The Impact of Monotonicity Formulas in Regularity of Free Boundaries

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Abstract. In this note we give a survey on recent developments in the regularity of free boundaries of obstacle type in absence of the obstacle, giving rise to solutions that may change sign. The focus is on two techniques, the monotonicity formulas and global versus local analysis.

1. Introduction

For a bounded domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ we assume that there exists a function u such that locally

$$(\Delta u - f)u = 0 \quad \text{in } B(x^0, r_0) \setminus \Omega, \quad x^0 \in \partial\Omega\,, \tag{1}$$

where $f > 0$ is Lipschitz, and the equation is satisfied in the sense of distributions.

Two main questions that comes up immediately are the following:

Q1) How smooth is u across $\partial\Omega$?
Q2) How smooth is $\partial\Omega$ in a neighborhood of x^0?

The problem described above has its origin in inverse problems of potential theory, also known as harmonic continuations of potentials from the free space into the domain of integration. To explain this in more detail let U denote the Newtonian potential of Ω (bounded set) with constant density (i.e., the convolution of the fundamental solution with χ_Ω) and with $x^0 \in \partial\Omega$. Then suppose (this is not necessarily the case in general) there exists a harmonic function w in $B(x^0, r)$ (r small) such that $w = U$ in Ω^c (the complement of Ω); observe that U is harmonic in Ω^c. This property is referred to as harmonic continuation of potentials.

The reader familiar with elliptic theory can immediately see that Q1) can be partially answered. Indeed, by elliptic estimate u is $C^{1,\alpha}$ in $B(x^0, r_0/2)$ for $\alpha < 1$. A more elaborated estimate may also be shown using the potential representation. This yields the estimate

$$|\nabla u(x) - \nabla u(x^0)| \leq C|x - x^0| \log |x - x^0|\,.$$

The problem here is to get rid of the log-term.

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In the case of obstacle problem, i.e., when $u \geq 0$ one may easily obtain that $u \in C^{1,1}(B(x^0, r_0/2))$. This can be done using Harnack's inequality or simple estimate based on the so called "Schwarz potential" (the difference between the paraboloid $|x-x^0|^2/2n$ and the fundamental solution with source at x^0). We refer to [9] for some details in the latter case. For the application of the Harnack's inequality we refer to [2].

Now the "no-sign-assumption" in problem (1) introduces a new and actually a very peculiar difficulty. To overcome this difficulty the authors in [8] have used the monotonicity formula developed by [1]. Unfortunately the technique works(ed) only for the case when $f \equiv 1$, since then $D_e u$, where D_e denotes partial derivative in direction e, will be a harmonic function and consequently $(D_e u)^{\pm}$ will be subharmonic functions with disjoint support. These are the exact condition in the monotonicity formula.

2. The Use of the Monotonicity Formula

To discuss the approach of [8] in answering Q1)–Q2) we state the following monotonicity lemma.

Lemma 2.1. (See [1, Lemma 5.1]**)** *Let h_1, h_2 be two non-negative continuous subsolutions of $\Delta u = 0$ in $B(x^0, R)$ $(R > 0)$. Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$. Then the following function is monotone in r $(0 < r < R)$*

$$\varphi(r) = \frac{1}{r^4} \left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2}{|x-x^0|^{n-2}} \right) \left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2}{|x-x^0|^{n-2}} \right).$$

Now in the above lemma one replaces h_i with $(D_e u)^{\pm}$. Since, heuristically $\varphi(0, D_e u) \approx D_{ij} u(0)$ one expects to obtain a uniform bound. Let us briefly explain this, for more detail we refer to [8, Theorem I]. First let us define

$$S_r = \sup_{B_r} |u(x)|,$$

where B_r denotes the ball of radius r, centered at the origin. Here we assume that $x^0 \in \partial\Omega$ is the origin. Since the Laplacian is rotation and translation invariant we can do the same argument for any point of $\partial\Omega$ other than the origin. Next we expect to have

$$S_r \leq Cr^2, \quad \forall\, r < 1/2,$$

and for some constant C depending on the sup-norm of u and the space dimension only. Therefore we claim that there exists C such that

$$S_{r/2} \leq \max\left(\frac{S_r}{4}, Cr^2\right) \quad \forall\, r < 1/2. \tag{2}$$

If this is true then we are done using iteration. The idea of introducing (2) (rather than the more difficult approach in [8]) is due to N. Uraltseva; see [11]). Now we suppose, towards a contradiction, that (2) fails. Then there exists a sequence $\{r_j\}$

(and if we want to have a uniformity for a class of solutions, also a sequence of solutions u_j to (1), but here for simplicity we do this for just one function) such that (2) fails. More exactly we have

$$S_{r_j/2} \geq \max\left(\frac{S_{r_j}}{4}, jr_j^2\right), \quad j = 1, 2, \dots \tag{3}$$

Now the reader may verify that (3) along with the properties of u implies that

$$u_j(x) = \frac{u(r_j x)}{S_{r_j}}$$

is uniformly bounded on B_1 and

$$|\Delta u_j| \leq \frac{4}{j}\,.$$

Consequently a subsequence of $\{u_j\}$ converges uniformly to a harmonic function u_0 in the unit ball and that $u_0(0) = |\nabla u_0|(0) = 0$. Moreover, from (3), it follows that

$$\max_{B_{1/2}} |u_0| = 1\,. \tag{4}$$

Next we use the monotonicity formula to obtain that

$$\varphi(1, D_e u_j) = \left(\frac{r_j^2}{S_{r_j}}\right)^4 \varphi(r_j, D_e u) \leq C' \left(\frac{r_j^2}{S_{r_j}}\right)^4 \to 0\,,$$

by (3).

In particular this implies that

$$\varphi(1, D_e u_0) = 0\,,$$

and consequently either of $(D_e u_0)^{\pm}$ is to be zero. In particular $D_e u_0$ is a harmonic function that does not change sign. Since also $|\nabla u_0(0)| = 0$ we must have $D_e u_0(0) = 0$ for all directions e and hence u_0 is constant. The constant is zero since $u_0(0) = 0$. This contradicts (4) and therefore (2) must be true. Now from the estimate in (2) one can further show that u is $C^{1,1}$ using classical elliptic regularity.

One thing that we have not been careful with is the convergence in $W^{2,2}$ in the monotonicity formula as $r_j \to 0$. However, the proof of this with all the details is given in [8, Theorem I]. For the general Lipschitz right hand sides the above monotonicity formula is out of use. However, newly developed monotonicity formulas by L. Caffarelli, D. Jerison, and C. Kenig [7] will still help us to obtain the $C^{1,1}$ estimate for solutions of (1). Here we formulate their result.

Lemma 2.2. (See [7]**)** *Recall the assumptions in lemma 2.1, and replace the subharmonicity assumption by the boundedness of the Laplacian of h_i, i.e., assume $|\Delta h_i| \leq 1$. Suppose moreover $|h_i(x)| \leq C|x|^\epsilon$ for some $\epsilon > 0$. Then, for $0 < r_1 \leq r_2 \leq R_0$,*

$$\varphi(r_1) \leq (1 + r_2^\epsilon)\varphi(r_2) + Cr_2^\epsilon\,.$$

The question that raises itself is: What is the best result in this direction?. Can we relax the Lipschitz condition on f and still obtain such a result. It is known that the regularity of f can be relaxed but has to be compensated somehow, e.g., it is known that if the complement of Ω is thick enough (in the sense of capacitary density) near the origin then we still obtain a quadratic growth for the function u; see [10]. Of course not a $C^{1,1}$ estimate, as it is obvious. However, this is enough to blow up solutions and work with global solutions, a notion that is almost inevitable in the context of regularity of free boundaries. Let us explain this in more detail.

A global solution to problem (1) is a solution u of

$$(\Delta u - 1)u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega\,,$$

with the extra assumption that

$$|u(x)| \leq (|x| + 1)^2\,.$$

In other words a solution in the entire space. In [8, Theorem II] the authors prove that global solutions are either quadratic polynomials or non-negative convex functions. In particular if the complement of the set Ω has non-empty interior then global solutions are convex, and consequently $\mathbb{R}^n \setminus \Omega$ is convex.

Now this classification is the core of the analysis of the regularity of free boundaries. Indeed, if one assumes an a priori thickness condition on the complement of Ω near a free boundary point x^0 then a blow up of u at x^0 will result in a global convex solution u_0, with x^0 on the free boundary $\partial\Omega(u_0)$. Here we choose to explain the technique for just one function and not a whole class of functions. The technique for a class is very much similar but becomes more technical. We avoid this here.

The best way is to think of Ω to have a truncated cone outside it with vertex at x^0. Then one immediately verifies that the same property is inherited by the blow-ups of u, whatever the sequence of blow-up be. The reader should notice that there might well be examples such that blow up w.r.t. two different sequences $\{r_j\}$ and $\{t_j\}$ are different functions. One of the main difficulties in the theory is to prove that this is not the case, provided the free boundary does not develop cusp singularities.

Next having a convex global solution one needs to prove that these solutions have locally C^1 (actually analytic) boundary. This is done more easily, since convexity of the complement implies Lipschitz regularity of $\partial\Omega$. One may also easily see that the boundary is C^1 by just blowing up u at a boundary point x^0. Now a blow up at a boundary point will give a homogeneous solution. Indeed we have

$$\varphi(s, D_e u_0) = \lim_{r\to 0} \varphi(s, D_e u_r) = \lim_{r\to 0} \varphi(sr, D_e u) = C_e\,, \tag{5}$$

for all $s > 0$.

The problem in (5) is that we cannot use a different sequence of functions u_j since then the limit C_e is not necessarily unique, and the problem becomes very involved in the case we consider a whole class of functions.

Now, going back to our simple situation of just one function, we have that global solutions with non-void $\mathbb{R}^n \setminus \overline{\Omega}$ are convex, and locally the free boundary is C^1. Now we use this idea for local solutions, by scaling the function u with u_{r_j} and assume that

$$B_{r_j} \setminus \Omega(u) \quad \text{is thick enough}\,. \tag{6}$$

Now u_{r_j} is a solution in B_{1/r_j} (almost a global solution, or which is known as approximate global solution) and by (6)

$$B_1 \setminus \Omega(u_{r_j}) \quad \text{is thick enough}\,; \tag{7}$$

the thickness property introduced by L. Caffarelli [4] is stable under scaling. Now (7), in conjunction with the classification of global solutions, imply that $\partial\Omega(u_{r_j})$ near the origin is almost C^1 (or flat). Now from here one wants to go further to prove that the boundary is actually C^1. One way of doing this, and this is the most simple technique developed by L. Caffarelli, is to apply the maximum principle to the function $w_j^e = Cr_j D_e u - u$ in $\Omega(u) \cap B_{r_j}$ to show that, for some e and small r_j, this function is non-negative in B_{r_j}, provided u_{r_j} is close enough to a global solution with $\partial\Omega(u_{r_j})$ flat enough in the unit ball; see [5] and [8] for the simple proof.

Once it is shown that w_j^e is nonnegative, we can integrate by parts to obtain $u \geq 0$ in $B_{r_j/4}$. Actually one can show that $w_j^e \geq 0$ for a whole set of vectors e, i.e., in B_{r_j}

$$w_j^e \geq 0 \quad \forall \quad e \in K_j := \{\nu\colon \nu \cdot e_0 \geq \frac{1}{j}\}\,,$$

where e_0 depends on u. From here one obtains the Lipschitz regularity of the free boundary without much efforts. The next step is to show that the Lipschitz norm gets better, if we choose smaller balls. One shows that for each $j > 0$ there exists r_j such that $w_j^e \geq 0$ in B_{r_j} for all $e \in K_j$. This gives the C^1-regularity of $\partial\Omega$.

Finally, to illuminate the importance of the monotonicity formula, we give an application of it in classification of global solutions, see [8, Theorem II]. Cf. also [11, Theorem B]. So let us assume u is a global solution, i.e., a solution to (1) in entire $\mathbb{R}^n$. To give a simplified example, suppose also that there exists a sequence $R_j \to \infty$ such that

$$\frac{\mathrm{vol}(B_{R_j} \setminus \Omega)}{\mathrm{vol}(B_{R_j})} \geq \epsilon_0 > 0, \quad \forall\, j\,.$$

The latter means that the set $\mathbb{R}^n \setminus \Omega$ has positive upper Lebesgue density at the infinity point. Now define the blow-up sequence u_{R_j} and apply the monotonicity formula as in (5), to obtain

$$\varphi(s, D_e u_0) = \lim_{j \to 0} \varphi(s, D_e u_{R_j}) = \lim_{j \to 0} \varphi(sR_j, D_e u) = C_e\,, \tag{8}$$

for all $s > 0$. Now according to the monotonicity formula (actually a stronger version of it; see [6]) $\varphi(s, D_e u_0)$ is either identically zero or strictly increasing, or the sets $\{(D_e u)^{\pm} > 0\} \cap B(0, s)$ are half spherical caps up to zero are. Now the latter case is not possible due to condition (8), and therefore the only possibility is that $\varphi(s, D_e u_0) \equiv 0$. Now this in turn implies that $\varphi(r, D_e u) \leq \varphi(\infty, D_e u) = \varphi(1, D_e u_0) = C_e = 0$. We thus arrive at the fact that at least one of the functions $(D_e u)^{\pm}$ is identically zero. Hence $D_e u > 0$ (say). The strict inequality is due to the maximum principle, since $D_e u$ is harmonic (we look at the case $f \equiv 1$ for global solutions). From here it is not hard to show that u is one dimensional and consequently $u = (\max(x_1, 0))^2/2$ in some rotated system.

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