

Moduli, Motives, Mirrors

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1. Introduction

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1.1. Brief history

Mathematical history of Mirror Symmetry started in 1991, when an identity of a new type was discovered in the ground-breaking paper by four physicists [20] (it was reproduced in [85] where earlier works are also described and motivated).

The left hand side (or *A*-side) of this identity was a generating series for the numbers $n(d)$ of rational curves of various degrees d lying on a smooth quintic hypersurface in \mathbf{P}^4 . The right hand side (*B*-side) was a certain hypergeometric function. The Mirror Identity states that the two functions become identical after an explicit change of variables which is defined as a quotient of two hypergeometric functions of the same type.

At the moment of discovery, not only the identity itself remained unproved, but even its *A*-side was not well defined: the correct way of counting rational curves was proposed by M. Kontsevich ([60]) only in 1994. In the same remarkable paper Kontsevich gave an explicit formula for $n(d)$ creatively using Bott's fixed point formula for torus actions at the target space. After the appearance of this paper one could hope that the Mirror Identity for quintics (and more general toric submanifolds) ought to be provable by algebraic manipulations with both sides. This turned out to be a difficult problem. A. Givental brought this program to a successful completion in 1996, by introducing a new torus action at the source space, stressing equivariant cohomology and inventing ingenious calculational strategy (see [40, 43, 45, 15, 90]). For subsequent important developments, see [66, 67, 14].

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This work however did not unveil the mystery of the Mirror Identity. The point is that the identity itself was discovered by the physicists as only one manifestation of a deeper principle. Physicists believe that with any Calabi-Yau manifold X one can associate two $N = (2, 2)$ Superconformal Field Theories (SCFT) which are the respective A and B models (see e.g. [105]). The Mirror Correspondence between X and Y supposedly interchanges their A and B models. In particular, in the case of quintics the hypergeometric functions involved are actually periods of the mirror partner family of our quintics, and B -models generally reflect properties of variations of periods and Hodge structures.

Unfortunately, a precise and complete mathematical definition of what constitutes an $N = (2, 2)$ ScFT is still lacking. Various components of this structure with varying degree of precision are described in the papers collected in [85] and [86]. In particular, a part of this structure is a modular functor in the sense of Segal, with possibly infinite dimensional Hilbert space. In turn, such theories are often constructed via representation theory of a vertex algebra. See [73] and [17, 18] for the most recent mathematical approach to this picture, achieving at least the construction of what seems to be the right vertex algebra.

The parts that are involved in the statement of Mirror Identity above refer correspondingly to the Quantum Cohomology (A -model, physicists' σ -model) and extended variations of Hodge structure. Both are now well understood mathematically: see [79] and [4] respectively. However, the Mirror partners are connected by much more ties than a mere Mirror Identity. These ties, in particular, relate Lagrangian and complex geometry in a remarkable way: see [96] and [58] for the basic conjectures to this effect.

Therefore now, more than decade after it was discovered, the Mirror Symmetry mathematically looks like a complex puzzle, some of the pieces of which have found their respective places, some are still lying in disorder, and some, most probably, are missing.

1.2. Plan of the paper

This puzzle metaphor guided the organization of this report.

Section 2 is devoted to the binary relation of *mirror partnership* between families of Calabi-Yau manifolds endowed with additional structures which we call here *cusps*. This relation consists in the isomorphism of two Frobenius manifolds, constructed by two different ways for the respective families. In turn, Frobenius manifold isomorphisms generalize the Mirror Identity of [20].

Section 3 explains various versions of another mirror partnership relation, this time between certain symplectic, on the one hand, and complex, on the other hand, manifolds, endowed with additional structure which in this case is a choice of a *fibration by real tori*. Here I have took as starting point a part of Kontsevich's package [58], with further detalization taken from [96, 93, 2], and other papers. I have chosen for these relations the word "partnership", or "duality", as opposed to "symmetry", because the definition of both of them is explicitly un-symmetric.

The next part of the section 3 restores the idea of Mirror Symmetry: according to [96], the Mirror Symmetry relation connects Calabi-Yau manifolds endowed with Kähler structure, and thus simultaneously with compatible Lagrangian and complex structures, so that Kontsevich's duality can be imposed simultaneously upon two crossover Lagrangian/complex pairs.

Although the Frobenius manifold duality and the Lagrangian/complex duality some day are expected to become parts of a unified picture, at present contours of the latter are rather vague.

One common part of both dualities is the prediction of *mirror isomorphisms* connecting the cohomology spaces of mirror partners (X, Y) . In particular, isomorphisms of the Frobenius manifolds restricted to their spaces of flat vector fields produces isomorphisms $\mu_{X,Y}: H^*(X, \mathbf{C}) \rightarrow H^*(Y, \wedge^*(T_Y) \otimes V_Y^{-2})$ where $V_Y = H^0(Y, \Omega_Y^{\max})$.

Actually, algebraic geometric model of the *A*-side, the theory of Gromov-Witten invariants, endows $H^*(X, \mathbf{C})$ with much stronger structure, which is motivic by its nature. It would be very interesting to understand the geometry of the mirror reflection of Calabi-Yau motives.

However, even when the intricate inner workings of Mirror Symmetry are understood, this will not be the end of the story.

1.3. Dualities in string theory

All this machinery emerged as an approximation in the quantum superstring theory whose aim is to provide a unified theory of matter and gravity (space-time). The first superstring revolution (1984–85) led to the belief that there are five consistent (perturbative, without ultraviolet divergencies) superstring theories, each on ten-dimensional space-time, or in other words, 10d Poincaré-invariant vacuum. For all of them, the low-energy approximation is an effective 10d supergravity theory. The second superstring revolution (1994–??) started with the Witten's suggestion that all five theories are limits of a single theory (see review in [94]). In other words, they are perturbative expansions of a single underlying theory about distinct points in the moduli space of quantum vacua. Moreover, a sixth special point in this space is a 11d Poincaré-invariant vacuum.

C. Vafa suggests to look at the underlying *M-theory* as patched up from five/six local descriptions and their compactifications, in much the same way as a manifold is patched up from coordinate neighborhoods. The transition functions are called *dualities*, and according to [96], Mirror Duality is one of them.

If this is true, the Mirror Symmetry acquires an incredibly high epistemological status as one of the building blocks of the ambitious Unified Quantum Superstring Theory.

For mathematicians, this means that the puzzle we are trying to assemble, is only a small piece of the still larger puzzle whose contours are yet barely visible.

2. Frobenius Manifolds and Mirror Partnership between Families of Calabi-Yau Manifolds

2.1. Calabi-Yau manifolds

In this section I will call a Calabi-Yau (CY) manifold *in the weak sense* any projective (or compact Kähler) complex manifold X with trivial canonical sheaf. Any such manifold admits a finite unramified covering \tilde{X} with the following property:

\tilde{X} is a direct product of a complex torus A , of a simply connected CYY with $h^{2,0}(Y) = h^{2,0}(\tilde{X})$, and of a simply connected CY Z with $h^{2,0}(Z) = 0$.

If the factors A and Y are absent in any finite unramified covering of X , then X is CY *in the strong sense*.

In dimension 1, the only CY manifolds are elliptic curves, in dimension 2, besides complex tori, there are K3-surfaces. In dimension 3, the first examples of CY in the strong sense appear. Quintics in \mathbf{P}^4 are the simplest of them.

More generally, anticanonical hypersurfaces in any compact toric manifold associated with a *reflexive polyhedron* are CY's: see [6]. This method produces 4319 families of K3-surfaces and 473 800 776 families of CY threefolds, among which at least 30 178 families can be distinguished by their Hodge numbers (see [63]). It is still unknown, whether the number of maximal families of CYs in any dimension ≥ 3 is finite or not.

The toric construction (and its generalization to complete intersections in arbitrary Fano manifolds) remain the most important testing ground for basic conjectures about CYs.

A general approach to the complex moduli spaces of CY manifolds is furnished by the deformation theory. The Kodaira-Spencer local versal deformation of an n -dimensional CY manifold X is unobstructed and has dimension $h^{1,n-1}(X)$.

2.2. Mirror partner families of CYs: preliminaries

The notion which we will describe in this section is interesting mainly for CYs in the strong sense. It develops the discovery made in [20].

This notion is an asymmetric binary relation between versal local families $\{X_s \mid s \in S\}$, $\{Y_t \mid t \in T\}$ of CYs, satisfying the condition $h^{1,1}(X) = h^{1,n-1}(Y) = r$ and endowed with some additional structure.

On the A -side, this additional structure consists in a choice of a basis $(\beta_1, \dots, \beta_r)$ of the group of numerically effective classes in $A_1(X_s)$. When $s \in S$ varies, elements of this basis must be horizontal with respect to the Gauss-Manin connection. Such a basis determines functions q_j^A on $H^2(X_s, \mathbf{C}/\mathbf{Z})$: $q_j^A(L) := e^{2\pi i(L, \beta_j)}$. We will refer to $H^2(X_s, \mathbf{C}/\mathbf{Z})$ together with q_j^A as a *Kähler cusp*.

On the B -side, this additional structure consists in the choice of a partial compactification $T \subset \bar{T}$ looking locally like the embedding of a product of pointed open unit discs in \mathbf{C} into the product of non-pointed unit discs. The variation of Hodge structures of the family Y_t must have maximal unipotent monodromy on this compactification: see [88, 89, 27]. Geometrically, \bar{T} contains a point of

“maximal degeneration” of the family Y_t and T parametrizes Calabi-Yau manifolds “with large complex structure”. The point of maximal degeneration is the transversal intersection of discriminantal divisors. Building upon [20] and [87, 88], Deligne has shown in [27] how to define a system of functions q_j^B on \overline{T} in terms of the variation of Hodge structures determined by Y_t . We will refer to the germ of \overline{T} at its point of maximal degeneration as *moduli cusp* of the relevant moduli space.

The relation of *mirror partnership* between such enhanced families X/S and Y/T , in particular, identifies q_j^A with q_j^B and thus establishes an isomorphism between a domain in $H^2(X, \mathbf{C}/\mathbf{Z})$ where q_j^A are sufficiently small and the respective domain in the moduli cusp. This isomorphism must identify two functions: *potential of the small quantum cohomology* at the A -side, and an integral involving a holomorphic volume form on the fibers Y_t at the B -side.

A fuller formulation of the mirror partnership relation consists in the identification of two *formal Frobenius manifolds (FM)*: quantum cohomology of any X_s at the A -side, and Barannikov-Kontsevich’s FM on a formal *extended moduli space* at the B -side.

Most of the remaining part of this section will be devoted to the description of the relevant Frobenius manifolds.

However, the reader must be aware of more global aspects of this essentially local picture. In fact, moduli stack of complex variations of Y may have many cusps; they are acted upon by the Teichmüller group $\text{Diff}(Y)/\text{Diff}_0(Y)$. One can speculate that mirror partnership is stable with respect to such moduli cusp changes. Then the question arises, what corresponds to them at the A -side.

Two partial answers were suggested. In [3] it was argued that different birational models of some X_s can produce canonically isomorphic cohomology groups, in particular H^2 , in which however Kähler cones will form a non-trivial fan (in the sense of toric geometry). Maximal cones of this fan support Kähler cusps that might correspond to different moduli cusps of the same family at the B -side. In this picture, one does not see what should correspond to the Teichmüller group. M. Kontsevich suggested in the framework of his conjectured Lagrangian/complex duality that it must be the autoequivalence group of the derived category of coherent sheaves on X_s . Some evidence for this was furnished by comparison of the stabilizing subgroups of the cusps: see [52, 95].

2.3. Frobenius manifolds

Let M be an analytic or formal supermanifold. A structure of the Frobenius manifold on it is given by a flat metric g (symmetric non-degenerate form on the tangent sheaf) and a function (potential) Φ with the following property. Let (x_a) be a local g -flat coordinate system, $\partial_a = \partial/\partial x_a$, $\Phi_{abc} = \partial_a \partial_b \partial_c \Phi$. Raise one index of Φ_{abc} using g and define an \mathcal{O}_M -bilinear multiplication \circ on \mathcal{T}_M by $\partial_a \circ \partial_b := \sum_c \Phi_{ab}{}^c \partial_c$. Then this multiplication must be associative (it is obviously (super)commutative). Additional structures that are present in the mirror picture are the flat identity e for \circ and an Euler vector field E satisfying the conditions $\text{Lie}_E(g) = Dg$ for some

constant D and $\text{Lie}_E(\circ) = \circ$. It expresses homogeneity properties of Φ : we have $E\Phi = (D+1)\Phi + \text{a polynomial in flat coordinates of degree } \leq 2$.

At the A -side, the relevant Frobenius manifold is formal: M is the formal completion of the linear space $H^*(X_s, \mathbf{C})$, with its Poincaré pairing as g and the potential Φ constructed as formal series whose Taylor coefficients are Gromov-Witten invariants of X_s . At the B -side the relevant Frobenius manifold can be conceived as a certain formal neighborhood of the classical moduli space T near the relevant cusp: *extended moduli space* of the B -family. Both formal spaces can be refined to germs of analytic spaces.

Here are some details.

2.4. Quantum cohomology

Potential Φ of the quantum cohomology can be defined for any projective complex (or compact symplectic) manifold X . After taking into account the relevant homogeneity properties of the Gromov-Witten invariants it can be written as a formal series in linear coordinates (x^a) on $\oplus_{k \neq 2} H^k(X, \mathbf{C})$ and their exponentials on $H^2(X)$:

$$\Phi(x) = \frac{1}{6} \left(\left(\sum x_a \Delta_a \right)^3 \right) + \sum_{\substack{\beta \neq 0 \\ (\beta, \sum |\Delta_b|=2 \sum x_b \Delta_b)}} e^{(\beta, \sum |\Delta_b|=2 \sum x_b \Delta_b)} \sum_{n \geq 0, (a_i): |\Delta_{a_i}| \neq 2} \langle \Delta_{a_n} \dots \Delta_{a_1} \rangle_{0,n,\beta} \frac{x_{a_1} \dots x_{a_n}}{n!}. \quad (1)$$

Here (Δ_a) is the basis of $H^*(X, \mathbf{C})$ dual to (x_a) , $\Delta_k \in H^{|\Delta_k|}(X)$, the first term in the rhs of (1) is the cubic self-intersection index, β runs over numerically effective 1-classes in X . Finally, the Gromov-Witten invariant $\langle \Delta_{a_n} \dots \Delta_{a_1} \rangle_{0,n,\beta}$ counts virtual number of stable maps of genus zero $(C; x_1, \dots, x_n; f: C \rightarrow X)$ such that $f_*([C]) = \beta$ and $f(x_i) \in D_{a_i}$ where D_{a_i} is a cycle representing homology class dual to Δ_{a_i} . Physically, Δ_a are called *the primary fields* of the respective Conformal Field Theory, and the Gromov-Witten invariants are their *correlators*.

The *small quantum cohomology potential* is obtained by restricting $\Phi(x)$ to H^2 , that is, putting $x_a = 0$ for $|\Delta_a| \neq 2$.

2.5. Barannikov-Kontsevich's construction

On the B -side, the relevant formal Frobenius potential is constructed on the completion at zero of the cohomology space $H^*(Y, \wedge^*(T_Y))$ interpreted as a formal moduli space \mathcal{M}_{A_∞} of A_∞ -deformations of Y . This construction was introduced in [4]; it refines the earlier proposal from [5]. Unlike the case of quantum cohomology, here it is essential to require Y to be a (weak) Calabi-Yau manifold. This condition will be used, in particular, through a choice of the global holomorphic volume form Ω on Y .

This geometric setup produces first of all an algebraic object $(\mathcal{A}, \delta, \Delta, f)$, *special differential Batalin-Vilkovyski algebra (dBV)*, consisting of the following data which we will describe in axiomatized form.

(i) \mathcal{A} is a supercommutative \mathbf{C} -algebra.

In the Calabi-Yau setup, $\mathcal{A} = \Gamma_{C^\infty}(Y, \wedge^*(\overline{T}_Y^*) \otimes \wedge^*(T_Y))$.

(ii) δ is an odd \mathbf{C} -derivation of \mathcal{A} , $\delta^2 = 0$.

In our case, $\delta = \overline{\partial}$, the operator defining the complex structure on Y and its tangent bundle, so that \mathcal{A} is the Dolbeault resolution of the exterior algebra of the tangent bundle.

Therefore, the δ -cohomology space $H = H(\mathcal{A}, \delta) = \text{Ker } \delta / \text{Im } \delta$ in our case is identified with coherent cohomology $H^*(Y, \wedge^*(T_Y))$. Generally, we assume it to be of finite dimension.

The space H plays the central role, because it will support the structure of the formal Frobenius manifolds.

We will denote by $K = \mathbf{C}[[x_a]]$ the ring of formal functions on H , (x_a) being coordinates on H dual to a basis (Δ_a) .

(iii) Δ is another odd differential, $\Delta^2 = 0$, which is a differential operator of order two with respect to the multiplication in \mathcal{A} .

More precisely, we assume that for any $a \in \mathcal{A}$ the formula

$$\partial_a b = (-1)^{\bar{a}} \Delta(ab) - (-1)^{\bar{a}} (\Delta a)b - a\Delta b$$

defines a derivation ∂_a . Moreover, we assume that $\delta\Delta + \Delta\delta = 0$.

In our case, Δ is obtained from the ∂ -operator on the complexified C^∞ de Rham complex of Y after the identification of this complex with \mathcal{A} with the help of Ω : $\Delta(a) := (\vdash \Omega)^{-1} \circ \partial \circ (a \vdash \Omega)$.

From the $\partial\overline{\partial}$ -Lemma in Kähler geometry, it follows that the two canonical embeddings of differential spaces

$$(\text{Ker } \Delta, \delta) \rightarrow (\mathcal{A}, \delta), \quad (\text{Ker } \delta, \Delta) \rightarrow (\mathcal{A}, \Delta) \quad (2)$$

are quasi-isomorphisms, and moreover, homology of all four differential spaces can be identified with $(\text{Ker } \Delta \cap \text{Ker } \delta) / \text{Im } \delta\Delta$.

As a part of this package, one also obtains the following formality property: the natural map $\text{Ker } \Delta \rightarrow H(\mathcal{A}, \Delta)$ induces surjection of differential Lie algebras which is a quasi-isomorphism:

$$(\text{Ker } \Delta, [\bullet], \delta) \rightarrow (H(\mathcal{A}, \Delta), 0, 0).$$

In the axiomatized situation, we impose these conditions as an additional axiom. This condition can be weakened: it suffices to require only that cohomology of differentials $\delta + \Delta$ and δ have the same dimension.

(iv) $\int: \mathcal{A} \rightarrow \mathbf{C}$ is a linear functional which must satisfy two integration by parts identities:

$$\int (\delta a)b = (-1)^{\bar{a}+1} \int a\delta b, \quad \int (\Delta a)b = (-1)^{\bar{a}} \int a\Delta b. \quad (3)$$

The integral is given by the formula

$$\int a = \int_Y (a \vdash \Omega) \wedge \Omega \quad (4)$$

where Ω means a holomorphic volume form on Y whose period over the unique monodromy invariant cycle at the chosen cusp is $(2\pi i)^d$, $d = \dim Y$.

(v) *Algebra grading* $\mathcal{A} = \oplus \mathcal{A}^n$, $\mathbf{C} \in \mathcal{A}^0$.

We assume that with respect to this grading, δ and Δ are of degree 1, and \int has a definite degree. (This is at variance with [78, 79], but agrees with [4].)

Grading produces an Euler field on H , whereas the image of $1 \in \mathcal{A}$ serves as flat identity.

In the Calabi-Yau setup, we can grade $\wedge^p \overline{T}_Y^* \otimes \wedge^q T_Y$ by $q - p$.

2.5.1. FROBENIUS STRUCTURE Having thus described the formal properties of a Batalin-Vilkovyski algebra $(\mathcal{A}, \delta, \Delta, \int)$, we can now explain the derivation of the Frobenius structure on H .

One starts with checking that the bilinear operation $[a \bullet b] = \partial_a b$, together with multiplication, endows \mathcal{A} by the structure of *Gerstenhaber, or odd Poisson superalgebra*, in which the Lie bracket is a parity changing operation, and all the usual axioms are valid after inserting appropriate signs.

The basic ingredient of the construction from [4] is a certain exponential map Φ^W . In the Calabi-Yau setup it is an A_∞ -analog $\mathcal{M}_{A_\infty} \rightarrow H^*(Y, \mathbf{C})[[\hbar^{-1}, \hbar]][d]$ of the classical period map. Roughly speaking the map Φ^W is described by the formula

$$\Phi^W(x_a, \hbar) = \left[\exp \frac{1}{\hbar} \tilde{\Gamma} \right]$$

where $\tilde{\Gamma} \in \mathcal{A} \hat{\otimes} K[[\hbar]]$ is a W -normalized generic solution to the Maurer-Cartan equation $(\delta + \hbar \Delta) \tilde{\Gamma} + \frac{1}{2} [\tilde{\Gamma} \bullet \tilde{\Gamma}] = 0$ and $[a]$ denotes the cohomology class with respect to the differential $\delta + \hbar \Delta$. Here δ and Δ are assumed to be extended to $\mathcal{A} \hat{\otimes} K[[\hbar]]$ by linearity and $\tilde{\Gamma}$ is supposed to be W -normalized generic in the following sense: firstly, $\left[\exp \frac{1}{\hbar} \tilde{\Gamma} \right] \in 1 + L_W$, where L_W , $\hbar^{-1} L_W \subset L_W$ is semi-infinite subspace associated with an increasing isotropic filtration on cohomology of $\delta + \Delta$, and, secondly, the map $(\Phi^W - 1) \bmod (\hbar^{-1} L_W) : H \rightarrow L_W / \hbar^{-1} L_W$ is linear and is an isomorphism. In the Calabi-Yau setting W is the monodromy weight filtration associated with the relevant cusp. Existence of such solution $\tilde{\Gamma}$ for W satisfying certain transversality condition can be proved by induction on the order of coefficients of Taylor expansion.

As a matter of fact, at this stage this construction exhibits certain common features with the K. Saito's construction of FM structures on unfolding spaces of singularities. It seems that if one chooses for W a certain special filtration then the primitive form from the K. Saito theory can be identified with an analog of $\Phi^W(x_a, \hbar)$. The existence of a primitive form in K. Saito's theory is a nontrivial fact which follows in general from the theory of mixed Hodge modules of M. Saito.

Let us put now $\Gamma = \tilde{\Gamma}(x^a, \hbar = 0)$ and $\delta_\Gamma := id \otimes \delta + [\Gamma \bullet]$. The operator δ_Γ is a homological differential acting on $\mathcal{A}_K := K \hat{\otimes} \mathcal{A}$. By continuity, one can canonically identify $H(\mathcal{A}_K, \delta_\Gamma)$ with $K \otimes H$. On the other hand, multiplication in \mathcal{A}_K induces

a multiplication on $H(\mathcal{A}_K, \delta_\Gamma)$. This is our \circ . The map $\Phi^W(x_a, \hbar)$ induces a pairing on the tangent sheaf to H :

$$\langle \partial_a, \partial_b \rangle^W := \int \partial_a \Phi^W(x_a, \hbar) \partial_b \Phi^W(x_a, -\hbar).$$

The properties of the map Φ^W imply that this pairing is constant: $\langle \partial_a, \partial_b \rangle^W = g_{ab}$. This is our flat metric.

2.5.2. MIRROR IDENTITIES FOR COMPLETE INTERSECTIONS IN PROJECTIVE SPACES After these preparations, Barannikov’s proof runs as follows. Barannikov invokes the famous Givental’s result ([40, 43, 66]) establishing the mirror identity on the level of “small quantum cohomology” (restriction to H^2) replacing A -model, and classical moduli space replacing B -model. This furnishes identification of a part of Gromov-Witten invariants as coming from the relevant Picard-Fuchs equations. Now, Kontsevich-Manin’s “First reconstruction theorem” from [61] shows that this part suffices for the identification of the remaining invariants as soon as we know that Associativity Equations (= Frobenius structure) hold. In dimension 3 the latter supply no additional information, but the larger dimension is, the more important Associativity Equations become.

2.5.3. EXTENDED MODULI SPACES The context of Mirror Symmetry served to increase awareness of the importance of *extended moduli spaces* in many other contexts of algebraic geometry. Roughly speaking, any classical deformation problem is governed by a cohomology group H^k classifying infinitesimal extensions and the next cohomology group H^{k+1} classifying obstructions. In the stable and unobstructed case, H^k is the tangent space to the base of versal deformation. Extended moduli space in the unobstructed case has total cohomology H^* as tangent space. Barannikov-Kontsevich’s B -model is such an extended moduli space for Calabi-Yau manifolds.

See [62, 23, 74] for a discussion of this matter in general, and [84] for interesting constructions, related to the Frobenius structure.

2.6. Other mirror isomorphisms

There exist isomorphisms of auxiliary Frobenius manifolds connecting certain unfolding spaces of singularities (B -model) and moduli spaces of curves with spin structure (A -model) respectively, as was suggested by Witten [106] and mathematically developed in [55, 56]. See also [78] about possible relations to the Calabi-Yau mirror picture, developing the context in which the Mirror Symmetry was first discussed in [36, 37].

3. Lagrangian/Complex Duality and Mirror Symmetry

3.1. Classical phase spaces

Consider a C^∞ symplectic manifold (X, ω) , endowed with a submersion $p_X: X \rightarrow U$ whose fibers are Lagrangian tori, and a Lagrangian section $0_X: U \rightarrow X$. This is

the classical setup of action-angle variables in the theory of completely integrable systems.

The form ω identifies the bundle of Lie algebras of the tori $p_X^{-1}(u)$, $u \in U$, with the cotangent bundle T_U^* . Hence T_U^* can be seen as fiberwise universal cover of X , and we have a canonical isomorphism $X = T_U^*/H$ where H is a Lagrangian sublattice in T_U^* with respect to the lift of ω which is the standard symplectic form on the cotangent bundle. There exists also a canonical flat symmetric connection on T_U^* for which H is horizontal.

Put $H^t = \text{Hom}(H, \mathbf{Z})$. This local system is embedded as a sublattice into T_U , and we can define the *mirror partner* of $(p_X: X \rightarrow U, \omega, 0_X)$ as the toric fibration $Y := T_U/H^t$ endowed with the projection to the same base $p_Y: Y \rightarrow U$ and the zero section 0_Y .

3.2. Complex structure on Y

Passing from X to Y we have lost the symplectic form. To compensate for this loss, we have acquired a complex structure $J: T_Y \rightarrow T_Y$ which can be produced from $(p: X \rightarrow U, \omega, 0_X)$ in the following way. The flat connection on T_U obtained by the dualization from T_U^* produces a natural splitting $T_Y = p_Y^*(T_U) \oplus p_Y^*(T_U)$. With respect to this splitting, J acts as $(t_1, t_2) \mapsto (-t_2, t_1)$.

Conversely, suppose that we have a complex manifold Y endowed with a fibration by real tori $Y \rightarrow U$ with zero section, such that the operator of complex structure along the zero section identifies T_U with the bundle of Lie algebras of fibers. Then we can consecutively construct the lattice $H^t \subset T_U$, the dual fibration $X := T_U^*/H$ and the symplectic form on X coming from the cotangent bundle.

3.3. Fourier-Mukai transform and further relationships between Lagrangian and complex geometry

Consider first a pair of dual real tori $\mathbf{T} = H_{\mathbf{R}}/H$ and $\mathbf{T}^t = H_{\mathbf{R}}^t/H^t$ where H is a free abelian group of finite rank, H^t the dual group. Denote by $\langle \cdot, \cdot \rangle$ the scalar product $H^t \times H \rightarrow \mathbf{Z}$ and its real extensions. Each point $x^t \in \mathbf{T}^t$ can be interpreted as a local system of one dimensional complex vector spaces with monodromy $\pi_1(\mathbf{T}) = H \rightarrow S^1: h \mapsto e^{2\pi i \langle x^t, h \rangle}$. Hence \mathbf{T}^t becomes the moduli space of such systems on \mathbf{T} , and similarly with roles of \mathbf{T} and \mathbf{T}^t reversed.

This can be conveniently expressed by introducing the *Poincaré bundle* $(\mathcal{P}, \nabla_{\mathcal{P}})$ on $\mathbf{T} \otimes \mathbf{T}^t$ which is rank one complex bundle with connection. The connection is flat along both projections, but has curvature $2\pi i \langle \partial^t, \partial \rangle$ on $(\partial^t, \partial) \in H^t \times H$.

Using $(\mathcal{P}, \nabla_{\mathcal{P}})$, we can extend the correspondence between points of \mathbf{T} and local systems on \mathbf{T}^t in the following way. Call a skyscraper sheaf \mathcal{F} on \mathbf{T} a sheaf consisting of a finite number of vector spaces F_i supported by points x_i . We can define a functorial map

$$\mathcal{F} \mapsto p_{\mathbf{T}^t*}(p_{\mathbf{T}}^*(\mathcal{F}) \otimes \mathcal{P}) \quad (5)$$

whose image, if one takes in account the induced connection, is a *unitary local system on \mathbf{T}^t* , that is, a complex vector bundle with flat connection and semisimple monodromy with eigenvalues in S^1 .

Let now X and Y be mirror partners in the sense of 3.1–3.2. The construction above shows first of all that points y of Y bijectively correspond to pairs consisting of a Lagrangian torus $L = p_X^{-1}(p_Y(y))$ and a unitary local system of rank one on it.

Moreover, $X \times_U Y$ carries the relative Poincaré bundle which we again will denote $(\mathcal{P}, \nabla_{\mathcal{P}})$: connection is extended in an obvious way in the horizontal directions. An appropriate relative version of skyscraper sheaves is played by pairs (L, \mathcal{L}) consisting of a Lagrangian submanifold of X transversal to the tori and a unitary local system \mathcal{L} on L . The Fourier transform (5) of such a system is defined by

$$(L, \mathcal{L}) \mapsto p_{Y*}(p_L^* \mathcal{L} \otimes (i \times \text{id})^* \mathcal{P}) \quad (6)$$

where we denote by $i: L \rightarrow X$ the Lagrangian immersion, and $p_Y: L \times_U Y \rightarrow Y$, $i \times \text{id}: L \times_U Y \rightarrow Y$, $p_L: L \times_U Y \rightarrow L$. The image of (6) also carries the induced connection. We can calculate the $\bar{\partial}$ -component of it in the complex structure of Y and find out that it is flat. In other words, the rhs of (6) is canonically a holomorphic vector bundle on Y .

3.3.1. AN EXAMPLE: MIRROR DUALITY BETWEEN COMPLEX OR p -ADIC ABELIAN VARIETIES In this subsection we propose a definition of mirror duality for abelian varieties which works uniformly well over arbitrary complete normed fields K . We will represent such a variety \mathcal{A} as a quotient (in the analytic category) of an algebraic K -torus T by a discrete subgroup B of maximal rank. Such a “multiplicative uniformization” goes back to Jacobi. The passage to the algebraic-geometric picture is mediated by the classical or p -adic theta-functions which are defined as analytic functions on T with the usual automorphic properties with respects to shifts by elements of B , see e.g. [80] for details. The choice of multiplicative uniformization adequately models the choice of a cusp in the moduli space of abelian varieties.

To be precise, algebraic torus T with the character group H over a field K is the spectrum of the group ring of H . The dual torus T^t , as above, has the character group H^t .

Consider now any diagram of the form

$$(j, j^t): T(K) \leftarrow B \rightarrow T^t(K) \quad (7)$$

where B is free abelian group of the same rank as H and j , resp. j^t , are its embeddings as discrete subgroups into $T(K)$, resp. $T^t(K)$.

We will say that pairs $(\mathcal{A} := T(K)/j(B), j^t)$ and $(\mathcal{B} := T^t(K)/j^t(B), j)$ are mirror dual to each other. The quotient spaces \mathcal{A}, \mathcal{B} not always have the structure of abelian varieties, but this is not important for the following.

In order to motivate this definition, we will show that for $K = \mathbf{C}$, we can produce from (7) a pair of dual real toric fibrations over a common base.

We have the Lie group isomorphism $\mathbf{C}^* \rightarrow S^1 \times \mathbf{R}: z \mapsto (z/|z|, \log |z|)$. This induces an isomorphism

$$(\alpha, \lambda): T(\mathbf{C}) \rightarrow \mathrm{Hom}(H, S^1) \times \mathrm{Hom}(H, \mathbf{R}). \quad (8)$$

Since $j(B)$ is discrete of maximal rank, then $\lambda \circ j(B)$ is an additive lattice in the real space $\mathrm{Hom}(H, \mathbf{R})$. Thus (8) produces a real torus fibration of $T(\mathbf{C})$ over the base which is as well a real torus of the same dimension:

$$0 \rightarrow \mathrm{Hom}(H, S^1) \rightarrow T(\mathbf{C})/j(B) \rightarrow \mathrm{Hom}(H, \mathbf{R})/\lambda \circ j(B) \rightarrow 0. \quad (9)$$

Similarly, we have

$$0 \rightarrow \mathrm{Hom}(H^t, S^1) \rightarrow T^t(\mathbf{C})/j^t(B) \rightarrow \mathrm{Hom}(H^t, \mathbf{R})/\lambda^t \circ j^t(B) \rightarrow 0 \quad (10)$$

where λ^t is defined for T^t in the same way as λ for T . Let us identify linear real spaces $H_{\mathbf{R}}$ with $H_{\mathbf{R}}^t$ in such a way that lattice points $\lambda \circ j(b)$ and $\lambda^t \circ j^t(b)$ are identified for all $b \in B$. Then (2.5) and (2.6) become dual real torus fibrations over the common base.

The relevant complex structures in our context come from covering tori. They produce symplectic forms as was explained above.

3.4. Kontsevich's package

We now return to the general mirror dual toric fibrations. With some stretch of imagination, one can see the following pattern in the picture described above: *Lagrangian cycles with local systems on X , whose projection to U have real dimension k , must correspond to coherent sheaves on Y with support of complex dimension k .*

Kontsevich in [58] suggested a considerably more sophisticated conjecture. Namely, let X be a *compact* symplectic manifold with $c_1(X) = 0$, and Y some compact complex Calabi-Yau manifold.

Then the relation of mirror partnership between X and Y consists in an equivalence between the Fukaya triangulated category $D(\mathrm{Fuk}_X)$ concocted out of Lagrangian cycles with local systems on the one side, and (a subcategory of) $D^b(\mathrm{Coh}_X)$ on the other side.

Briefly, to construct $D(\mathrm{Fuk}_X)$ one proceeds in three steps: first, one constructs an A_∞ -category Fuk_Y , then one produces from it another A_∞ -category of twisted complexes, and finally, one passes to the homology category of the latter.

Objects $\Lambda = (L, \mathcal{L}, \lambda)$ of Fuk_Y are Lagrangian submanifolds L in X with unitary local systems \mathcal{L} , endowed with a lifting λ to the fiberwise universal cover of the Lagrangian Grassmannian of X .

Morphism space between a pair of such objects admits a transparent description in the case when their Lagrangian submanifolds L_1, L_2 intersect transversally. In this case it is simply $\mathrm{Hom}(\mathcal{L}_\infty, \mathcal{L}_2)$ in the category of sheaves on X . This space is \mathbf{Z} -graded with the help of a construction using λ and Maslov index.

However, the composition of morphisms is not at all the composition of these morphisms of sheaves. In fact, a modification of Floer's construction using summation over pseudoholomorphic parametrized discs in X produces a series of polynomial maps

$$\begin{aligned} m_1 &: \operatorname{Hom}(\Lambda_1, \Lambda_2) \rightarrow \operatorname{Hom}(\Lambda_1, \Lambda_2), \\ m_2 &: \operatorname{Hom}(\Lambda_1, \Lambda_2) \otimes \operatorname{Hom}(\Lambda_2, \Lambda_3) \rightarrow \operatorname{Hom}(\Lambda_1, \Lambda_3), \end{aligned}$$

and generally

$$m_r: \operatorname{Hom}(\Lambda_1, \Lambda_2) \otimes \cdots \otimes \operatorname{Hom}(\Lambda_{r-1}, \Lambda_r) \rightarrow \operatorname{Hom}(\Lambda_1, \Lambda_r).$$

If the respective sums converge, m_1 endows the graded Hom-spaces with the structure of a complex, m_2 becomes the morphism of complexes, and higher multiplications are interrelated by the A_∞ -identities ensuring that the associativity constraints for the composition of morphisms are valid up to explicit homotopies.

For more detailed discussion, see [58, 93, 35], and the literature quoted therein. In particular, the case of elliptic curves is rather well understood thanks to Polishchuk and Zaslow, and Fukaya started treating abelian varieties and complex tori.

Both categories involved in the Kontsevich's conjecture generally have non-trivial discrete symmetries, induced in the CY-context by monodromy at the Lagrangian side and by derived correspondences at the complex side. Thus some additional data have to be chosen in order to pinpoint the expected functor. The awareness of symmetries led Kontsevich to beautiful predictions about the correspondence between monodromy actions and automorphisms of derived categories: see [52, 95, 97]. We mentioned these predictions above, when we discussed the global properties of the Frobenius partnership relations.

Kontsevich was vague about both the origin of the equivalence functor and exact geometric relation between X and Y . One can interpret the picture described in 3.1–3.3 which emerged later as a precise guess about the nature of several data left implicit in Kontsevich's presentation:

(i) *The character of additional data to be chosen: dual toric fibrations of X , Y over a common base.*

We will see below how this choice at the complex side is related to the notion of cusp of the relevant moduli space which we introduced in the context of Frobenius mirror partnership.

(ii) *The structure of the restriction of the equivalence functor acting on the simple objects: Fourier-Mukai transform corresponding to the choice (i).*

With exception of the case of complex tori, there is not much chance that X or Y would admit a global fibration by real tori: degenerate fibers are generally unavoidable, and their geometry and influence on the global geometry of the mirror picture are poorly understood. The case of $K3$ -surfaces offers some testing ground, because $K3$ -surfaces are hyperkähler, and Lagrangian tori can be transformed into a pencil of elliptic curves by an appropriate rotation of the complex structure.

Recently M. Kontsevich and A. Todorov came up with a conjectural limiting metric picture of the maximally degenerating family of CY manifolds of dimension d (private communication). Namely, fix a cohomology class of Kähler forms and a moduli cusp. Deform the complex structure by moving to the maximal degeneration point, and the Calabi-Yau metric in the chosen class by multiplying it by a real number in such a way that the diameter of the space remains 1.

Todorov and Kontsevich expect that the limit \mathcal{X} in the Hausdorff-Gromov sense of this family of metric spaces will be a real d -dimensional manifold with a Riemannian metric which might have singularities in codimension two. Moreover, the remnants of the special real torus fibration consist in the following additional data: affine structure and a sublattice in the tangent bundle. In local affine coordinates, the metric must be the second derivative of a convex function H , and the volume form of the metric must be constant.

Conjecturally, mirror dual family (endowed with appropriate cusps) produces the same limiting metric space $\mathcal{Y} = \mathcal{X}$, but with a different affine structure and sublattice in the lattice bundle.

3.5. Mirror Symmetry between Calabi-Yau manifolds

Let now X, Y be two C^∞ -manifolds each of which is endowed by a symplectic form, real toric fibration over a common base, and a complex structure, (ω_X, p_X, J_X) and (ω_Y, p_Y, J_Y) respectively. We will say that they are related by Mirror Symmetry, if (X, p_X, ω_X) is the mirror partner of (Y, p_Y, J_Y) and (X, p_X, J_X) is the mirror partner of (Y, p_Y, ω_Y) in the sense of Lagrangian/complex duality. An example of this setup is described in 3.3.1.

The structures J and ω at each side, of course, can be related. The most rigid connection between them is the presence of the Riemann metric g producing the Kähler package (J, ω, g) . In the case of Calabi-Yau manifolds, the natural choice is Yau's Ricci-flat metric g .

The program of [96] develops this setup, in particular, supplying the topological and the metric characterization of the basic toric fibrations. Namely, the cohomology class of any toric fiber in X , resp. Y must be the generator of the cyclic group of invariant cycles in the middle cohomology with respect to the local monodromy action at the chosen cusp of moduli space. Moreover, non-degenerate toric fibers (and other relevant Lagrangian submanifolds) must be not simply Lagrangian, but *special* Lagrangian. This produces a version of Lagrangian geometry whose rigidity is comparable to that of complex one, and makes it fit for comparison with the complex picture: see [48, 49, 98, 99] for many details.

It would be important to develop a version of Fukaya's category in this rigid context where the usual tools of homological algebra might work better.

3.6. Motives in the looking glass

One of the most basic expressions of the Mirror Symmetry of the Calabi-Yau manifolds is the existence of highly nontrivial isomorphisms between their cohomology

spaces: the relation of mirror partnership between X and Y is expected to produce, roughly speaking, an isomorphism $H^*(X) \rightarrow H^*(Y)$.

More precisely, any isomorphism between the quantum cohomology of X and Barannikov-Kontsevich formal Frobenius manifold of Y produces an identification of their spaces of flat vector fields, that is a mirror isomorphism of the cohomology spaces

$$\mu_{X,Y}: H^*(X, \mathbf{C}) \rightarrow H^*(Y, \wedge^*(\mathcal{T}_Y)) \otimes V_Y^{-2}, \quad V_Y := H^0(Y, \Omega_Y^{\max}). \quad (11)$$

Near a cusp in the moduli space of Y , V_Y can be trivialized by the choice of a volume form Ω having period $(2\pi i)^{\dim Y}$ along the invariant cycle. Then (11) becomes a ring isomorphism. Trace functionals and flat metrics on both sides are identified via (4). Comparing Euler fields, one sees that $H^{p,q}(X)$ is identified with $H^q(Y, \wedge^p(\mathcal{T}_Y))$. In particular, $H^{1,1}(X)$ becomes $H^2(Y, \mathcal{T}_Y)$, and the induced integral structure on the latter space (exponential coordinates near the cusp) are described in [27].

Notice now that that the Frobenius structure at the left hand side of (11) is essentially motivic, in the sense that numerical Gromov-Witten invariants of X come from algebraic correspondences between X^n and $\overline{M}_{0,n}$, $n \geq 3$. More generally, theory of Gromov-Witten invariants can be conceived as a chapter of algebraic and/or non-commutative geometry *over the category of motives*, replacing the more common category of linear spaces. This geometry deals, for example, with affine groups whose function rings are Hopf algebras in the category of *Ind*-motives. P. Deligne developed basics of this geometry in [25, 26], in order to clarify the notion of motivic fundamental group. Further examples come from or are motivated by physics: besides Gromov-Witten invariants, one can mention Nakajima's theory of Heisenberg algebras related to Chow schemes of surfaces, and a recent paper [72].

It makes sense to ask then, what can be the mirror reflection of this motivic geometry. Since the mirror maps are highly transcendental, developing the adequate language presents an interesting challenge. Starting with the category of motives in the sense of [1] generated by Calabi-Yau manifolds, we can try to extend it by adding mirror isomorphisms as new motivated morphisms. In this context, Kontsevich's correspondence between CY Teichmüller groups and autoequivalences of derived categories might have an analog, saying that the mirror isomorphisms connect the motivic fundamental groups (see [26]) and motivic automorphism groups of CYs whose Lie algebras were studied in [71]. For abelian varieties, this phenomenon is stressed in [46].

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