. *Assume that X supports a p-Poincaré inequality, $p > 1$. Then*

$$N^{1,p}(X) = H^{1,p}(X) = P^{1,p}(X)$$

coincide as sets. Moreover, $N^{1,p}(X)$ is a reflexive Banach space, and the usual Sobolev inequalities hold for functions in $N^{1,q}(X)$, $q \geq p$.

By Sobolev inequalities we mean the results from theorem 2.4 with the exponent p there any exponent larger or equal to the exponent p in theorem 3.1, except that for $p < s$ we allow $q = ps/(s-p)$ in the first of the inequalities. Other than for this improvement, the Sobolev inequalities directly follow from theorem 2.4. For the above endpoint result, one uses truncation arguments, see [10]. The reflexivity is not easy to establish. This is due to Cheeger [6]; his definition for the Sobolev space is slightly different from the above definitions but, as shown by Shanmugalingam [26], the resulting space is isometrically equivalent to $N^{1,p}(X)$. For the coincidence of the three spaces above see [7, 10, 26]. There are also other possible definitions for a Sobolev class. The class introduced by Korevaar and Schoen [18] coincides with the above Sobolev spaces under the Poincaré inequality assumption but the space introduced by Hajłasz only for exponents $q > p$, see [20].

We began by considering upper gradients that form a substitute for the length of a gradient. Very surprisingly, the Poincaré inequality guarantees that even the differential of a function makes sense. This is a remarkable result of Cheeger [6] and it is a crucial part of his proof for the reflexivity. Another amazing conclusion of Cheeger is that the "minimal upper gradient" of a Sobolev function can be realized as the point-wise Lipschitz constant. Unfortunately, we have no space to discuss this in detail here, and we have to confine ourselves to referring to his paper.

3.2. Sobolev mappings

We originally got interested in Sobolev functions on metric spaces because of the need for tools suitable to handle quasiconformal mappings. For the basic questions, like theorem 1.1, the real-valued theory was sufficient: the trick to be used was to replace "locally" a mapping f with $u(x) = d_Y(f(x), f(x_0))$, where x_0 is a "locally" fixed base point. The limits in this approach showed in that we were not able to obtain an analytic definition for quasiconformality under the Q-Poincaré inequality assumption. The above switch from a mapping f to a real-valued function u can be used to define the class of (local) Sobolev mappings from X to Y, but a better definition is obtained using post-composition with Lipschitz functions. This replaces the mapping with a family of real-valued functions. Our obstacle was the lack of Lipschitz approximations to our quasiconformal mapping.

The key for us was to embed Y isometrically into $\ell^\infty(Y)$ to gain linear structure; this can be done for any metric space Y. This allows us to conveniently invoke the vector-valued integration theory of Bochner and Pettis. The nice thing here is that the validity of a real-valued p-Poincaré inequality on X is equivalent with that for the $\ell^\infty(X)$-valued case, see [15]. Even though usual convolutions cannot be used, one can still use certain "discrete" convolutions to approximate our quasiconformal mapping to show that the mapping belongs to the desired Sobolev class. As an application to the theory of quasiconformal mappings we have the following consequence.

Theorem 3.2. *Let X be a proper, Q-regular metric space that supports a Q-Poincaré inequality and f a self-homeomorphism of X. Then f is quasiconformal if and only if $f \in N^{1,Q}_{loc}(X,X)$ and*

$$Lip\, f(x)^Q \leq K \mu_f(x)$$

for a.e. $x \in X$.

Here $\operatorname{Lip} f(x)$ is a local Lipschitz constant of f, this time defined as

$$\operatorname{Lip} f(x) = \limsup_{r\to 0^+} \sup_{\{y:d(x,y)\leq r\}} \frac{d(f(x), f(y))}{r}$$

and

$$\mu_f(x) = \limsup_{r\to 0^+} \frac{\mu(fB(x,r))}{\mu(B(x,r))}$$

is the volume derivative of f. For this result see [15].

4. Minimization Problems and other Related Topics

By the reflexivity of the Sobolev space $N^{1,p}(X)$ under the p-Poincaré inequality assumption, direct methods from the calculus of variations are available. Thus one has the existence and uniqueness for solutions to the Dirichlet problem (c.f. [6]).

What about the regularity of the solutions? It seems plausible that one could use the Moser iteration scheme to obtain Harnack's inequalities for positive solutions and as a consequence Hölder continuity of the solutions. There is, however, a hidden danger which surfaces when trying to establish Caccioppoli-type inequalities: the norm on the differentials need not necessarily be smooth. This problem can be overcome by using De Giorgi's method, and the solutions are indeed Hölder continuous; see [16]. There are also recent results on the boundary continuity [3].

Another area where Poincaré inequalities in our sense have appeared is geometric measure theory. This relates to the borderline case $p = 1$ of Sobolev functions where functions of bounded variation occupy the theory. Ambrosio [1] studies properties of sets of finite perimeter in Ahlfors regular metric spaces that support a 1-Poincaré inequality. For applications to the setting of the Heisenberg groups see [8]. It is not a big suprpise that the 1-Poincaré inequalities surface here: there is an intimate connection between isoperimetric and 1-Poincaré inequalities as observed already years ago by Maz'ya, Federer and Fleming.

Let us close this note by going back to the origin for our motivation to look at quasiconformal mappings and Sobolev functions in the metric setting. The quasiconformal mappings appeared for the first time in non-Riemannian setting in the work of Mostow on rigidity of symmetric spaces. It would thus be desireable that the metric theory of quasiconformal mappings also would result in rigidity results. This is indeed the case: Bourdon and Pajot [5] have very recently obtained ridigity results for hyperbolic buildings using the metric theory of quasiconformal mappings.

References

[1] L. Ambrosio, *Some fine properites of sets of finite perimeter in Ahlfors regular metric measure spaces*, Scuola Normale Superiore Pisa, Preprint (2000).

[2] A. Beurling and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142

[3] J. Björn, *Boundary continuity for quasi-minimizers on metric spaces*, Linköping University, Department of Mathematics, Preprint #1 (2000).

[4] M. Bourdon and H. Pajot, *Poincaré inequalities and quasiconformal structure on the boundaries of some hyperbolic buildings*, Proc. Amer. Math. Soc. **127** (1999), 2315–2324.

[5] M. Bourdon and H. Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings*, preprint (1999).

[6] J. Cheeger, *Differentiability of Lipschitz functions on metric spaces*, Geom. Funct. Anal. **9** (1999), 428–517.

[7] B. Franchi, P. Hajłasz and P. Koskela, *Definitions of Sobolev classes on metric spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1903–1924.

[8] B. Franchi, R. Serapioni and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, preprint (1999).

[9] P. Hajłasz and P. Koskela, *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris **320** (1995), 1211–1215.

[10] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, to appear in Mem. Amer. Math. Soc., Max Planck Institute, Leipzig Preprint #41 (1998).

[11] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. math. **120** (1995), 61–79

[12] J. Heinonen and P. Koskela, *From local to global in quasiconformal structures*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), 554–556.

[13] J. Heinonen and P. Koskela, *Quasiconformal maps on metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[14] J. Heinonen and P. Koskela, *A note on Lipschitz functions, upper gradients and the Poincaré inequality*, New Zealand J. Math. **28** (1999), 37–42.

[15] J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, University of Jyväskylä, Department of Mathematics, Preprint #218 (2000).

[16] J. Kinnunen and N. Shanmugalingam, *Quasi-minimizers on metric spaces*, Mittag-Leffler Institute Preprint #6 (1999/2000).

[17] A. Korányi and H. M. Reimann, *Foundations for the theory of quasiconformal mappings on the Heisenberg group*, Adv. Math **111**, (1995), 1–87.

[18] N. J. Korevaar and R. M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), 561–659.

[19] P. Koskela, *Upper gradients and Poincaré inequalities*, lectures in Trento in June, 1999.

[20] P. Koskela and P. MacManus, *Quasiconformal mappings and Sobolev spaces*, Studia Math. **131** (1998), 1–17.

[21] T. Laakso, *Ahlfors Q-regular spaces with arbitrary Q admitting weak Poincaré inequality*, Geom. Funct. Anal. **10** (2000), 111–123.

[22] G. A. Margulis and G. D. Mostow, *The differential of a quasiconformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. **5** (1995), 402–433

[23] G. D. Mostow, *A remark on quasiconformal mappings on Carnot groups*, Michigan Math. J. **41** (1994), 31–37.

[24] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. Math. **129** (1989), 1–60.

[25] S. Semmes, *Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities*, Selecta Math. **2**, (1996), 155–295.

[26] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces*, to appear in Rev. Mat. Iberoamericana.

[27] K. T. Sturm, *Diffusion processes and heat kernels on metric spaces*, Ann. Probab. **26** (1998), 1–55.

[28] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Math. **5** (1980), 97–114.

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