Contact Structures, Rational Curves and Mori Theory

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Abstract. This paper intends to report on the recent progress in the classification of contact structures on complex projective and compact Kähler manifolds. In particular we explain how to apply the geometry of rational curves and Mori theory to investigate projective contact manifolds whose second Betti number is at least 2.

1. Introduction

Given a projective or compact Kähler manifold X, a *special* subbundle or coherent subsheaf $E \subset T_X$ in the holomorphic tangent bundle T_X often carries significant geometric information about the underlying manifold. One important special property is certainly integrability. Integrable subbundles E, i.e. subbundles closed under the Lie bracket, define a foliation and one might hope —under additional assumptions— to find compact leaves and therefore to obtain strong geometric informations on X. At the moment there are however only a very few results when the leaves are compact. In sharpe contrast are contact structures, i.e. subbundles of corank 1 which are maximally non-integrable. Contact structures play an important role in real differantial geometry and attracted rather recently interest in complex geometry via the theory of quaternionic-Kähler manifolds: given a quaternionic Kähler manifold (M, g) with positive scalar curvature, the twistor space Z is a Fano manifold (i.e. has a metric with positive Ricci curvature) with a contact structure and classifying those Fano manifolds means to classify the quaternionic-Kähler manifolds with positive scalar curvature (Salamon, LeBrun).

Therefore one is very much interested in classifying projective contact manifolds, and apart from the connection with quaternionic Kähler manifolds, the interest in complex geometry really lies in being "opposite" to foliations.

This article mainly intends to report the recent progress in this topic and to discuss the main open problems. It is hopefully written in such a way to make it accessible also to differential geometers.

2. Complex Contact Structures: Basic Facts, State of the Art and Conjectures

Given a complex manifold X with sheaf $\mathcal{O}_X$ of holomorphic functions we shall denote T_X its tangent bundle, and denote by Ω^1_X the sheaf of holomorphic 1-forms, i.e. the sheaf of holomorphic sections of T^*_X. Moreover $K_X = \det T^*_X$ is the canonical bundle on X and $c_1(X) = c_1(T_X)$ denotes the first Chern class of X. A line bundle L on a (usually compact) manifold is *positive* or *ample* if it carries a metric of positive curvature. A line bundle L on a *projective* manifold X is called *nef*, if $c_1(L) \cdot C = c_1(L \mid C) \geq 0$ for any irreducible compact curve $C \subset X$. If L carries a metric of semipositive curvature, then L is nef but the converse is false. Finally we denote by $\kappa(L)$ the Kodaira dimension of L and let $\kappa(L) = \kappa(K_X)$ be the Kodaira dimension of X. So $\kappa(X) = -\infty$ if no multiple $mL = L^{\otimes m}$ has a section; $\kappa(L) = 0$, if some mL has a section but $\dim H^0(X, mL) \leq 1$ for all m and $\kappa(L) = k > 0$, if $\dim H^0(X, mL)$ grows as m^k.

Definition 2.1. *A compact complex manifold X of dimension $2n+1$ together with a subbundle $F \subset T_X$ of rank $2n$ is a* contact manifold *if the pairing $\omega\colon F \times F \to T_X/F =: L$ induced by the Lie bracket is everywhere non-degenerate.*

An equivalent definition is as follows: we require that K_X is divisible by $n+1$ ($\dim X = 2n+1$); if we write $-K_X = (n+1)L$, then we must have a section $\theta \in H^0(X, \Omega^1_X \otimes L)$ and the non-degeneracy condition can be reformulated as follows: an elementary calculation shows that $\theta \wedge (d\theta)^{\wedge k}$, although computed in a local trivialisation of L, gives a global section of $\Omega^{2k+1}_X \otimes L^k$. Then $\theta \wedge (d\theta)^n$ has no zeroes. The subbundle $F \subset T_X$ is now given by $F = \ker(\theta)$.

Remark 2.2. *As already mentioned, the anticanonical bundle is a multiple of the contact line bundle L: we have $-K_X = (n+1)L$. This already restricts the class of possible contact manifolds significantly. E.g. a quadric hypersurface in projective space can never carry a contact structure.*

For a detailed discussion of contact manifolds we refer to [12] and [4].

Examples 2.3. (1) *Let $X = \mathbb{P}_{2n+1}$ and take a non-zero form $\omega \in H^0(\Omega^1_X \otimes \mathcal{O}(2))$. Then ω has no zeroes and therefore defines a bundle epimorphism $T_X \to \mathcal{O}(2)$. The kernel now defines a contact structure on $\mathbb{P}_{2n+1}$. In the theory of vector bundles on projective space, this kernel is known (up to a twist with a line bundle) as null-correlation bundle, see* [17].

(2) *Let Y be any complex manifold of dimension $n+1$ and let $X = \mathbb{P}(T_Y)$. Then X has a contact form as follows.*
$H^0(X, \pi^*(\Omega^1_Y \otimes \mathcal{O}_X(1))) = H^0(Y, \Omega^1_Y \otimes T_Y)$ has a canonical element, namely the identity endomorphism on T_Y. So letting $L = \mathcal{O}_X(1)$ we obtain an element $\omega \in H^0(X, \Omega^1_X \otimes \mathcal{O}(1))$ which easily is seen to give a contact structure.

(3) *Let G be a complex simple Lie group with Lie algebra $\mathcal{G}$. Then there is an unique closed orbit $X_{\mathcal{G}}$ for the adjoint action of G on $\mathbb{P}(\mathcal{G})$. Now $X_{\mathcal{G}}$*

can be seen to be a Fano contact manifold; for details see [3] *and* [4]*. We say that the contact structure is induced by a simple Lie group. Of course projective space can be rediscovered in that way* ($G = \mathrm{Sp}(2n+2)$)*.*

(4) *Let* (M, g) *be a quaternion-Kähler manifold of positive scalar curvature. Then its twistor space* $p\colon Z \to M$*, a* S^2*-submersion over* M*, is a Fano contact manifold (and even Kähler-Einstein). We refer to* [19] *and* [12]*.*

Now a standard conjecture says that these should be the only examples.

Conjecture 2.4. *Let* X *be a compact complex contact Kähler manifold. Then* X *is induced by a simple Lie group or* $X = \mathbb{P}(T_Y)$ *for some compact manifold* Y*.*

Notice that $\mathbb{P}(T_{\mathbb{P}_{n+1}})$ occurs in both lists ($G = \mathrm{Sl}(n+1)$), but all other manifolds coming from simple Lie groups have second Betti number $b_2(X) = 1$. There is a lot of evidence that the conjecture should be true in the projective category (and then also in the Kähler category), but all the tools used fail in the general complex setting so that the term "conjecture" might be not very appropriate here.

In particular this conjecture would imply a classification of quaternion-Kähler manifolds via the twistor construction.

In dimension 3 the conjecture has been proved by Ye [21], whereas in dimension 5 it is proved by Druel [7] up to possible contact 5-folds with nef canonical bundles. We next state the general results known so far to support the conjecture.

Proposition 2.5. (Druel [7]**)** *A compact Kähler contact manifold* X *has Kodaira dimension* $\kappa(X) = -\infty$*, i.e.* $H^0(X, \mathcal{O}(mK_X)) = 0$ *for all positive* m*.*

The Kähler assumption is needed to conclude that a holomorphic 1-form on X is closed.

Proposition 2.6. (Ye [21]**)** *A compact Kähler contact manifold* X *cannot have numerically trivial canonical bundle.*

In fact, such a manifold has $\kappa(X) = 0$ by virtue of the decomposition theorem [2].

Very recently, Demailly [6] has generalised these two results and made the following important contribution.

Proposition 2.7. (Demailly) *Let* X *be a compact Kähler contact manifold. Then* K_X *is never pseudo-effective, i.e.* K_X *does not carry a (possibly singular) hermitian metric whose curvature current is non-negative.*

If X is projective, then K_X is pseudo-effective if and only if K_X is contained in the closure of the cone generated by the (classes of the) effective divisors, i.e. of the irreducible hypersurfaces in X. See theorem 4.1 for a more general version of proposition 2.7.

We shall distinguish the cases $b_2(X) = 1$ and $b_2(X) \geq 2$. If $b_2(X) = 1$, then X, being Kähler, is automatically projective and propositions 2.5/2.6 imply that X is a Fano manifold, i.e. $-K_X$ is ample. In that case X should come from a simple Lie group according to the conjecture. In fact, one has

Theorem 2.8. (Beauville [3]**)** *Let X be a Fano contact manifold with $b_2(X) = 1$ and —as usual— with contact line bundle L. Assume that*

(a) *the rational map defined by the linear system $|L|$ (i.e. by the global sections of L) is generically finite;*
(b) *the group G of contact automorphisms is reductive.*
Then the Lie algebra $\mathcal{G}$ of G is simple and X is of the form $X_{\mathcal{G}}$ (example 2.3(3)).

The contact automorphisms are the automorphisms of X preserving F; notice also that the Lie algebra of G can be identified naturally with $H^0(X, L)$. In order to prove the conjecture, one now has to construct sufficiently many sections in L, which might be quite hard. In [4] Beauville removed the assumption that G is reductive but instead had to assume that L is very ample, i.e. the map given by the global sections of L is an embedding.

We next turn to the case $b_2(X) \geq 2$. From now on we will assume X projective and say only a few words on the Kähler case later. The aim is to rediscover the projective bundle structure X is expected to have according to the conjecture. In that case the canonical bundle K_X is negative on the projective spaces, in particular K_X is not nef, i.e. $K_X \cdot C < 0$ for some irreducible curve C. Therefore it is natural to use Mori theory for the investigation of X. We refer to [9, 15] for basic informations on Mori theory. In general, Mori theory is applicable if K_X is not nef, this being guaranteed by proposition 2.7. Therefore the main result of [10], discussed at length in section 3, yields.

Theorem 2.9. *Let X be a projective contact manifold with K_X not nef and $b_2(X) \geq 2$. Then X is of the form $X = \mathbb{P}(T_Y)$ with a projective manifold Y.*

The case "K_X nef" is potentially also ruled out by standard conjectures in minimal model theory.

Abundance Conjecture. *Let X be a projective manifold with K_X nef. Then mK_X is spanned by global sections for suitable large m. In particular $\kappa(X) \geq 0$.*

This conjecture has been proved by Miyaoka [13, 15] and Kawamata [8] in dimension 3, but it is completely open in higher dimensions. Of course we need only a "small" part, namely that $\kappa(X) \geq 0$, i.e. some multiple of K_X has a section.

Remark 2.10. *In the Kähler case we would basically need the corresponding results on Mori theory to prove the conjecture which do not exist yet. In dimension* 3 *how there are some results which suffice to settle conjecture 2.4 except for the mysterious case of "simple threefolds with $\kappa(X) = -\infty$". We refer to* [18].

It would be nice to have a direct proof of the following conjecture which of course would be a direct consequence of conjecture 2.4 and which also would rule out the nef case immediately.

Conjecture 2.11. *Let X be a compact complex contact Kähler manifold of dimension $2n + 1$. Then $c_1(X)^n \neq 0$.*

Notice that one can not do better: let Y be a 2-dimensional complex torus and let X be the projectivised tangent bundle: $X = \mathbb{P}(T_Y)$ (we take hyperplanes in the fibers!). Then $X \simeq \mathbb{P}_1 \times Y$ and clearly $c_1(X)^2 = 0$.

Given a contact manifold X, it is natural to ask how many contact structures can exist on X. Of course two contact forms define the same contact structure if and only if they differ only by a scalar. In [10] it is proved

Theorem 2.12. *The contact structures on a complex manifold $X = \mathbb{P}(T_Y)$ are in natural $1:1$-correspondence with the space $H^0(Y, \mathrm{End}(\Omega^1_Y))$ of endomorphisms of Ω^1_Y.*

If Ω^1_Y is simple, i.e. there are no endomorphisms except for multiples of the identity, then it follows that the contact structure is unique. This is in particular the case when X is projective and stable with respect to some ample polarisation. For example the Fano manifold $\mathbb{P}(T_{\mathbb{P}_{n+1}})$ has only one contact structure.

Concerning Fano manifolds with $b_2(X) = 1$, LeBrun proved in [12] that a Fano manifold which admits a Kähler-Einstein metric carries more than one contact structure if and only if X is projective space. It is natural to conjecture that this should be true without the Kähler-Einstein assumption; in fact conjecture 2.4 implies that a Fano contact manifold is Kähler-Einstein, because it is rational-homogeneous.

It is interesting to notice that the contact sequence never splits in the Kähler case:

Proposition 2.13. *Let X be a compact Kähler contact manifold. Then the contact sequence $0 \to F \to T_X \to L$ never splits.*

Proof. Suppose the contact sequence splits. Consider the (proportional) Chern classes

$$c_1(L), c_1(F) \in H^1(X, \Omega^1_X) = H^1(X, L^*) \oplus H^1(X, F^*) .$$

Then, not surprisingly, actually $c_1(L) \in H^1(X, L^*)$ and $c_1(F) \in H^1(X, F^*)$ (see [5]). Hence the proportionality $c_1(F) = nc_1(L)$ yields $c_1(X) = c_1(L) = c_1(F) = 0$ which is impossible as we already saw.

LeBrun ([12]) already noticed that the contact sequence never splits in case X is Fano. He also proved that the image of $c_1(L)$ under the canonical morphism

$$H^1(X, \Omega^1_X) \to H^1(X, F^*) \to H^1(X, F \otimes L^*)$$

is (up to $2\pi i$) the extension class of the contact sequence (the second map comes from the isomorphism $F^* \simeq F \otimes L^*$ provided by the non-degenerate map $F \times F \to L$).

3. The Use of Mori Theory

In this section we explain the methods of the proof of theorem 2.9. So we fix for this section a projective contact manifold X of dimension $2n+1$ with contact line bundle L; we also suppose $b_2(X) \geq 2$. The final goal is to show that X is of the form $X = \mathbb{P}(T_Y)$. The main problem is to rediscover a $\mathbb{P}_n$-bundle structure on X: once we have this, it is not so difficult to show that X is a projectivised tangent bundle.

Let us suppose first that K_X is not nef. By Mori theory we obtain a surjective holomorphic map

$$\phi\colon X \to Y$$

to a normal projective variety Y such that $b_2(X) = b_2(Y) + 1$ and such that

$$-K_X \cdot C > 0$$

for one and hence for all irreducible algebraic curves C contracted by ϕ. Now we shall study this map ϕ and we would like to see that ϕ defines a $\mathbb{P}_n$-bundle structure. There are three main steps.

Lemma 3.1. $\dim Y < \dim X$*, i.e.* ϕ *is not birational.*

Lemma 3.2. *The general fiber of* ϕ *is* $\mathbb{P}_n$*.*

Lemma 3.3. *All fibers of* ϕ *are smooth of the same dimension.*

The **proof of lemma 3.1** is very technical and relies on a detailed study of rational curves of small degree with respect to L, actually of rational curves C with $L \cdot C = 1$. Such curves exist: by Mori's breaking technique a projective manifold X of dimension m with K_X not nef always carries a rational curve C with $0 > K_X \cdot C \geq -m - 1$. Via $-K_X = (n+1)L$ in our situation, we obtain a rational curve C with $L \cdot C = 1$ or $L \cdot C = 2$. If we cannot find C with $L \cdot C = 1$, then one can see that X must be Fano with $b_2 = 1$ (we have large families of these lines covering X), so we always have "L-lines". Now one main point is to study the deformations of such a curve. In this context a main point is to study the restriction of T_X to such lines.

We postpone (some of) the technical details to the sketch of proof of lemma 3.3; instead we present here a simplified version (not using any projectivity assumption):

Simplified Lemma. *Let* X *be a compact complex manifold,* $\phi\colon X \to Y$ *the blow-up of a submanifold* B *in the compact manifold* Y*. Then* X *cannot carry a contact structure.*

Proof. Take any non-trivial fiber X_y of ϕ. Let $k = \dim X_y$. Then $T_X \mid l$ is of the following form

$$T_X \mid l = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus k-1} \oplus \mathcal{O}^{\oplus n-k-1} \oplus \mathcal{O}(-1)\,.$$

Here the factor $\mathcal{O}(2)$ comes from the tangent bundle T_l of l. In particular $c_1(X) \cdot l = k$. Now $c_1(X)$ is divisible by $n+1$, hence $k = n+1$. Thus $c_1(L) \cdot l = 1$. The tangent map $T_l \to T_X \mid l$ composed with the contact map $T_X|l \to L|l$ yields a map

$$\mu\colon T_l \to L \mid l\,.$$

But T_l has degree 2, whereas $L \mid l$ has degree 1, so $\mu = 0$. We conclude that therefore $T_l \subset F_l$ for all l. Since the tangents of the lines in X_y generate the tangent space of X_y at every point, it follows that $T_{X_y} \subset F_{X_y}$. Since $\dim X_y = n+1$, this contradicts the non-integrability of F.

In some sense, birational Mori contractions are the proper generalisations of blow-ups in classification theory, so that the simplified lemma at least gives strong evidence for lemma 3.1.

Next we discuss the **proof of lemma 3.2**. We claim that the general fiber X_y must be $\mathbb{P}_n$. First notice that the contact structure defines an isomorphism $F^* \otimes L \to F$ which can be extended to a map (with 1-dimensional kernel)

$$\alpha \colon T_X^* \otimes L \to T_X \,.$$

We then consider the composition

$$\beta \colon \phi^* T_Y^* \otimes L \mid X_y \to T_X^* \otimes L \to T_X \mid X_y \,,$$

where the first arrow is given by the differential of ϕ. Observe that clearly $\beta \neq 0$ and that $\phi^* T_Y^* \otimes L \mid X_y = L^{\oplus m} \mid Y_y$, which is ample (having in mind that ϕ is a Mori contractions so that $-K_X$ is positive on the fibers of ϕ, hence L is positive on the fibers of L). Composing β with $T_{X_y} \to N_{X_y/X}$ (where N denotes the normal bundle) we obtain a map $\gamma \colon L^{\oplus m} \mid X_y \to N_{X_y}$. Since N_{X_y} is trivial, $\gamma = 0$. Hence we obtain an injective map $L_{X_y} \to T_{X_y}$. But a remarkable theorem of Wahl [20] says that then X_y must be projective space. By divisibility reasons, $X_y \simeq \mathbb{P}_n$. The theorem of Wahl, in a slightly weaker version due to Mori and Sumihiro [16], says that a projective manifold admitting a vector field which vanishes on an ample divisor, must be projective space.

The **proof of lemma 3.3** is again rather technical. The main point is to prove that all fibers of ϕ have the same dimension n, then one can apply a result of Fujita to see that ϕ is a $\mathbb{P}_n$-bundle, then again it is not so difficult to conclude $X = \mathbb{P}(T_Y)$. We decribe one important proposition (= (2.9) in [10]). We consider the space $\mathrm{Hom}(\mathbb{P}_1, X))$ of holomorphic maps $f \colon \mathbb{P}_1 \to X$. Geometrically any (non-constant) map f determines an irreducible rational curve in X. Now consider a component V of $\mathrm{Hom}(\mathbb{P}_1, X)$. We says that V is unsplit if the curves from V cannot be deformed into a sum of rational curves (possibly with multiplicities). Now if V is unsplit and if $\deg f^*(L) = 1$, for one (hence for all) $f \in V$, then the curves from V fill up X (up to closure) and the curves from V passing through a fixed point form a subvariety of dimension n in X_{2n+1}. This is based on a careful analysis of the differential of the map

$$F \colon \mathbb{P}_1 \times \mathrm{Hom}(\mathbb{P}_1, X) \to X,\; F(x, f) = f(x) \,.$$

By Mori theory V always exist and so one can apply these considerations: they yield immediately that φ cannot be birational (lemma 3.1); equidimensionality requires further considerations.

4. The "Nef" Case

Demailly's contribution to the classification of contact Kähler manifolds (proposition 2.7) is a special case of his more general

Theorem 4.1. *Let X be a compact Kähler manifold carrying a pseudo-effective line bundle L. Let $\eta \in H^0(X, \Omega_X^p \otimes L^*)$ be a non-zero L^*-valued holomorphic p-form for some $1 \leq p \leq \dim X$. Let $S \subset T_X$ be the coherent subsheaf of vector fields v such that the contraction $i_v(\eta)$ vanishes. Then S is integrable.*

The proof uses of course the theory of currents together with some ("singular") integration by parts. Specialising to $p = 1$, S defines a meromorphic foliation of codimension 1, i.e. $\eta \wedge d\eta = 0$. In the contact situation this is applied to $S = F$ to show that L^* and hence K_X cannot be pseudo-effective.

The partial results in [10] to exclude projective contact manifolds with K_X nef lead to some interesting questions on nef subsheaves in Ω_X^1. In principle one is interested in the problem how positive a subsheaf in Ω_X^1 can be. We shall restrict ourselves to rank 1 subsheaves $\mathcal{E}$. Then Bogomolov has shown that $\kappa(\mathcal{E}) \leq \dim X - 1$. We will now make the following assumptions.

(*) X_n is a projective manifold, $\mathcal{E} \subset \Omega_X^1$ locally free of rank 1, $\mathcal{E}$ is nef and there exists a positive rational number α such that $\alpha\mathcal{E} = K_X$ (as $\mathbb{Q}$-divisors).

Proposition 4.2. *Assuming (*), we have $c_1(X)^2 = K_X^2 = 0$.*

Proof. We choose general hyperplane sections $H_1, \ldots, H_{n-1}$ to obtain a smooth surface $S = H_1 \cap \ldots \cap H_{n-1}$. Then consider the restricted sequence

$$0 \to \mathcal{E}_S \to \Omega_X^1 \mid S \to Q_S \to 0\,.$$

Via the map $\Omega_X^1 \mid S \to \Omega_S^1$, we obtain a map $\varphi\colon \mathcal{E} \mid S \to \Omega_S^1$. This map is non-zero, hence injective; in fact, otherwise we would have a map $\mathcal{E} \mid S \to N_S^*$ which has to vanish since $\mathcal{E} \mid S$ is nef and the normal bundle N_S is a direct sum of ample line bundles (actually one can see that the general choice of S already enforces $\varphi \neq 0$). Now the already mentioned theorem of Bogomolov yields $\kappa(\mathcal{E} \mid S) \leq 1$. On the other hand, the nefness of $\mathcal{E} \mid S$ implies that $c_1(\mathcal{E} \mid S) \geq 0$. If however $c_1(\mathcal{E} \mid S)^2 > 0$, then the Riemann-Roch theorem gives $\kappa(\mathcal{E} \mid S) = 2$; so we must have $c_1(\mathcal{E} \mid S)^2 = 0$. Since $\mathcal{E} \mid S$ is proportional to $K_X \mid S$, we obtain

$$K_X^2 \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0$$

for any choice of ample line bundles H_i. Now some standard considerations (see [10]) show that actually $K_X^2 = 0$.

As a consequence, it is easily shown that if $\kappa(X) \geq 0$, then either $K_X \equiv 0$ or $\alpha \geq 1$. The abundance conjecture predicts that $\kappa(X) \geq 0$ should always hold since K_X is nef. Assuming $\kappa(X) \geq 0$, we are reduced to study two cases: $K_X \equiv 0$ and $\alpha \geq 1$. If $K_X \not\equiv 0$, then $K_X^2 = 0$ predicts that we should have $\kappa(X) = 1$. If this is really true, then we have

Theorem 4.3. *Suppose (*) and $\kappa(X) = 1$. Then K_X is semi-ample, i.e. some mK_X is spanned. Let $f\colon X \to C$ be the Iitaka fibration and let B denote the divisor part of the zeroes of $f^*(\Omega^1_C) \to \Omega^1_X$. Then there exists an effective divisor D such that*

$$\mathcal{E} = f^*(\Omega^1_C) \otimes \mathcal{O}_X(B - D).$$

Turning to the case that $\kappa(X) = 0$, one of course expects that $K_X \equiv 0$, and therefore the decomposition theorem [2] says that there exists a finite étale cover $f\colon \tilde{X} \to X$ such that $\tilde{X} = A \times Y$ with A abelian and Y simply connected and that $f^*(\mathcal{E}) = \mathcal{O}_{\tilde{X}}$.

There is the following well-known *"conjecture K"* of Ueno

Conjecture 4.4. *Let X be a projective manifold with $\kappa(X) = 0$. Then the Albanese map is birational to an étale fiber bundle over its Albanese torus which is trivialised by an étale base change.*

Then we have [11]

Theorem 4.5. *In the situation of (*) suppose that $\kappa(X) = 0$. If conjecture 4.4 holds, then $K_X \equiv 0$, and therefore there exists a finite étale cover $f\colon \tilde{X} \to X$ such that $\tilde{X} = A \times Y$ with A abelian and Y simply connected and that $f^*(\mathcal{E}) = \mathcal{O}_{\tilde{X}}$.*

Since Conjecture K is known to be true for $q(X) \geq 2$, we conclude that if in (*) we have $\kappa(X) = 0$ and if $\dim X \leq 4$, then $K_X \equiv 0$.

As a conclusion, nef rank 1 subsheaves $\mathcal{E} \subset \Omega^1_X$ which are proportional to K_X will give some very precise geometric information on X. If we introduce

$$\mathcal{F} = (\Omega^1_X/\mathcal{E})^* \subset T_X\,,$$

then in case $\kappa(X) = 1$, the leaves of $\mathcal{F}$ are just the fibers of the Iitaka fibration $f\colon X \to C$, and in case $\kappa(X) = 0$, possibly after finite étale cover, $\mathcal{F} = pr^*(T_Y,)$ in the notation of theorem 4.5, i.e. the leaves of $\mathcal{F}$ are the fibers of the projection $X \to A$.

References

[1] W. Barth, C. Peters and A. Van de Ven, *Compact complex surfaces,* Springer 1984.

[2] A. Beauville, *Variétés dont la premiere classe de Chern est nulle*, J. Diff. Geom., **18** (1983), 755–782.

[3] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comm. Math. Helv., **73** (1998), 566–583.

[4] A. Beauville, *Riemannian holonomy and Algebraic Geometry*, Duke/alg-geom Preprint 9902110 (1999).

[5] A. Beauville, *Complex manifolds with split tangent bundle*, in: *Complex analysis and algebraic geometry*, volume in honour of M. Schneider, ed. T. Peternell, F. O. Schreyer, 61–70. Birkhäuser 2000.

[6] J. P. Demailly, *Frobenius integrability of certain holomorphic p-forms*, Preprint (2000).

[7] S. Druel, *Contact structures on* 5*-dimensional manifolds*, C. R. Acad. Sci. Paris, **327** (1998), 365–368.

[8] Y. Kawamata, *Abundance theorem for minimal threefolds*, Inv. Math., **108** (1992), 229–246.

[9] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Mat., **10** (1987), 283–360.

[10] S. Kebekus, T. Peternell, A. J. Sommese and J. Wisniewski, *Projective contact manifolds*, to appear in Inv. Math. (2000).

[11] S. Kebekus, T. Peternell, A. J. Sommese and J. Wisniewski, *Nef subsheaves in the cotangent bundle*, Preprint (2000).

[12] J. LeBrun, *Fano manifolds, contact structures and quaternionic geometry*, Int. Journ. Math., **6** (1995), 419–437.

[13] Y. Miyaoka, *On the Kodaira dimension of minimal threefolds*, Math. Ann., **281** (1988), 325–332.

[14] Y. Miyaoka, *Abundance conjecture for* 3*-folds - case* $\nu = 1$, Comp. Math., **68** (1988), 203–220.

[15] Y. Miyaoka and T. Peternell, *Geometry of higher dimensional algebraic varieties*, DMV Seminar, vol. 26. Birkhäuser 1997.

[16] S. Mori and Y. Sumihiro, *On Hartshorne's conjecture*, J. Math. Kyoto Univ., **19** (1978), 523–533.

[17] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*; Birkhäuser 1980.

[18] T. Peternell, *Towards a Mori theory on compact Kähler threefolds, III*, Preprint (1999).

[19] S. M. Salamon, *Quaternionic-Kähler manifolds*, Inv. Math., **67** (1982), 143–171.

[20] J. Wahl, *A cohomological characterisation of* $\mathbb{P}_n$, Inv. Math., **72** (1983), 315–322.

[21] Y.-G. Ye., *A note on complex projective threefolds admitting holomorphic contact structures*, Inv. Math., **115** (1994), 311–314.

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