The Dynamics of Algebraic $\mathbb{Z}^d$-Actions

Klaus Schmidt

Abstract. An *algebraic $\mathbb{Z}^d$-action* is a $\mathbb{Z}^d$-action by automorphisms of a compact abelian group. By Pontryagin duality, there is a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions and modules over the ring R_d of Laurent polynomials with integer coefficients in d commuting variables.

This correspondence establishes a close connection between algebraic and arithmetical properties of R_d-modules and dynamical properties of algebraic $\mathbb{Z}^d$-actions, which is the subject of this article.

1. Algebraic $\mathbb{Z}^d$-Actions and Their Dual Modules

An *algebraic $\mathbb{Z}^d$-action* is an action $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ of $\mathbb{Z}^d$, $d \geq 1$, by continuous automorphisms of a compact abelian group X with Borel field $\mathcal{B}_X$ and normalized Haar measure λ_X. Two algebraic $\mathbb{Z}^d$-actions α and β on compact abelian groups X and Y are *algebraically conjugate* if there exists a continuous group isomorphism $\phi\colon X \longrightarrow Y$ with

$$\phi \cdot \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \cdot \phi \tag{1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$. If the map ϕ in (1) is a homeomorphism then α and β are *topologically conjugate*. Finally we call α and β *measurably conjugate* if there exists a measure space isomorphism $\phi\colon (X, \mathcal{B}_X, \lambda_X) \to (Y, \mathcal{B}_Y, \lambda_Y)$ satisfying (1) λ_X-*a.e.* for every $\mathbf{n} \in \mathbb{Z}^d$.

In [4] and [13], Pontryagin duality was shown to imply a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions (up to algebraic conjugacy) and modules over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ with integral coefficients in the commuting variables $u_1, \dots, u_d$ (up to module isomorphism).

In order to explain this correspondence we write a typical element $f \in R_d$ as

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \tag{2}$$

with $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $c_f(\mathbf{m}) = 0$ for all but finitely many $\mathbf{m}$. If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \widehat{\alpha^{\mathbf{m}}}(a) \tag{3}$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{m}}}$ is the automorphism of $M = \widehat{X}$ dual to $\alpha^{\mathbf{m}}$. In particular,

$$u^{\mathbf{m}} \cdot a = \widehat{\alpha^{\mathbf{m}}}(a) \tag{4}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in M$. Conversely, any R_d-module M determines an algebraic $\mathbb{Z}^d$-action α_M on the compact abelian group $X_M = \widehat{M}$ with $\alpha_M^{\mathbf{m}}$ dual to multiplication by $u^{\mathbf{m}}$ on M for every $\mathbf{m} \in \mathbb{Z}^d$ (cf. (4)). Note that X_M is metrizable if and only if its *dual module* M is countable.

Examples 1.1. (1) *Let $M = R_d$. Since R_d is isomorphic to the direct sum $\sum_{\mathbb{Z}^d} \mathbb{Z}$ of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}^d$, the dual group $X = \widehat{R_d}$ is isomorphic to the Cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}})$ with $x_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and choose the following identification of $X_{R_d} = \widehat{R_d}$ and $\mathbb{T}^{\mathbb{Z}^d}$: for every $x \in \mathbb{T}^{\mathbb{Z}^d}$ and $f \in R_d$,*

$$\langle x, f \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}}, \tag{5}$$

where f is given by (2). Under this identification the $\mathbb{Z}^d$-action α_{R_d} on $X_{R_d} = \mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action

$$(\alpha_{R_d}^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}. \tag{6}$$

(2) *Let $I \subset R_d$ be an ideal and $M = R_d/I$. Since M is a quotient of the additive group R_d by an $\widehat{\alpha_{R_d}}$-invariant subgroup (i.e. by a submodule), the dual group $X_M = \widehat{M}$ is the closed α_{R_d}-invariant subgroup*

$$\begin{aligned} X_{R_d/I} &= \{x \in X_{R_d} = \mathbb{T}^{\mathbb{Z}^d} : \langle x, f \rangle = 1 \text{ for every } f \in I\} \\ &= \Bigl\{ x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = 0 \pmod 1 \\ &\qquad\qquad \text{for every } f \in I \text{ and } \mathbf{m} \in \mathbb{Z}^d \Bigr\}, \end{aligned} \tag{7}$$

and $\alpha_{R_d/I}$ is the restriction of the shift-action α_{R_d} in (6) to the shift-invariant subgroup $X_{R_d/I} \subset \mathbb{T}^{\mathbb{Z}^d}$.

Conversely, let $X \subset \mathbb{T}^{\mathbb{Z}^d} = \widehat{R_d}$ be a closed subgroup, and let

$$X^{\perp} = \{f \in R_d : \langle x, f \rangle = 1 \text{ for every } x \in X\}$$

be the annihilator of X in $\widehat{R_d}$. Then X is shift-invariant if and only if $X^{\perp}$ is an ideal in R_d.

The correspondence between algebraic $\mathbb{Z}^d$-actions $\alpha = \alpha_M$ and R_d-modules M yields a correspondence (or 'dictionary') between dynamical properties of α_M and algebraic properties of the module M (cf. [16]). It turns out that some of the principal dynamical properties of α_M can be expressed entirely in terms of the prime ideals associated with the module M, where a prime ideal $\mathfrak{p} \subset R_d$ is *associated with* M if

$$\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$$

for some $a \in M$. The set of all prime ideals associated with M is denoted by $\operatorname{asc}(M)$; if M is Noetherian, then $\operatorname{asc}(M)$ is finite.

Figure 1 provides a small illustration of this correspondence; all the relevant results can be found in [16]. In the third column we assume that the R_d-module $M = \widehat{X}$ defining α is of the form $R_d/\mathfrak{p}$, where $\mathfrak{p} \subset R_d$ is a prime ideal, and describe the algebraic condition on $\mathfrak{p}$ equivalent to the dynamical condition on $\alpha = \alpha_{R_d/\mathfrak{p}}$ appearing in the second column. In the fourth column we consider a countable R_d-module M and state the algebraic property of M corresponding to the property of $\alpha = \alpha_M$ in the second column.

	Property of α	$\alpha = \alpha_{R_d/\mathfrak{p}}$	$\alpha = \alpha_M$
(1)	α is expansive	$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$	M is Noetherian and $\alpha_{R_d/\mathfrak{p}}$ is expansive for every $\mathfrak{p} \in \operatorname{asc}(M)$
(2)	$\alpha^{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$	$u^{k\mathbf{n}} - 1 \notin\subset \mathfrak{p}$ for every $k \geq 1$	$\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic for every $\mathfrak{p} \in \operatorname{asc}(M)$
(3)	α is ergodic	$\{u^{k\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \not\subset \mathfrak{p}$ for every $k \geq 1$	$\alpha_{R_d/\mathfrak{p}}$ is ergodic for every $\mathfrak{p} \in \operatorname{asc}(M)$
(4)	α is mixing	$u^{\mathbf{n}} - 1 \notin \mathfrak{p}$ for every non-zero $\mathbf{n} \in \mathbb{Z}^d$	$\alpha_{R_d/\mathfrak{p}}$ is mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$
(5)	α is mixing of every order	Either $\mathfrak{p}$ is equal to pR_d for some rational prime p, or $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ and $\alpha_{R_d/\mathfrak{p}}$ is mixing	For every $\mathfrak{p} \in \operatorname{asc}(M)$, $\alpha_{R_d/\mathfrak{p}}$ is mixing of every order
(6)	$h(\alpha) > 0$	$\mathfrak{p}$ is principal and $\alpha_{R_d/\mathfrak{p}}$ is mixing	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for at least one $\mathfrak{p} \in \operatorname{asc}(M)$
(7)	$h(\alpha) < \infty$	$\mathfrak{p} \neq \{0\}$	If M is Noetherian: $\mathfrak{p} \neq \{0\}$ for every $\mathfrak{p} \in \operatorname{asc}(M)$
(8)	α has completely positive entropy (or is Bernoulli)	$h(\alpha^{R_d/\mathfrak{p}}) > 0$	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for every $\mathfrak{p} \in \operatorname{asc}(M)$

FIGURE 1: A POCKET DICTIONARY

The notation in figure 1 is as follows. In (1),

$$V_{\mathbb{C}}(\mathfrak{p}) = \{c \in (\mathbb{C} \smallsetminus \{0\})^d : f(c) = 0 \text{ for every } f \in \mathfrak{p}\}$$

is the *variety* of $\mathfrak{p}$, and $\mathbb{S} = \{c \in \mathbb{C} : |c| = 1\}$. From (2)–(4) it is clear that α is ergodic if and only if $\alpha^{\mathbf{n}}$ is ergodic for *some* $\mathbf{n} \in \mathbb{Z}^d$, and that α is mixing if and only if $\alpha^{\mathbf{n}}$ is ergodic for *every nonzero* $\mathbf{n} \in \mathbb{Z}^d$. In (5), α is mixing of order $r \geq 2$ if

$$\lim_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_r \in \mathbb{Z}^d \\ \|\mathbf{n}_i - \mathbf{n}_j\| \to \infty \text{ for } 1 \leq i < j \leq d}} \lambda_X \left(\bigcap_{i=1}^r \alpha^{-\mathbf{n}_i} B_i \right) = \prod_{i=1}^r \lambda_X(B_i)$$

for all Borel sets $B_i \subset X$, $i = 1, \dots, r$. In (6)–(8), $h(\alpha)$ stands for the topological entropy of α (which coincides with the metric entropy $h_{\lambda_X}(\alpha)$). In [8] and [16] there is an explicit entropy formula for algebraic $\mathbb{Z}^d$-actions. In the special case where $\alpha = \alpha_{R_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R_d$ this formula reduces to

$$h(\alpha) = \begin{cases} |\log \mathsf{M}(f)| & \text{if } \mathfrak{p} = (f) = fR_d \text{ is principal}, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\mathsf{M}(f) = \begin{cases} \exp\bigl(\int_{\mathbb{S}^d} \log |f(\mathbf{s})| \, d\mathbf{s}\bigr) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0, \end{cases}$$

is the *Mahler measure* of the polynomial f. Here $d\mathbf{s}$ denotes integration with respect to the normalized Haar measure on the multiplicative subgroup $\mathbb{S}^d \subset \mathbb{C}^d$.

For background, details and proofs of these and further results we refer to [16] and the original articles cited there. The remainder of this note is devoted to two particular problems: the higher order mixing behaviour and the conjugacy problem for algebraic $\mathbb{Z}^d$-actions.

2. Higher Order Mixing Properties of Algebraic $\mathbb{Z}^d$-Actions

In this section we describe the connection between higher order mixing properties of algebraic $\mathbb{Z}^d$-actions and certain diophantine results on additive relations in fields due to Mahler ([9]), Masser ([10, 5]) and Schlickewei, W. Schmidt and van der Poorten ([1, 17]). In the discussion below we shall use the following elementary consequence of Pontryagin duality:

Lemma 2.1. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module M. Then X is connected if and only if* no *prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant, and X is zero-dimensional if and only if* every *$\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant.*

Let $\mathfrak{p} \subset R_d$ be a prime ideal, and let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action with dual module $M = R_d/\mathfrak{p} = \widehat{X}$. If α is not mixing, then there exist Borel sets $B_1, B_2 \subset X$ and a sequence $(\mathbf{n}_k,\, k \geq 1)$ in $\mathbb{Z}^d$ with $\lim_{k\to\infty} \mathbf{n}_k = \infty$ and

$$\lim_{k \to \infty} \lambda_X(B_1 \cap \alpha^{-\mathbf{n}_k} B_2) = c$$

for some $c \neq \lambda_X(B_1)\lambda_X(B_2)$. Fourier expansion implies that the latter condition is equivalent to the existence of nonzero elements $a_1, a_2 \in M$ such that

$$a_1 + u^{\mathbf{n}_k} \cdot a_2 = 0$$

for infinitely many $k \geq 1$. In particular,

$$(u^{\mathbf{m}} - 1) \cdot a_2 = 0 \tag{8}$$

for some nonzero $\mathbf{m} \in \mathbb{Z}^d$ (cf. figure 1(4)). A very similar argument shows that α is not mixing of order $r \geq 2$ if and only if there exist elements $a_1, \ldots, a_r$ in M, not all equal to zero, and a sequence $((\mathbf{n}_k^{(1)}, \ldots, \mathbf{n}_k^{(r)}),\ k \geq 1)$ in $(\mathbb{Z}^d)^r$ such that $\lim_{k\to\infty} \|\mathbf{n}_k^{(i)} - \mathbf{n}_k^{(j)}\| = \infty$ for all i, j with $1 \leq i < j \leq r$, and with

$$u^{\mathbf{n}_k^{(1)}} \cdot a_1 + \cdots + u^{\mathbf{n}_k^{(r)}} \cdot a_r = 0 \tag{9}$$

for every $k \geq 1$.

Below we shall see that higher order mixing of an algebraic $\mathbb{Z}^d$-action α on a compact abelian group X can break down in a particularly regular way (cf. examples 2.7 and 2.10). We call a nonempty finite subset $S \subset \mathbb{Z}^d$ *mixing* under α if

$$\lim_{k\to\infty} \lambda_X\Bigl(\bigcap_{\mathbf{n}\in S} \alpha^{-k\mathbf{n}} B_{\mathbf{n}}\Bigr) = \prod_{\mathbf{n}\in S} \lambda_X(B_{\mathbf{n}}) \tag{10}$$

for all Borel sets $B_{\mathbf{n}} \subset X$, $\mathbf{n} \in S$, and *nonmixing* otherwise. If α is r-mixing, then every set $S \subset \mathbb{Z}^d$ with cardinality $|S| = r$ is obviously mixing. The validity of the reverse implication for algebraic $\mathbb{Z}^d$-actions is an open problem (cf. problem 2.11 and conjecture 2.12).

As in (10) one sees that a nonempty finite set $S \subset \mathbb{Z}^d$ is nonmixing if and only if there exist elements $a_{\mathbf{n}} \in M$, $\mathbf{n} \in S$, not all equal to zero, such that

$$\sum_{\mathbf{n}\in S} u^{k\mathbf{n}} \cdot a_{\mathbf{n}} = 0 \tag{11}$$

for infinitely many $k \geq 1$.

The higher order mixing behaviour of an algebraic $\mathbb{Z}^d$-action α with dual module M is again completely determined by that of the actions $\alpha_{R_d/\mathfrak{p}}$ with $\mathfrak{p} \in \operatorname{asc}(M)$.

Theorem 2.2. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$.*

(1) *For every $r \geq 2$, the following conditions are equivalent:*
   (a) *α is r-mixing,*
   (b) *$\alpha_{R_d/\mathfrak{p}}$ is r-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

(2) *For every nonempty finite set $S \subset \mathbb{Z}^d$, the following conditions are equivalent:*
   (a) *S is α-mixing,*
   (b) *S is $\alpha_{R_d/\mathfrak{p}}$-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

In order to exhibit the connection between mixing properties and additive relations in fields we begin with a theorem by Mahler.

Theorem 2.3. ([9]) *Let K be a field of characteristic 0, $r \geq 2$, and let $x_1, \dots, x_r$ be nonzero elements of K. If we can find nonzero elements $c_1, \dots, c_r$ such that the equation*

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

has infinitely many solutions $k \geq 0$, then there exist integers $s \geq 1$ and i, j with $1 \leq i < j \leq r$ such that $x_i^s = x_j^s$.

We denote by K the field of fractions of the integral domain $R_d/\mathfrak{p}$, choose a finite set $S = \{\mathbf{n}_1, \dots, \mathbf{n}_r\} \subset \mathbb{Z}^d$ with $r \geq 2$, and set $x_i = u^{\mathbf{n}_i}$ for $i = 1, \dots, r$. In view of figure 1(4)–(5), lemma 2.1, (8), (11) and theorem 2.2, theorem 2.3 implies (and is, in fact, equivalent to) the following statement:

Theorem 2.4. ([14]) *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then every nonempty finite subset $S \subset \mathbb{Z}^d$ is mixing.*

If an algebraic $\mathbb{Z}^d$-action α is not mixing of every order, then there exists a smallest integer $r \geq 2$ such that α is not r-mixing. As a consequence of lemma 2.1 and (9) one obtains the equivalence of the theorems 2.5 and 2.6 below.

Theorem 2.5. ([1, 17]) *Let K be a field of characteristic 0 and G a finitely generated multiplicative subgroup of $K^\times = K \smallsetminus \{0\}$. If $r \geq 2$ and $(c_1, \dots, c_r) \in (K^\times)^r$, then the equation*

$$\sum_{i=1}^{r} c_i x_i = 0 \tag{12}$$

has only finitely many solutions $(x_1, \dots, x_r) \in G^r$ such that no sub-sum of (12) vanishes.

Theorem 2.6. ([15]) *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then α is mixing of every order.*

The 'absolute' version of the S-unit theorem in [1] contains a bound on the number of solutions of (12) without vanishing subsums which is expressed purely in terms of the integer r and the rank of the group G (in our setting: the order of mixing and the rank of the group $\mathbb{Z}^d$). This bound could be used, for example, to obtain quite remarkable uniform statements on the speed of multiple mixing for all irreducible and mixing algebraic $\mathbb{Z}^d$-actions (cf. definition 3.1).

For algebraic $\mathbb{Z}^d$-actions on disconnected groups the situation is considerably more complicated due to the possible presence of nonmixing sets (cf. (10)).

Example 2.7. ([7]) *Let $\mathfrak{p} = (2, 1+u_1+u_2) = 2R_2 + (1+u_1+u_2)R_2$, $M = R_2/\mathfrak{p}$, and let $\alpha = \alpha_M$ be the algebraic $\mathbb{Z}^2$-action on $X = X_M = \widehat{M}$ defined in example 1.1(2). Then α is mixing by figure 1(4), but not three-mixing.*

Indeed, $(1 + u_1 + u_2)^{2^n} \cdot a = 0$ for every $n \geq 0$ and $a \in M$. For $a = 1 + (2, 1 + u_1 + u_2) \in M$ our identification of M with $\widehat{X}$ in example 1.1(2) implies *that* $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0 \pmod 1$ *for every* $x \in X$ *and* $n \geq 0$. *For* $B = \{x \in X : x_{(0,0)} = 0\}$ *it follows that*

$$B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B) = B \cap \alpha^{-(2^n,0)}(B)\,,$$

and hence that

$$\lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B \cap \alpha^{-(2^n,0)}(B)) = 1/4$$

for every $n \geq 0$. *If* α *were three-mixing, we would have that*

$$\lim_{n\to\infty} \lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B)^3 = 1/8\,.$$

By comparing this with (10) *we see that the set* $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ *is nonmixing.*

A mixing algebraic $\mathbb{Z}^d$-action α on a disconnected compact abelian group X has nonmixing sets if and only if it is not Bernoulli (cf. figure 1(8), [5] and [16, Section 27]). In particular, if α is an ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X with zero entropy, then α has nonmixing sets. The description of the nonmixing sets of such an action α is facilitated by a Theorem of Masser ([5, 10]), which should be seen as an analogue of theorem 2.3 in positive characteristic.

Theorem 2.8. *Let* K *be an algebraically closed field of characteristic* $p > 0$, $r \geq 2$, *and let* $(x_1, \dots, x_r) \in (K^\times)^r$. *The following conditions are equivalent:*

(1) *There exists an element* $(c_1, \dots, c_r) \in (K^\times)^r$ *such that*

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

for infinitely many $k \geq 0$;

(2) *There exists a rational number* $s > 0$ *such that the set* $\{x_1^s, \dots, x_r^s\}$ *is linearly dependent over the algebraic closure* $\bar{F}_p \subset K$ *of the prime field* $F_p = \mathbb{Z}/p\mathbb{Z}$.

Corollary 2.9. *Let* $\mathfrak{p} \subset R_d$ *be a prime ideal containing a rational prime* $p > 1$, *and let* $\alpha = \alpha_{R_d/\mathfrak{p}}$ *be the algebraic* $\mathbb{Z}^d$-*action on* $X = X_{R_d/\mathfrak{p}}$ *defined in example 1.1(2). We denote by* $K = Q(R_2/\mathfrak{p}) \supset R_2/\mathfrak{p}$ *the quotient field of* $R_d/\mathfrak{p}$, *write* $\bar{K}$ *for its algebraic closure, and set* $x_{\mathbf{n}} = u^{\mathbf{n}} + \mathfrak{p} \in R_d/\mathfrak{p} \subset K \subset \bar{K}$ *for every* $\mathbf{n} \in \mathbb{Z}^d$. *If* $S \subset \mathbb{Z}^d$ *is a nonempty finite set, then the following conditions are equivalent:*

(1) S *is not* α-*mixing;*

(2) *There exists a rational number* $s > 0$ *such that the set* $\{x_1^s, \dots, x_r^s\} \subset \bar{K}$ *is linearly dependent over* $\bar{F}_p \subset K$.

Examples 2.10. ([5])

(1) *In the notation of examples 2.7 and 1.1(2) we set* $f = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 \in R_2$ *and put* $\mathfrak{p} = (2, f) \subset R_2$, $M = R_2/\mathfrak{p}$, $\alpha = \alpha_M$

and $X = X_M = \widehat{M}$. *We claim that the set* $S = \{(0,0),(1,0),(0,1)\}$ *is nonmixing.*

In order to verify this we define $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\} \subset K = Q(R_2/\mathfrak{p})$ *as in corollary 2.9 and choose* $\omega \in \bar{F}_2 \subset \bar{K}$ *with* $1 + \omega + \omega^2 = 0$. *Since*

$$f = (1 + \omega u_1 + \omega^2 u_2)(1 + \omega^2 u_1 + \omega u_2)\,,$$

we obtain that $x_{(0,0)} + \omega x_{(1,0)} + \omega^2 x_{(0,1)} = 0$, *so that* S *is nonmixing by corollary 2.9.*

Since the element $\omega' = \frac{1+u_1}{u_1+u_2} + \mathfrak{p} \in K$ *satisfies that* $1 + \omega' + \omega'^2 = 0$, *we can recover* (11) *from the fact that*

$$(u_1 + u_2) + (1 + u_2)u_1^{3k} + (1 + u_1)u_2^{3k} \in \mathfrak{p}$$

for every $k \geq 0$.

(2) *Let* $g = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 + u_1^3 + u_1^2u_2 + u_1u_2^2 + u_2^3$ *and* $\mathfrak{q} = (2, g) \subset R_2$, $M = R_2/\mathfrak{q}$, $\alpha = \alpha_M$ *and* $X = X_M = \widehat{M}$. *We claim that the set* $S = \{(0,0),(1,0),(0,1)\}$ *is again nonmixing.*

In example (1) above we used the fact that f *is irreducible over* F_2, *but not over* $\bar{F}_2$. *Here the polynomial* g *is irreducible over* $\bar{F}_2$; *however, the polynomial* $g(u_1^3, u_2^3)$ *turns out to be divisible by* $1 + u_1 + u_2$, *which can be translated into the statement that the set* $\{x_{(0,0)}^{1/3}, x_{(1,0)}^{1/3}, x_{(0,1)}^{1/3}\}$ *is linearly dependent over* $\bar{F}_2$.

The main open question concerning higher order mixing is the following:

Problem 2.11. *Let* α *be an algebraic* $\mathbb{Z}^d$*-action on a compact abelian group* X, *and let* $r \geq 2$. *If every subset* $S \subset \mathbb{Z}^d$ *of cardinality* r *is mixing, is* α r*-mixing?*

A positive answer to problem 2.11 would be equivalent to the following analogue of theorem 2.5 in characteristic $p > 0$:

Conjecture 2.12. *Let* K *be an algebraically closed field of characteristic* $p > 0$, $G \subset K^\times = K \smallsetminus \{0\}$ *a finitely generated multiplicative group,* $r \geq 2$, *and* $(c_1, \dots, c_r) \in (K^\times)^r$. *Let us call a solution* $(x_1, \dots, x_r) \in G^r$ *of the equation*

$$\sum_{i=1}^{r} c_i x_i = 0 \tag{13}$$

regular *if there exists a rational number* $s > 0$ *such that* $\{x_1^s, \dots, x_r^s\}$ *is linearly dependent over* $\bar{F}_p \subset K$, *and* irregular *otherwise.*

Then the equation (13) *has only finitely many irregular solutions.*

3. Conjugacy of Algebraic $\mathbb{Z}^d$-Actions

Every algebraic $\mathbb{Z}^d$-action α with completely positive entropy is measurably conjugate to a Bernoulli shift (cf. figure 1(8)). Since entropy is a complete invariant for measurable conjugacy of Bernoulli shifts by [11], α is measurably conjugate to the $\mathbb{Z}^d$-action

$$\alpha^A\colon \mathbf{n} \mapsto \alpha^{A\mathbf{n}}$$

for every $A \in \mathrm{GL}(d,\mathbb{Z})$, since the entropies of all these actions coincide. In general, however, α and α^A are not topologically conjugate.

Every algebraic $\mathbb{Z}^d$-action α with positive entropy has Bernoulli factors by [8] and [12], and two such actions may again be measurably conjugate without being algebraically or topologically conjugate. For zero entropy actions, however, there is some evidence for a very strong form of isomorphism rigidity. Let us begin with a special case.

Definition 3.1. *An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is* irreducible *if every closed, α-invariant subgroup $Y \subsetneq X$ is finite.*

Irreducibility is an extremely strong hypothesis: if α is mixing it implies that $\alpha^{\mathbf{n}}$ is Bernoulli with finite entropy for every nonzero $\mathbf{n} \in \mathbb{Z}^d$. If β is a second irreducible and mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group Y such that $h(\alpha^{\mathbf{n}}) = h(\beta^{\mathbf{n}})$ for every $\mathbf{n} \in \mathbb{Z}^d$, then $\alpha^{\mathbf{n}}$ is measurably conjugate to $\beta^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. However, if $d > 1$, then the actions α and β are generally nonconjugate.

Theorem 3.2. ([2, 6]) *Let $d > 1$, and let α and β be irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. If $\phi\colon X \longrightarrow Y$ is a measurable conjugacy of α and β, then ϕ is λ_X-*a.e. *equal to an affine map (a map $\phi\colon X \longrightarrow Y$* affine *if it is of the form $\phi(x) = \psi(x) + y$ for every $x \in X$, where $\psi\colon X \longrightarrow Y$ is a continuous group isomorphism and $y \in Y$). In particular, measurable conjugacy implies algebraic conjugacy.*

If the irreducible actions α and β in theorem 3.2 are of the form $\alpha = \alpha_{R_d/\mathfrak{p}}$ and $\beta = \alpha_{R_d/\mathfrak{q}}$ for some prime ideals $\mathfrak{p}, \mathfrak{q} \subset R_d$, then measurable conjugacy implies that $\mathfrak{p} = \mathfrak{q}$. This allows the construction of algebraic $\mathbb{Z}^d$-actions with very similar properties which are nevertheless measurably nonconjugate.

Example 3.3. *Consider the algebraic $\mathbb{Z}^2$-actions α, α', α'' on $X = \mathbb{T}^3$ generated by the matrices*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 8 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 8 & 4 \end{pmatrix},$$

$$A' = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -6 & 9 & 2 \end{pmatrix} \quad \textit{and} \quad B' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -6 & 9 & 4 \end{pmatrix},$$

$$A'' = \begin{pmatrix} -3 & 4 & 0 \\ -3 & 3 & 1 \\ -10 & 11 & 2 \end{pmatrix} \quad \textit{and} \quad B'' = \begin{pmatrix} -1 & 4 & 0 \\ -3 & 5 & 1 \\ -10 & 11 & 4 \end{pmatrix},$$

respectively. In [2] *it was shown that these actions are not measurably conjugate, although it appears difficult to distinguish them with the usual invariants of measurable conjugacy.*

Example 3.4. (Nonconjugacy of $\mathbb{Z}^2$-actions with positive entropy) *Let*

$$\begin{aligned} f_1 &= 1 + u_1 + u_1^2 + u_1u_2 + u_2^2\,, \\ f_2 &= 1 + u_1^2 + u_2 + u_1u_2 + u_2^2\,, \\ f_3 &= 1 + u_1 + u_1^2 + u_2 + u_2^2\,, \\ f_4 &= 1 + u_1 + u_1^2 + u_2 + u_1u_2 + u_2^2\,, \end{aligned}$$

in R_2, put $\mathfrak{p}_i = (2, f_i) \subset R_2$, $J_i = (4, f_i) \subset R_2$, $M_i = R_2/J_i$, and define the algebraic $\mathbb{Z}^2$-actions $\alpha_i = \alpha_{R_2/J_i}$ on $X_i = X_{R_2/J_i}$ as in example 1.1(2). Then $h(\alpha_{R_2/\mathfrak{q}}) = \log 2$ and $h(\alpha_{R_2/\mathfrak{p}_i}) = 0$, and [8, Theorem 6.5] *implies that the Pinsker algebra $\pi(\alpha_i)$ of α_i is the sigma-algebra $\mathcal{B}_{X_i/Y_i}$ of Y_i-invariant Borel sets in X_i, where $Y_i = N_i^{\perp}$ and*

$$N_i = \{a \in M_i : \mathfrak{p}_i \cdot a = 0\} = 2M_i \cong R_2/\mathfrak{p}_i\,.$$

In other words, the $\mathbb{Z}^2$-action β_i induced by α_i on the Pinsker algebra $\pi(\alpha_i)$ is measurably conjugate to $\alpha_{R_2/\mathfrak{p}_i}$.

Since any measurable conjugacy of α_i and α_j would map $\pi(\alpha_i)$ to $\pi(\alpha_j)$ and induce a conjugacy of β_i and β_j, theorem 3.2 implies that α_i and α_j are measurably nonconjugate for $1 \le i < j \le 4$.

The basic idea of the proof of theorem 3.2 in [2] and [5] was suggested by Thouvenot: if $\phi\colon X \longrightarrow Y$ is a measurable conjugacy of α and β, then there exists a unique probability measure ν on the graph $\Gamma(\phi) = \{(x, \phi(x)) : x \in X\} \subset X \times Y$ which projects to λ_X and λ_Y, respectively, and which is invariant under the product-action $\alpha \times \beta\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}} \times \beta^{\mathbf{n}}$ of $\mathbb{Z}^d$ on $X \times Y$. Since $\alpha \times \beta$, acting on $(X \times Y, \nu)$, is measurably conjugate both to α and to β, the measure ν is mixing and has positive entropy under $\alpha^{\mathbf{n}} \times \beta^{\mathbf{n}}$ for every nonzero $\mathbf{n} \in \mathbb{Z}^d$. The proof of theorem 3.2 consists of showing that ν is a translate of the Haar measure of some closed $(\alpha \times \beta)$-invariant subgroup of $X \times Y$ (this obviously implies that ϕ is affine). If X and Y are connected, the relevant property of ν follows from [3], and if X and Y are zero-dimensional, the nonmixing sets of ν provide the necessary tool in [6].

Since there are considerable difficulties in extending either of these techniques to general algebraic $\mathbb{Z}^d$-actions with zero entropy, the following conjecture may seem a little premature, but I would still like to risk stating it:

Conjecture 3.5. *Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. If $h(\alpha) = 0$, and if $\phi\colon X \longrightarrow Y$ is a measurable conjugacy of α and β, then ϕ is λ_X-a.e. equal to an affine map. In particular, measurable conjugacy implies algebraic conjugacy.*

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Mathematics Institute,
University of Vienna,
Strudlhofgasse 4,
A-1090 Vienna, Austria
E-mail address: `klaus.schmidt@univie.ac.at`

Erwin Schrödinger Institute for Mathematical Physics,
Boltzmanngasse 9,
A-1090 Vienna, Austria