New Families of Solutions in N-Body Problems

Carles Simó

Abstract. The N-body problem is one of the outstanding classical problems in Mechanics and other sciences. In the Newtonian case few results are known for the 3-body problem and they are very rare for more than 3 bodies. Simple solutions, as the so called *relative equilibrium solutions*, in which all the bodies rotate around the center of mass keeping the mutual distances constant, are in themselves a major problem. Recently, the first example of a new class of solutions has been discovered by A. Chenciner and R. Montgomery. Three bodies of equal mass move periodically on the plane along the same curve. This work presents a generalization of this result to the case of N bodies. Different curves, to be denoted as *simple choreographies*, have been found by a combination of different numerical methods. Some of them are given here, grouped in several families. The proofs of existence of these solutions and the classification turn out to be a delicate problem for the Newtonian potential, but an easier one in strong force potentials.

1. Introduction

The classical N-body describes the motion of N punctual masses under the action of Newton's gravitation law of attraction. Let $z_j \in \mathbb{R}^d$, $j = 1, \dots, N$ the positions of the bodies and $m_j > 0$, $j = 1, \dots, N$ the respective masses. For most of this work we shall consider the *planar problem* $d = 2$. The equations of motion are

$$\ddot{z}_j = \Sigma_{i=1,\, i\neq j}^{N} m_i (z_i - z_j) r_{i,j}^{-3}, \qquad (1)$$

where $r_{i,j} = |z_i - z_j|$, $|\ \ |$ being the Euclidean norm and where the gravitational constant is taken equal to 1. The system (1) has the trivial integrals of the center of mass: $\Sigma_{i=1}^{N} m_i z_i$ moves on a straight line with constant velocity. It is not restrictive to assume that the center of mass is kept fixed at the origin and we shall assume this from now on: $\Sigma_{i=1}^{N} m_i z_i = 0$. Furthermore we have the angular momentum, $c = \Sigma m_i z_i \wedge \dot{z}_i$, and the energy, $H = K - U$, first integrals. No more first integrals exist in general. Here K and $-U$ denote the kinetic and potential energy

$$K = \frac{1}{2} \Sigma_{i=1}^{N} m_i |\dot{z}_i|^2, \qquad U = \Sigma_{1\leq i<j\leq N} m_i m_j r_{i,j}^{-1}. \qquad (2)$$

Very few solutions are known for the general N-body problem. Some results are available for the 3-body, for some *restricted* problems, where some masses are infinitesimal, or for some very special subproblem.

Simple solutions can be obtained from *central configurations*. A central configuration is defined as a configuration of the N bodies such that $\ddot{z}_j = \lambda z_j$, $j = 1, \dots, N$ for some $\lambda < 0$ independent of j. Then, if the velocities are properly chosen, that is, if $|\dot{z}_j| = \gamma |z_j|$ with the same γ for all j and the angle between $\dot{z}_j$ and z_j is the same for all the bodies, the motion of the N bodies takes place on conics, all the conics being similar. In particular every body can move on a circle around the common center of masses. In these solutions the motion behaves as if the bodies form a *rigid body*. These solutions are also denoted as *relative equilibrium solutions*, being a fixed point of (1) if we use a rotating frame. They can also be obtained using a different approach. Let $I = \Sigma_{j=1}^{N} m_j |z_j|^2$ the moment of inertia of the N bodies around the center of mass. The set of configurations with a fixed value $I > 0$ defines a sphere $\mathcal{S}$ in $\mathbb{R}^{2N-2}$. We can restrict U to $\mathcal{S}$. Due to the homogeneous character of I and U we can always assume the value $I = 1$. Then the central configurations correspond to the critical points of $U|_{\mathcal{S}}$. The problem of finding the number of central configurations for a given N and how it depends on the masses is still an open question. See [7] for general results and [8] for a numerical study for $N = 4$ and arbitrary masses.

Now we consider the special case of all equal masses, taking $m_j = 1,\ j = 1, \dots, N$. The simplest relative equilibrium is the *regular N-gon*. It is obvious that all the bodies move then on the same circle with a periodic motion. This suggests the following question:

Are there other *periodic solutions* of the N-body problem such that all the bodies *travel along the same path* in the plane?

This is the main topic of the present work. It has been prompted by the recent discovery by A. Chenciner and R. Montgomery [4] of one such solution for $N = 3$, the bodies moving on a figure eight curve, and by a similar solution found by J. Gerver with four bodies [5]. For historical details about these quite new solutions I refer to [3]. Looking for solutions with N equal mass bodies on the same curve poses several problems: a) Existence proofs; b) The admissible geometries of the supporting curves; c) Computation of the solutions; d) Study of the dynamical properties; e) Generalizations to other potentials.

All these topics are strongly related. In fact existence proofs are only available for some class of potentials which excludes the Newtonian case. The difficulties with the Newtonian and other potentials are related to the possible existence of *collisions*, i.e., values t^* of time such that there exist i and j, $i \neq j$ such that $\lim_{t \to t^{*-}} r_{i,j}(t) = 0$. This, in turn, is related to the *admissible curves* and their *time parameterization*. The basic method for the proof relies on a *variational approach*. This is also useful, but not enough, for the computation. The study of the *geometrical and dynamical properties* of the solutions we are looking for, requires local information on orbits which are *far away* from any curve which could be described by analytical means. So the numerical approach seems to be the only possible way.

It is instructive to look at the motion of the N bodies along the same path on the plane by means of an animation. The bodies are seen to *dance* on a somewhat complicated way. This suggests the name *choreographies* to denote this kind of motions. To be precise we should name them as *simple* choreographies, because all the bodies are on the same curve. One can imagine also *multiple* choreographies, the bodies travelling on $k > 1$ different curves.

In section 2 we give some results about the *figure eight* choreography for $N = 3$. Section 3 is devoted to introduce the required notation about simple choreographies. The variational approach is presented in section 4. To this end we generalize the problem to potentials of the form r^{-a}, $a > 0$, instead of using only $a = 1$. The existence proof is sketched for $a \geq 2$, the case known as *strong force*. Section 5 shows different kinds of choreographies found up to now. The changes on the behavior of the choreographies as a function of a are displayed in section 6. This illustrates, also, the difficulties to be faced in the proofs for *weak forces*. Finally section 7 gives a short description of the numerical methods used.

2. The Figure Eight Solution

The first choreography for $N = 3$ was found by Lagrange in 1772: the celebrated equilateral triangular solution. It is only in December 1999 that the next one was found. The three bodies travel on a figure eight curve (see Fig 1.1). The period has been fixed equal to 2π for the periodic solution represented here and for all the solutions in this work. This fixes the size. Other periods are scaled to 2π by Kepler's third law $\ell^3 T^{-2} =$ constant, where ℓ is the length scale and T the period. It is also clear that $\mathbb{S}^1$ invariance allows to have the curve symmetrical with respect the horizontal axis. Denote as P_j the body located at z_j. One can take initially the bodies on a collinear (Eulerian) configuration, as in the figure, with P_3 between P_1 (on the left) and P_2, with P_3 moving upwards. For $t = \pi/6$ they are in isosceles configuration and for $t = \pi/3$ they are again in collinear configuration with P_2 in the middle. In a full period the bodies pass twice for each one of the 3 collinear configurations.

The proof of the existence (see [4]) involves a variational argument. Let $q(t)$ a parameterization of the solution such that if $z_1(t) = q(t)$ then $z_2(t) = q(t - 2\pi/3)$, $z_3(t) = q(t - 4\pi/3)$. The variational formulation of classical mechanics assures that any classical 2π-periodic solution is an extremal of the *action functional*

$$A = \int_0^{2\pi} L(t)dt, \quad L(t) = K(\dot{z}_1, \dot{z}_2, \dot{z}_3) - U(z_1, z_2, z_3), \tag{3}$$

the integral of the Lagrangian L. In fact the figure eight choreography is a minimum of A. They key point is to show that a minimization of A, inside the desired class of topologically figure eight curves, do not leads to collisions. Using estimates in [2] and a test path, the level curve of $U|_{\mathcal{S}}$ passing through the collinear points and travelled with constant speed with a suitable value of I, the exclusion of collisions is reduced to the evaluation of an integral.

The eight can be seen also as *a chain with two links*. Fig. 1 shows also the next simplest *chains* (3- and 4-chains) with 4 and 5 bodies. For completeness, initial data allowing the reader to reproduce the plots are given in Table 1. They have been rounded to 10^{-6} and, hence, should be refined for accurate purposes. Note that for the eight one has to rotate slightly Fig. 1.1 to agree with the initial conditions. The components of z_j are denoted as (x_j, y_j).

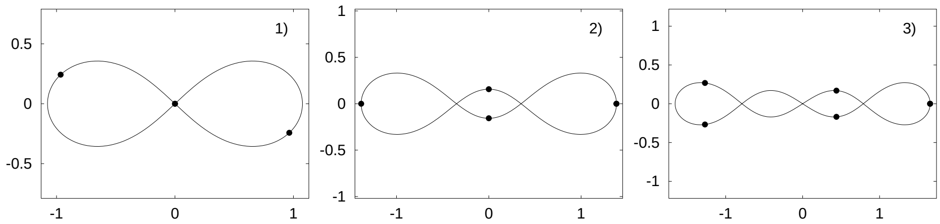

FIGURE 1. Chains with 3, 4 and 5 bodies.

$N=3$	$x_2 = 0.995492$	$\dot{x}_3 = 0.695804$	$\dot{y}_3 = 1.067860$	$x_1 = -x_2$, $\dot{z}_1 = \dot{z}_2$
$N=4$	$x_1 = 1.382857$	$\dot{y}_1 = 0.584873$	$y_2 = 0.157030$	$\dot{x}_2 = 1.871935$
	$x_3 = -x_1$	$y_4 = -y_2$	$\dot{y}_3 = -\dot{y}_1$	$\dot{x}_4 = -\dot{x}_2$
$N=5$	$x_2 = 0.439775$	$y_2 = -0.169717$	$\dot{x}_2 = 1.822785$	$\dot{y}_2 = 0.128248$
	$x_3 = -1.268608$	$y_3 = -0.267651$	$\dot{x}_3 = 1.271564$	$\dot{y}_3 = 0.168645$
	$x_4 = x_3$, $y_4 = -y_3$	$\dot{x}_4 = -\dot{x}_3$, $\dot{y}_4 = \dot{y}_3$	$x_5 = x_2$, $y_5 = -y_2$	$\dot{x}_5 = -\dot{x}_2$, $\dot{y}_5 = \dot{y}_2$

TABLE 1. Initial conditions for Fig. 1. The data not given, and not following from the center of mass condition, are zero.

This solution has many remarkable properties. Beyond obvious symmetries,

- It seems that it is unique, letting aside rotations. This is not proved in [4]. I am based on an extensive numerical search with zero angular momentum and $\dot{z}_1 = \dot{z}_2$.
- The eight lives on $c = 0$, the zero angular momentum level. When $c \neq 0$ it is possible to use a rotating frame (with frequency ω) and to look for figure eight periodic solutions in this frame. This has been done by M. Hénon [6]. The loops become asymmetrical, but the general pattern is similar. These rotating solutions give rise to 2D tori if $\omega \notin \mathbb{Q}$, while they give new *satellite* choreographies if $\omega \in \mathbb{Q}$.
- The orbit is *linearly stable*. It can be seen as a fixed point of a Poincaré map. The eigenvalues of the differential of that map are $\lambda = \exp(\pm 2\pi i \nu_j)$, with $\nu_1 = 0.00842272$, $\nu_2 = 0.29809253$. This comes as a surprise, because of the contrast with systems with two degrees of freedom, for which periodic orbits minimizing the action are unstable. Quoting Birkhoff ([1] p. 130) "*Doubtless analogous results hold for any number of degrees of freedom, and can be obtained by means of classical methods in the calculus of variations.*"

- To obtain non linear information a representation of the Poincaré map to high order is needed. Some methods are proposed in [9], by using higher order variationals. A different approach is given in section 7. In particular this allows to obtain the *Normal Form* around the periodic solution. The torsion matrix is indefinite and KAM theorem is applied to show the existence of invariant tori. Hence, most of the points close to the eight on $c = 0$ are stable. Furthermore, from the variation of the actions follows the existence of other classes of *satellite* choreographies, associated to periodic points, with suitable period, of the Poincaré map around the fixed point.
- It is possible to continuate the periodic solution to other nearby masses, each moving then in a slightly different "eight". But stability is only preserved for relative variations of the order of 10^{-5}.
- Keeping equal masses we can explore different potentials of the form

$$U(z_1, \ldots, z_N) = \Sigma_{1\leq i<j\leq N} f(r_{i,j}), \quad f(r) = r^{-a}, \quad a > 0. \tag{4}$$

  The eight can be continuated to all $a > 0$ and even to the limit case $f(r) = \log r$ and beyond. However it is found to be linearly stable only in a short range around $a = 1$.

3. Choreographies

We pass to $N > 3$. We look for 2π-periodic functions $q\colon \mathbb{S}^1 \mapsto \mathbb{R}^2$ such that if

$$z_j(t) = q(t - (j-1)2\pi/N),\ j = 1, \ldots, N, \tag{5}$$

we find a solution to (1). $\mathbb{Z}/N\mathbb{Z}$ acts on the set of bodies and in $\mathbb{S}^1$ by shifting to the next body (or to the next vertex of an N-gon). This can be used for theoretical and computational purposes. Note that $q(t)$ satisfies a differential equation with delays multiple of $2\pi/N$. This remark do not seems to reduce the difficulty.

A collision occurs if there exists a *double point* $q(t_1) = q(t_2)$ with $t_2 - t_1$ multiple of $2\pi/N$. We consider the class of collision-free functions. It has to be taken analytical (the potentials being analytical if $r \neq 0$), despite for the variational approach it is enough to consider the Sobolev space $H^1(\mathbb{S}^1, \mathbb{R}^2)$ (or H^1 for shortness) of functions with square integrable first derivative. Let $\Delta \subset H^1$ be the functions associated to collisions. We would like to see that in each connected component of $H^1 \setminus \Delta$ there is a solution minimizing the action. Unfortunately this seems not to be true for the Newtonian potential.

To begin with we give in Fig. 2 most of the choreographies known till present for 5 bodies, beyond the regular pentagon and the 4-chain of Fig. 1.3. They are numbered by increasing value of the action. The pentagon has an action less than all other choreographies and the 4-chain is located between cases 4 and 5 of Fig. 2. All of them have some symmetry.

Most of them can be seen as *linear chains* having loops of different size, with some of the loops eventually *folded*. Number 1 consists of a large loop and a small one. In the small loop there are either 1 or 2 bodies for all t.

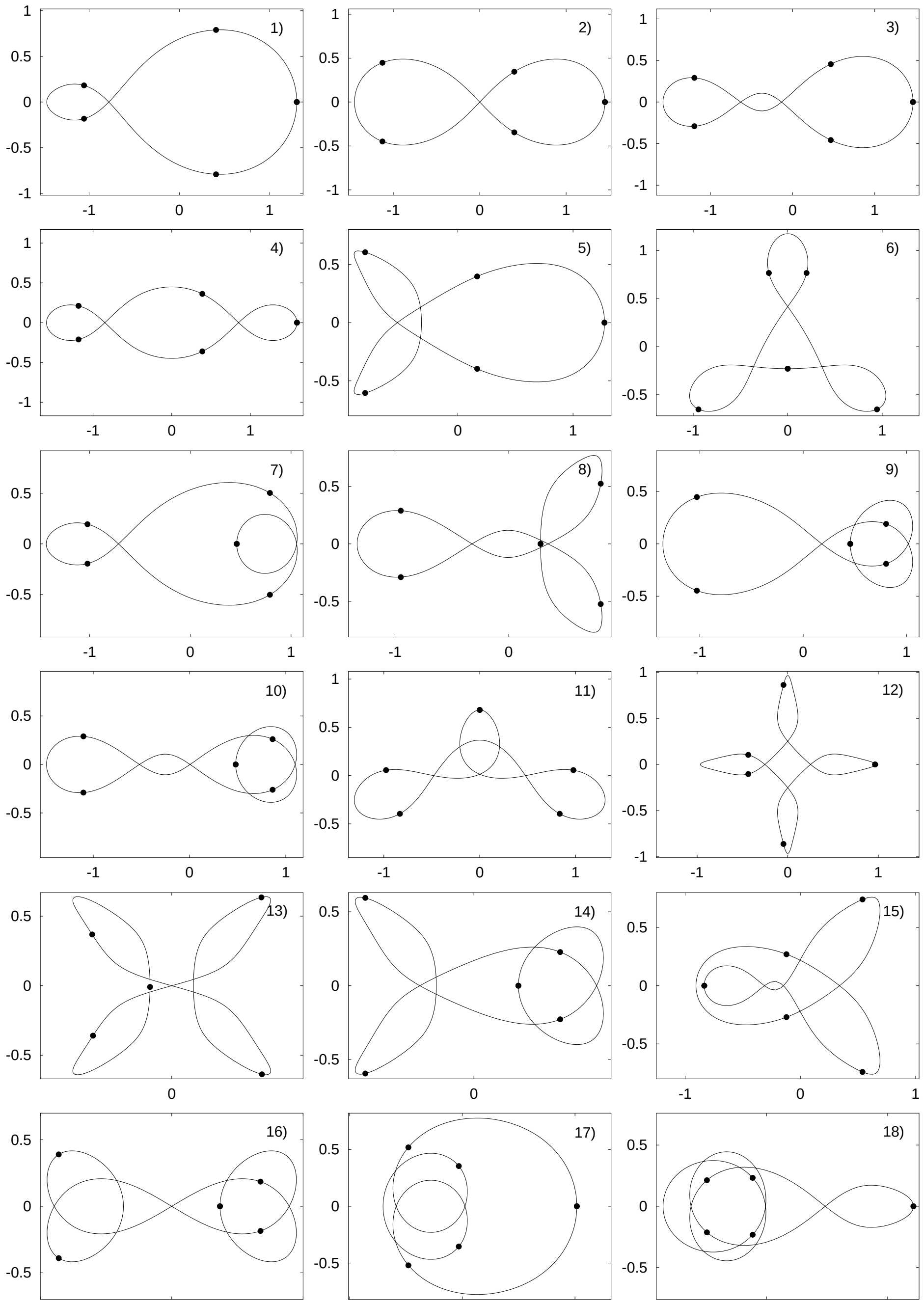

Figure 2. Choreographies found for 5 bodies. The dots denote initial conditions.

case	action	momentum	energy	$\min r_{i,j}(t)$	$\max \lambda_L$
circle	58.308755	-6.186751	-3.093376	1.307660	0.939150
1	68.851604	-3.454467	-3.652691	0.364076	0.999344
2	71.331244	0.000000	-3.784240	0.690443	1.225034
3	75.184575	2.462827	-3.988666	0.213061	1.573730
4	77.158798	0.879793	-4.093401	0.423223	1.323903
4-chain	80.366525	0.000000	-4.263577	0.339434	1.727383
5	85.474715	2.631679	-4.534574	0.326152	1.639335
6	86.051192	-0.664730	-4.565158	0.399373	1.819673
7	88.439746	-2.562017	-4.691874	0.389835	1.757461
8	89.255582	-0.333911	-4.735156	0.236213	2.544799
9	90.108332	0.646869	-4.780395	0.381350	2.362498
10	93.864859	-1.839421	-4.979685	0.212068	2.189505
11	96.798960	-0.207731	-5.135344	0.280668	2.196531
12	102.751489	-0.484014	-5.451136	0.207233	3.381876
13	103.201740	0.000000	-5.475022	0.246947	3.001603
14	105.954031	-1.753301	-5.621036	0.266926	2.272570
15	108.787904	2.243734	-5.771378	0.073286	2.730382
16	109.636187	0.000000	-5.816380	0.355173	2.417178
17	109.882868	-4.476957	-5.829467	0.447649	2.628803
18	119.318405	2.002793	-6.330038	0.311516	3.177711

TABLE 2. Numerical data for 5-body choreographies. $\min r_{i,j}(t)$ is taken over all $i \neq j$, $t \in \mathbb{S}^1$. λ_L means Lyapunov exponent.

Definition 3.1. *Given a double point being the image of t_1 and t_2 by q, the images of the two arcs going from t_1 to t_2 in $\mathbb{S}^1$ are denoted as* loops associated to the point. *Assume* $0 \leq t_1 < t_2 < 2\pi$. *The* lengths of these loops *are* $\ell = (t_2 - t_1)N/(2\pi)$ *and the complement* $\ell_c = N - \ell$. *This extends in a simple way to multiple points.*

A key role is played by the *integer lengths* $[\ell]$ and $[\ell_c]$, where [ ] denotes the integer part. As for a collision-free function $[\ell] + [\ell_c] = N - 1$, we usually refer to the minimum between $[\ell]$ and $[\ell_c]$ as the *integer length* associated to the point. It is clear that if we deform q without passing through collisions, the integer length cannot change. But *small loops* of length less than 1 (integer length zero) can be created/destroyed without problem. Also two nearby double points on $q(\mathbb{S}^1)$ can collapse by deformation of q to the some point, if this one has $\ell \notin \mathbb{N}$, and the points can disappear. In a similar way new loops can be created.

Going back to number 1 in Fig. 2, the integer length associated to the double point is 1. One can ask for a similar choreography having the small loop *inside* the larger one. For instance, number 7 has two small loops (inside and outside the circle), both with $[\ell] = 1$. A discussion on the existence of loops of $[\ell] = 1$ inside a large loop will be given in section 6. Beyond topological constraints there are also the dynamical ones. Take, for instance, number 15 in Fig. 2. There is a small region near the center with two double points very close. For both of them one has $[\ell] = 1$. They could be removed by deformation. But then the passage near collision, which is essential to change strongly the motion of the two bodies which have an encounter, would give a completely different path.

4. Variational Methods

A standard way to prove the existence of periodic orbits is the use of a variational approach. In this way we can obtain the

Theorem 4.1. *Consider problem (1) for a strong force potential as defined in (4) for* $a \geq 2$ *and* $z_j(t)$ *related to* q *as given by (5). Then in every class of choreographies, i.e., in a component of* $H^1 \backslash \Delta$, *there is a solution minimizing* $A = \int_0^{2\pi} L(t)dt$, L *as in (3) and* $K(\dot{z}_1, \ldots, \dot{z}_N)$, $U(z_1, \ldots, z_N)$ *given in (2).*

Proof. We sketch the key point: why collisions are avoided. We start at some point in the interior of a class of choreographies. It is enough to show that approaching Δ the action becomes unbounded. Let us consider a weak force, that is, with $a < 2$. Let $r_{i,j}$ be the distance going to zero. Then, a local computation close to the collision gives that, the dominant term of the contribution of this collision to the action when $r_{i,j}$ goes from r to 0 is

$$A_{\text{binary collision}} = \sqrt{8}\, r^{(2-a)/2}/(2-a). \tag{6}$$

When a tends to 2 it becomes unbounded. This is a fortiori true for $a \geq 2$. Hence one should remain always at the interior of the choreography class. □

For a detailed proof see [3].

At this point is also interesting to stress that the problem can be formulated as a variational condition on q, instead of considering the intermediate passage though the z variables: $q \to z \to A$. This leads to a *variational problem with delays*, a class of problems which seems not widely known.

5. Different Kinds of Choreographies

A selection of choreographies with different shapes is displayed in Fig. 3. They illustrate the many possibilities which appear. A quite simple family are the *chains*. We have seen before the cases of 3, 4 and 5 bodies. Fig. 1, Fig. 2 cases 1 to 4 and Fig. 3 cases 1 to 7 are *linear direct chains* (*ldc*), with $\mathbb{Z}_2$ symmetry and without *folding* the loops. They can be considered as bounding a concatenation of k topological discs. Each piece of the chain between two consecutive double points will be denoted as a *bubble*. The double points have a linear order, say, from left to right, and we can consider the lengths of the loops associated to them such that $0 \equiv [\ell_0] < [\ell_1] < \ldots [\ell_{k-1}] < [\ell_k] \equiv N-1$, where we take the length or the complementary length to obtain an increasing sequence. A rotation by π gives an equivalent chain. The first 5 plots in Fig. 3 have $N = 11$, and show only a few of the possible cases for this N. One can ask for the rate of increase of the number, $\gamma(N)$, of *ldc* as a function of N. We include also the N-gon as a *ldc*. The consecutive differences $\{b_j := [\ell_j] - [\ell_{j-1}],\ j = 1, \ldots, k,\ 1 \leq k \leq m\}$ give a partition of $m = N-1$. The value b_j measures the jth bubble. Let $D(m)$ be the total number of partitions, where, e.g., $b_1 = 1$, $b_2 = 2$ is seen as different from $b_1 = 2$, $b_2 = 1$. For instance $D(1) = 1$, $D(2) = 2$, $D(3) = 4$.

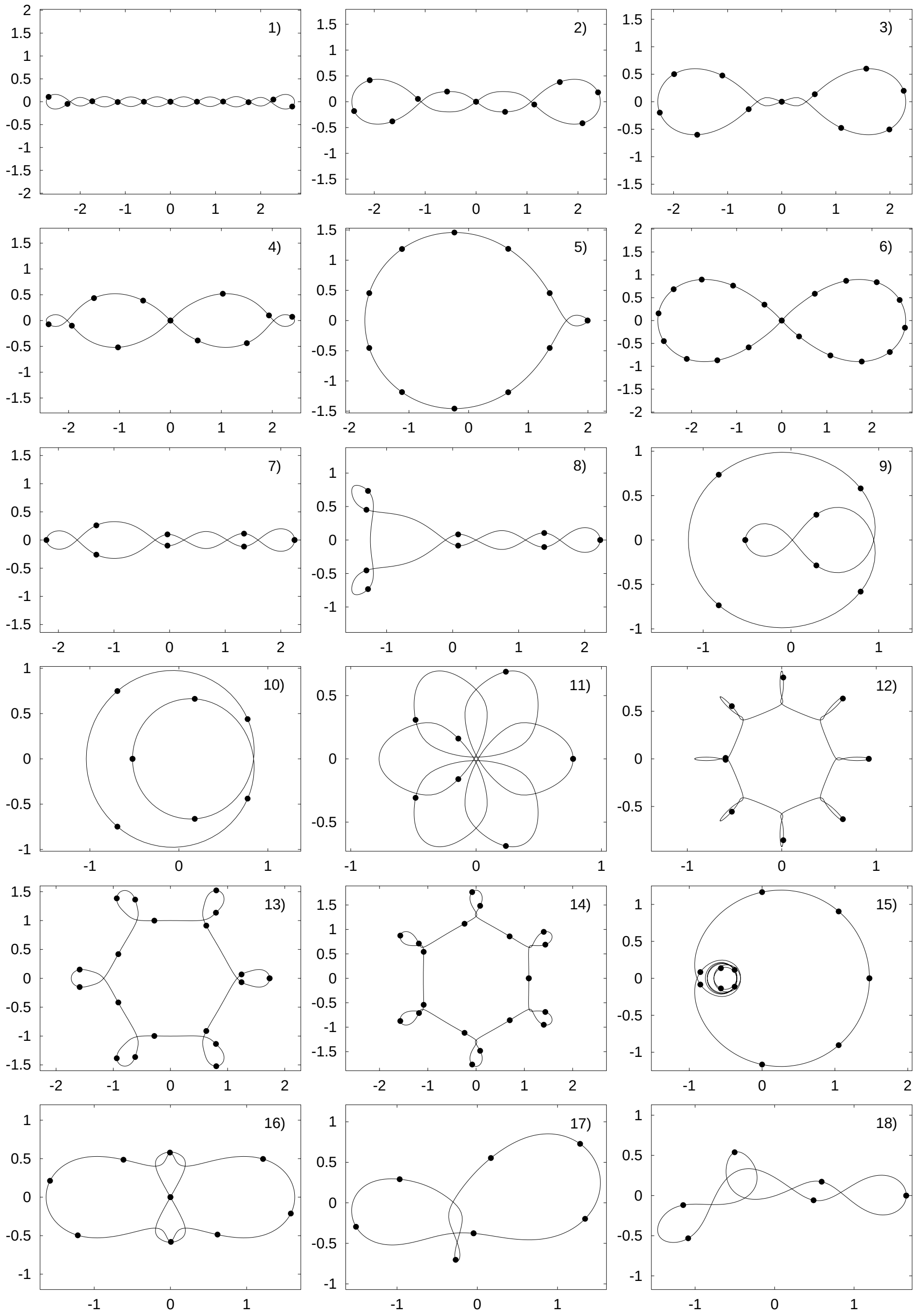

FIGURE 3. A sample of different choreographies.

Given $b_1, \dots, b_k$, a partition of m, then $b_1, \dots, b_k, 1$ and $b_1, \dots, b_k+1$ are partitions of $m+1$, all of them different. We obtain in this way all the partitions of $m+1$. Hence, $D(m)=2^{m-1}$. Let $D_s(m)$ be the symmetrical partitions: $b_1=b_k$, $b_2=b_{k-1}, \dots$ with $b_{k/2}=b_{k/2+1}$ if k even, $b_{(k+1)/2}$ arbitrary if k odd. If m is even then $D_s(m)=D(m/2)=2^{m/2-1}$, while for m odd $D_s(m)=1+D(1)+\dots+D((m-1)/2)=2^{(m-1)/2}$. As $\gamma(N)=\frac{1}{2}(D(N-1)-D_s(N-1))+D_s(N-1)$, we have proved

Proposition 5.1. *For $N \geq 3$ one has $\gamma(N)=2^{N-3}+2^{[(N-3)/2]}$.*

Other choreographies contain inner loops (like 9 and 10 in Fig. 3); have bifurcated chains, 8; look like flowers (11 to 14), or like a flower inside a circle, 15. More exotic are cases 17 and 18, with $N=7$ and $N=6$, respectively, which do not have any symmetry. Cases 13 and 14 are similar, but in 14 there are very small loops, where a passage near collision takes place. Note that one such solution, but with the loops inside, has not been found.

All the choreographies found, except the eight, are unstable.

6. Changing the Potential

An interesting question is to study the behavior of a given choreography, a periodic solution of the N-body problem when changing a in (4). Consider first a four body problem with a small loop, of length $[\ell]=1$, inside a larger loop. We know that it exists for strong force, but it seems not to exist for the Newtonian potential. Fig. 4.3 displays the evolution with a when potentials in r^{-a} are studied. Looking at the inner loop a decrease in a produces, first, a decrease in the size of the loop. The first 4 curves (in the sense of decreasing the size of the inner loop) correspond to $a=2$, 1.4, 1.1 and 1.03445. Near that value a *saddle-node* (*s-n*) bifurcation is produced and the family can be continuated, but for increasing values of a. Next curves correspond to $a=1.1$, 1.2, 1.4 and 1.5373. Close to this last value a new *s-n* appears and the family continues with a decreasing. Later on, it seems to approach a binary collision. Note the cusp for $a \simeq 1.2$, developed to a very small extra loop for $a=1.4$ and that for $a=1.5373$ gives almost a small loop travelled twice.

Hence, it seems that this solution is unable to reach $a=1$.

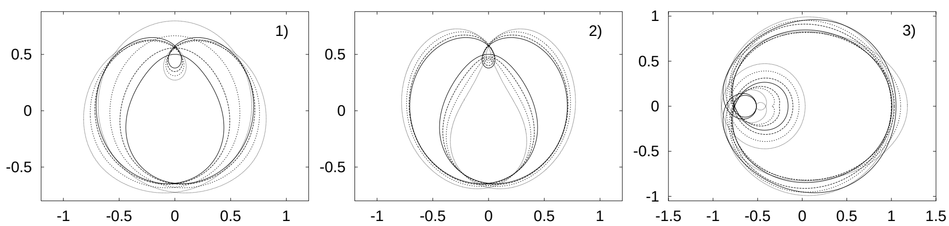

FIGURE 4. Several families for different potentials.

On the other side, consider also $N=4$ as displayed in Fig. 4.1. Looking at the small loop in the upper part, the values $a=2$, 1.4, 1.1 and 1 have been used.

Decreasing a the size of that loop decreases. The family can be continuated by decreasing a again as shown in Fig. 4.2, but for $a \simeq 0.98267$ it has a s-n and the continuation requires an increase of a. The values shown are $a = 1$, $0.98267, 1$ and 1.05. In particular there are two very close solutions for $a = 1$ and the solution for $a = 1.05$ is extremely close to a collision.

Therefore, any proof of existence or non-existence of a particular kind of solution, should be able to distinguish between very close values of a with different properties.

All the solutions displayed in Fig. 4 are minima of the action, but continuation has produced saddles for $N = 5$ (see [3]). An interesting question is to study what happens to a path giving a local minimum when we travel it several times. In particular we can consider a regular N-gon. The simplest example I have found appears for $N = 7$ travelled twice. Hence we consider the same path with period 4π instead of 2π. For $a = 1$ (the Newtonian case) it is no longer a local minimum. But a nearby orbit (Fig. 3.10), with two slightly different loops, is a local minimum. This subsists for a large range of values of a which contains $[1, 2]$ and values beyond $a = 2$. However, decreasing a there is again a s-n bifurcation close to $a = -0.3046$ and the continuation of the family requires to increase a.

All these difficulties are related to passage close to collisions. Assume that $r_{i,j}$ becomes small. Then it is better to study the relative motion, that is, the behavior of $r_{i,j}(t)$ (or $r(t)$ for short) and the contribution of that passage to the action. For simplicity we assume that all the other bodies are at finite distance. Going to a collision $r(t) = \alpha t^{\beta} \times (1 + o(1))$, where $\beta = 2/(2 + a)$, α depending also on a, and the contribution to the action, going from $r = r^*$ to $r = 0$ is given in (6). In the Newtonian case, after passing close to a collision the two bodies (moving near a degenerate ellipse) essentially “bounce”. This can be seen, using blow up, as a change by 2π in the argument of $r(t)$. For a general $0 < a < 2$ the variation of the argument is $1/(2 - a)$ revolutions. This shows why for $a = 2 - 1/n$, $n \in \mathbb{N}$ the collision can be regularized by bouncing and for $a = 2 - 2/(2n + 1)\, n \in \mathbb{N}$ by “crossing” through the collision. In the limit the bodies give an integer or half-integer number of revolutions around the common center of mass. Approaching $a = 2$ the number of revolutions increases. The cases $a \geq 2$ behave as a “black hole” concerning collisions.

A local analysis of a passage close to collision shows the following:

Assume we have to connect two points P_1 and P_2 with $|P_1| = |P_2|$ small by means of a solution $r_{i,j}(t)$ of the two-body problem with potential r^{-a}, $a \in (0, 2)$. Let δ be the variation (in revolutions) of the argument of $r_{i,j}(t)$ from one point to the other. Let A_δ be the minimum of the action along this path when we consider all values of the energy. We can also connect these points going from P_1 to collision and then to P_2 by ejection, matching two solutions of the problem. Then the action takes the value $A_c = 2A_{\text{binary collision}}$ as given in (6) with $r = |P_1|$ in this formula.

Proposition 6.1. $A_c < A_\delta$ *if and only if* $\delta > 1/(2 - a)$.

That is, the passage through collision is cheaper in terms of action only for large variations of the argument of r. Furthermore, if $\delta < 1/(2-a)$ the value A_δ is achieved for $h = 0$.

This seems to explain why a small loop inside a large one cannot exists for the Newtonian potential, as shown in Fig. 4.3. The shape of the curve forces a variation of the argument larger than admissible without collisions. Of course, a complete prove requires estimates on the effect of all other bodies. It also suggest the

Conjecture 6.2. *All linear direct chains exists for* $a = 1$.

There are some special chains. For N odd we can look for a symmetric eight shape curve having initially one body at the the origin. An example is Fig. 3.8 with $N = 19$. Another special case is a chain with $\ell_j = j$, $j = 1, \ldots, N-2$. In both examples it seems that the curves tend (after suitable scalings) to some limit.

7. Numerical Methods

We are faced to several problems: approximation of a choreography, refinement, continuation with varying exponent a and also the computation of good approximations of the Poincaré map around a periodic solution. In fact last topic has been applied only to the eight with $N = 3$. The tools for continuation are standard. I refer to [9] for the description, applications and analysis of the bifurcations.

7.1. Implementing the variational method

The function q can be approximated by a function $\hat{q}$, either a trigonometrical polynomial, the values at a set of equiespaced points or some other method. Let us denote as $\mathcal{P} \in \mathbb{R}^M$ the finite set of parameters needed for the approximation. Then the action A as in (3) is approximated by a *discretized* map $\hat{A}\colon \mathbb{R}^M \to \mathbb{R}$, where the integral is computed by a numerical quadrature formula. Due to the action of $\mathbb{Z}/N\mathbb{Z}$ (which must be preserved by the discretization), it is enough to do the integrals from 0 to $2\pi/N$. Taking into account that collision-free solutions are analytical, the trapezoid rule is highly convenient.

Then we proceed to minimize $\hat{A}$ using a combination of variants of the gradient method. The method has several

- Advantages: It is quite robust, and one can start from very rough approximations, like a few harmonics or a hand drawn curve. It is easy to program, to use any potential, etc. Furthermore the gradient can be obtained from the set $\mathcal{P}$ without need of numerical differentiation. It allows for checks, looking for the invariance of the energy and for the value of the *residual acceleration*: the difference between the acceleration of the masses computed from $\hat{q}$ and using (1).
- Inconveniences: Except in quite simple cases M has to be selected large, specially if there are passages not too far from collision. Typical values of M range in $[10^3, 10^4]$. Furthermore $\hat{A}$ is a very flat function, with lots of extrema. The number of iterations to achieve a good approximation is also in $[10^3, 10^4]$. This slows down the process.

The method detects clearly passages close to saddle points of $\hat{A}$. This can be used in the future to try to locate these solutions. An alternative approach is to implement numerically the *mountain pass lemma*, by following an arc between two local minima under the gradient flow.

The variational technique do not provides direct information on the stability properties of the periodic solutions.

7.2. Refinement of the solutions

Having some starting point, either provided by the variational method or from a solution for a nearby value of the exponent a, we can try to use a Newton method. Consider the values of z_j, $j = 1, \ldots, N$. The time $2\pi/N$ flow transports body j to body $j-1$, where the indices have to be considered in $\mathbb{Z}/N\mathbb{Z}$. This can be converted immediately to the search of a solution of the equation $G(Z) := \Phi(2\pi/N, Z) - S(Z) = 0$, where Z denotes all the z_j and $\dot{z}_j$, Φ is the flow of (1) and S is the shift of indices.

To solve $G(Z) = 0$ by Newton's method one requires $D_Z G(Z)$. This is obtained by simultaneous integration of the system of ODE given by (1) and the first variational equations. This can be cumbersome if N is large. For instance, $N = 100$ leads, essentially, to a system of dimension 160, 000. The use of parallel computers allows a dramatic reduction in the computing time, because the first variational equations can be integrated by blocks.

One of the problems is that many solutions are quite unstable for N large and suffer from passages close to collision. Also the bassin of convergence of the method can be small due to the existence of many nearby solutions. This can be solved by looking not for the value of Z at time $t = 0$ but also at some intermediate values $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 2\pi/N$. Let Z_m the value at t_m. Then, instead of the equation $G = 0$, one has more equations, by requiring $\Phi(t_{m+1}; t_m, Z_m) - Z_{m+1} = 0$. This is the well known *parallel shooting method* (see, e.g., [10]). The system has increased size, but the equations do not require additional effort and are better conditioned.

As a last comment, there is some freedom in the choice of initial time and the system is invariant by rotation. As presented here the method will fail because any solution has a related 6D family of solutions. So $D_Z G(Z)$ will be singular. This difficulty is skipped by selecting that, at $t = 0$, some of the bodies is on the x axis with $\dot{x} = 0$ and by using the center of mass reduction.

This method allows for very accurate solutions, can be implemented with arithmetics of large number of digits, in case of need, and, as the variational equations are solved simultaneously, allows to obtain the stability properties as a byproduct.

7.3. Computing the Poincaré map around a periodic solution

I want only comment that a routine providing an accurate computation of the Poincaré map (using the suitable arithmetics and a high order Taylor recurrent integration scheme) allows to obtain the coefficients of the polynomial expansion of the Poincaré map around the fixed point. It is enough to compute the coefficients by numerical differentiation, using the suitable number of points for the higher order derivatives. This procedure is also highly parallelizable. It is essential to select the optimal step size for the differentiation, which depends on the order.

Acknowledgements

This work has been possible thanks to the information that A. Chenciner, R. Montgomery and J. Gerver shared with me, and to the multiple discussions we are having on the topic. I am also indebted to M. Hénon, R. Martínez and R. Ortega for many useful suggestions. Most of the work has been done during a sabbatical leave that I spent at the Institute de Mécanique Celeste (Observatoire de Paris), thanks to the support of CNRS. To all my colleagues at that institution my recognition for the hospitality and interest on the work. The parallel computing facilities of the UB Grup de Sistemes Dinàmics have been widely used. The support of grants DGICYT PB 94-0215 (Spain) and CIRIT 1998SGR-00042 is also acknowledged.

References

[1] G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc., 1927.

[2] A. Chenciner, N. Desolneux, *Minima de l'intégrale d'action et équilibres relatifs de n corps*, C.R.A.S. Paris, **326**, Série I (1998), 1209–1212. Correction in C.R.A.S. Paris, **327**, Série I (1998), 193.

[3] A. Chenciner, J. Gerver, R. Montgomery, C. Simó, *Simple choreographic motions of N bodies with strong forces,* to appear in *Geometry, Mechanics, and Dynamics*, Springer.

[4] A. Chenciner, R. Montgomery, *A remarkable periodic solution of the three body problem in the case of equal masses,* submitted to Annals of Mathematics, (2000).

[5] J. Gerver, private communication, (2000).

[6] M. Hénon, private communication, (2000).

[7] R. Moeckel, *On central configurations*, Math. Zeit., **205** (1990), 499–517.

[8] C. Simó, *Relative equilibrium solutions in the four–body problem*, Cel. Mechanics, **18** (1978), 165–184.

[9] C. Simó, *Analytical and numerical computation of invariant manifolds.* In D. Benest et C. Froeschlé, editors, *Modern methods in celestial mechanics*, 285–330, Editions Frontières, 1990.

[10] J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, Springer, 1983 (second printing).

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona,
Gran Via, 585, 08007 Barcelona, Spain [1]
E-mail address: `carles@maia.ub.es`

[1]Temporary address: Astronomie et Systèmes Dynamiques, IMC-CNRS UMR 8028, 77 av Denfert-Rochereau, 75014 Paris.