The Berry-Tabor Conjecture

Jens Marklof

Abstract. One of the central observations of *quantum chaology* is that statistical properties of quantum spectra exhibit surprisingly universal features, which seem to mirror the chaotic or regular dynamical properties of the underlying classical limit. I will report on recent studies of simple regular systems, where some of the observed phenomena can be established rigorously. The results discussed are intimately related to the distribution of values of quadratic forms, and in particular to a quantitative version of the Oppenheim conjecture.

Quantum Chaos

One of the main objectives of *quantum chaology* is to identify characteristic properties of quantum systems which, in the semiclassical limit, reflect the regular or chaotic features of the underlying classical dynamics.

Take for example the geodesic flow on the unit tangent bundle of a compact two-dimensional Riemannian surface M. The corresponding quantum system is described by the stationary Schrödinger equation

$$-\Delta\varphi_j = \lambda_j \varphi_j \,, \tag{1}$$

where Δ is the Laplacian of M, λ_j represent the quantum energy eigenvalues and φ_j the corresponding eigenfunctions. The spectrum of the negative Laplacian is a discrete ordered subset of the real line,

$$0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty \,. \tag{2}$$

According to Weyl's law, the number of eigenvalues below λ is asymptotically

$$\#\{j : \lambda_j \le \lambda\} \sim \frac{\operatorname{area}(M)}{4\pi}\lambda \tag{3}$$

as $\lambda \to \infty$. Hence the mean spacing between adjacent levels is asymptotically $4\pi/\operatorname{area}(M)$. For simplicity, we may assume in what follows that $\operatorname{area}(M) = 4\pi$.

One quantity measuring the "randomness" of such a sequence on the scale of the mean spacing is the *consecutive level spacing distribution*, which is defined as

$$P(s, N) = \frac{1}{N}\sum_{j=1}^{N} \delta(s - \lambda_{j+1} + \lambda_j) \,, \tag{4}$$

where $\delta(x)$ is the Dirac mass. If the deterministic sequence $\{\lambda_j\}$ is sufficiently "random" we may expect to find a limit distribution $P(s)$ of $P(s,N)$ for $N\to\infty$, that is,

$$\lim_{N\to\infty}\int_0^\infty P(s,N)\,h(s)\,ds=\int_0^\infty P(s)\,h(s)\,ds\,, \tag{5}$$

for any sufficiently nice test function h.

It was conjectured by Berry and Tabor in 1977 [1] that *if the corresponding classical dynamics is completely integrable, then $P(s)$ exists and is equal to the waiting time between consecutive events of a Poisson process*,

$$P(s)=\exp(-s)\,. \tag{6}$$

That is, $\{\lambda_j\}$ behaves like a sequence of independent random variables. There are a number of obvious (and less obvious) counter examples to this behaviour already known to Berry and Tabor [1], which is why one expects the above to hold only for "generic" systems. Let us compare this with the case of "generic" chaotic dynamical systems. Here, the statistical nature of the limit distribution is completely different: Bohigas, Giannoni, and Schmit [3] suggested in 1984 that *if the geodesic flow is hyperbolic, then $P(s)$ exists and is equal to the consecutive level spacing distribution of a suitable Gaussian ensemble of hermitian random matrices.* In the case of the Gaussian unitary ensemble, the distribution is for example (approximately)

$$P_{\mathrm{GUE}}(s)\approx\frac{32}{\pi^2}\,s^2\mathrm{e}^{-\frac{4}{\pi}s^2}\,. \tag{7}$$

The characteristic level repulsion (i.e., $P(s)\to 0$ as $s\to 0$) was observed already by McDonald and Kaufman [16] and Berry [2] in the case of chaotic billiards.

There has been some recent progress on the above conjectures in the case of integrable systems, which will be reported here. The chaotic chase is less well understood.

Lattice Point Problems

The eigenvalues λ_j of the Laplacian on a compact two-dimensional surface with integrable geodesic flow are asymptotically given by the values $F(\mathbf{m})\sim\lambda_j$ at lattice points $\mathbf{m}\in\mathbb{Z}^2$, where the function $F=F_2+F_0$ is piecewise analytic and F_2, F_0 homogeneous of degree 2 and 0, respectively [6]. The question of spectral statistics may therefore be reformulated as a lattice point problem of lattice points $\mathbf{m}\in\mathbb{Z}^2$ close to the curve $F(\mathbf{x})=\lambda$.

Sinai [22] and Major [11] proved that the statistics of λ_j converge to the Poisson limit if F is generic in a certain function space. Unfortunately, a "generic" function in that space is not twice differentiable, and hence there is no corresponding smooth classical dynamical system. It is still open, if the Poisson limit exists for smooth F. Let us have a look at a few examples.

THE SPHERE AND OTHER ZOLL SURFACES We begin by discussing an obviously non-generic example with respect to the above conjectures. Consider the geodesic flow on the unit tangent bundle of the sphere, which clearly is completely integrable. The eigenvalues of $-\Delta$ are (up to some constant factor) $l(l+1)$ with integer $l = 0, 1, 2, \ldots$, each with multiplicity $2l+1$. Label these numbers in increasing order by $\lambda_1, \lambda_2, \ldots$; due to the high multiplicity one thus has $P(s) = \delta(s)$. The same result holds for all Zoll surfaces, since the eigenvalues are strongly clustered around the values $l(l+1)$, compare [8, 24, 7, 23]. The reason for this clustering is that all geodesics are closed and have the same period. Zoll surfaces are clearly quite non-generic in this respect, and it is therefore of no surprise that their eigenvalues do not follow the expected statistical behaviour.

FLAT TORI We now turn to a more generic class of surfaces, flat two-dimensional tori $\mathbb{T}^2 = \mathbb{R}^2/\mathfrak{L}$, where $\mathfrak{L}$ is some lattice in $\mathbb{R}^2$. The Laplacian on $\mathbb{T}^2$ is the standard Euclidean Laplacian acting on functions which are invariant under the lattice $\mathfrak{L}$. The eigenvalues λ_j are easily seen to be values at integers $(m, n) \in \mathbb{Z}^2$ of the positive definite quadratic form

$$Q(m,n) = \alpha m^2 + \beta mn + \gamma n^2 \,, \tag{8}$$

where the coefficients α, β, γ depend on the lattice $\mathfrak{L}$. In the special case when $\mathfrak{L} = \mathbb{Z}^2$ one has

$$Q(m,n) = m^2 + n^2 \,. \tag{9}$$

The spacings are in this case clearly integer, so $P(s, N)$ cannot converge to the conjectured exponential distribution. In fact, one can show that the limit distribution is again $P(s) = \delta(s)$, which is in disagreement to what is expected generically. This result follows directly from a classical result of Landau, which says that the number of ways of representing an integer as a sum of two squares grows on average logarithmetically. The same holds when Q is proportional to a form with rational coefficients [21].

Results which support the Berry-Tabor conjecture on a rigorous level have so far only been obtained for a statistic which is easier to handle than the consecutive spacing distribution: the pair correlation density.

Pair Correlation

The pair correlation density of the sequence $\{\lambda_j\}$ is defined as

$$R_2(s,N) = \frac{1}{N} \sum_{j,k=1}^{N} \delta(s - \lambda_j + \lambda_k) \,, \tag{10}$$

where one therefore considers the distribution of *all* spacings, instead of just nearest neighbours. According to the Berry-Tabor conjecture, we expect that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} R_2(s,N)\,h(s)\,ds = \int_{-\infty}^{\infty} R_2(s)\,h(s)\,ds \tag{11}$$

exists for generic integrable systems, where $R_2(s)$ is the pair correlation density of a Poisson process[1]

$$R_{2\ \mathrm{Poisson}}(s) = \delta(s) + 1\,. \tag{12}$$

Sarnak [21] was able to prove the Berry-Tabor conjecture for the pair correlation of a almost all flat tori: almost all with respect to Lebesgue measure in the moduli space of two-dimensional flat tori. His proof uses averaging techniques to reduce the pair correlation problem to estimating the number of solutions of systems of diophantine equations. The almost-everywhere result then follows from a variant of the Borel-Cantelli argument.[2]

Eskin, Margulis and Mozes [10] recently strengthened Sarnak's result by giving explicit diophantine conditions on (α,β,γ) under which the Berry-Tabor conjecture holds. Admissible forms are for example

$$m^2 + \gamma n^2 \tag{13}$$

with γ diophantine. [γ is called diophantine if there exist constants $\kappa \geq 2, C > 0$ such that $|\alpha - \frac{p}{q}| > Cq^{-\kappa}$ for all rationals $\frac{p}{q}$.[3]] These diophantine conditions cannot be dropped, since one can construct an uncountable set of tori with area 4π (say), whose pair correlation density does not converge, as pointed out by Sarnak [21]. In some sense, such tori feel the degeneracies in the spectra of tori corresponding to rational forms. The above uncountable set is in fact a set of second Baire category, and hence "generic" in the topological sense. This illustrates the subtlety of the problem: "generic" tori —in the measure-theoretic sense— follow the Berry-Tabor conjecture, topologically "generic" tori do not.

The pair correlation problem for positive definite quadratic forms may be viewed as a special case of the quantitative version of the Oppenheim conjecture of quadratic forms of signature $(2,2)$, which is particularly difficult [9].

TORI WITH AHARONOV-BOHM FLUX We have seen that the eigenvalue statistics for the standard square torus with lattice $\mathfrak{L} = \mathbb{Z}$ are non-generic. In order to break

[1] Here, the delta mass $\delta(s)$ is a result of our definition which counts spacings between equal elements whose spacing is trivially zero. The interesting part is the "1", the mean density of the eigenvalue sequence.

[2] For further related examples of sequences whose pair correlation function converges to the uniform density almost everywhere in parameter space, see [17, 19, 25, 26, 28]. For recent results on higher correlations see [18, 20, 27].

[3] Almost all numbers are diophantine. An example of a diophantine number is $\gamma = \sqrt{2}$, where $\kappa = 2$.

the degeneracy, let us replace the periodicity conditions with quasi-periodicity conditions,

$$\varphi(x+k,y+l)=\mathrm{e}^{-2\pi\mathrm{i}(\alpha k+\beta l)}\varphi(x,y),\quad k,l\in\mathbb{Z}\,. \tag{14}$$

The eigenvalues of the negative Laplacian are then

$$(m-\alpha)^2+(n-\beta)^2\,. \tag{15}$$

Such an effect is observed for instance in the presence of Aharonov-Bohm flux lines threading the torus. This is a pure quantum-topological effect, the corresponding classical geodesic flow is not affected. Cheng and Lebowitz have studied such a system numerically, and found good agreement with the Berry-Tabor conjecture for generic α, β. Cheng, Lebowitz and Major [5] were able to prove the convergence of the pair correlation density to the expected Poisson distribution on average over $(\alpha,\beta)\in[0,1]^2$.

In [15] it is proved that the conjecture is in fact true for fixed α, β, provided α or β is diophantine (e.g. $\alpha=\sqrt{2},\beta=\sqrt{3}$). This diophantine condition is necessary: there is a set of second Baire category of $(\alpha,\beta)\in[0,1]^2$ for which the pair correlation density does not converge. The idea of the proof is as follows. Consider the Fourier transform of the pair correlation density,

$$K_2(\tau,N)=\int_{-\infty}^{\infty}R_2(s,N)\,\mathrm{e}^{2\pi\mathrm{i}\tau s}\,ds \tag{16}$$

hence

$$K_2(\tau,N)=\big|N^{-1/2}\sum_{j=1}^{N}\mathrm{e}^{2\pi\mathrm{i}\lambda_j\tau}\big|^2\,. \tag{17}$$

The function $K_2(\tau,N)$ is often called the *spectral form factor*. We are interested in the asymptotics of

$$\int_{-\infty}^{\infty}R_2(s,N)\,h(s)\,ds=\int_{-\infty}^{\infty}K_2(\tau,N)\,\hat{h}(\tau)\,d\tau$$

where $\hat{h}$ is the Fourier transform of the test function h. The trick is now that the exponential sum $\sum\mathrm{e}^{2\pi\mathrm{i}\lambda_j\tau}$ turns out to be a theta sum that can be identified with a function on a noncompact manifold Σ with finite measure, and that the integral over τ corresponds to an average over a unipotent orbit on Σ that becomes equidistributed on Σ, as $N\to\infty$. The equidistribution follows essentially from Ratner's classification of invariant measures of unipotent flows. The average over τ can therefore be replaced by the average over the whole space Σ, which luckily turns out to be exactly the expected limit. A crucial subtlety in the proof is that the theta sum is unbounded on Σ, which is where the diophantine conditions come in.

It is quite interesting to remark that, although the pair correlation statistics of the λ_j follow the Poisson prediction, the value distribution of the sum $\sum\mathrm{e}^{2\pi\mathrm{i}\lambda_j\tau}$ defining the form factor does not satisfy a central limit theorem in general. For example, in the case of the rectangular torus discussed above, where $\lambda_j = m^2+\gamma n^2$, a limiting distribution exists [14] but has a power-like tail, and is therefore not normal. The proof is based on the asymptotic distribution of values of theta sums [12, 13]. A similar deviation from the normal distribution is observed whenever the eigenvalues λ_j are values of quadratic forms at lattice points, e.g. also when $\lambda_j = (m-\alpha)^2 + (n-\beta)^2$. If, however, the quadratic form is replaced by a more generic smooth function F homogeneous of degree two, I expect the limiting distribution to be normal. This will be discussed in detail elsewhere.

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School of Mathematics,
University of Bristol,
Bristol BS8 1TW, U.K.
E-mail address: `j.marklof@bristol.ac.uk`