Irreducible Modular Representations of a Reductive p-Adic Group and Simple Modules for Hecke Algebras

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Abstract. Let R be an algebraically closed field of characteristic $l \neq p$, let F be a local non archimedean field of residual characteristic p, and let G be the group of rational points of a connected reductive group defined over F. The two main points in the search for a classification of the irreducible R-representations of G is to try to prove that any irreducible cuspidal representation is induced from an open compact subgroup and that the irreducible representations with a given inertial cuspidal support are classified by simple modules for the Hecke algebra of a type. Over a field R which is not the complex field new serious difficulties arise and the purpose of this article is to indicate a way to avoid them. The mirabolic trick used when the group is $GL(n,F)$ does not generalize but our new method is general and we can extend from the complex case to R the results of Morris and Moy-Prasad for level 0 representations.

Contents

1. Introduction

Let (K,σ) be an irreducible simple cuspidal R-type in $GL(n,F)$ as defined by Bushnell and Kutzko, or a cuspidal R-type of level 0 in G as defined by Morris. We consider also an irreducible extended maximal R-type (N,Λ). We denote by ind the compact induction. We will prove:

Theorem 1.1. 1) *The irreducible R-representations of G which contain (K, σ) are in bijection with the simple modules for the Hecke algebra of (K, σ) in G.*

2) $\mathrm{ind}_N^G \Lambda$ *is irreducible.*

It is known that the Hecke algebra of (K, σ) in G is very closed to products of affine Hecke R-algebras hence the construction of simple modules for affine Hecke R-algebras is strongly related to the construction of irreducible R-representations of G.

As a consequence one can extend to R-representations the complex theory of representations of level 0 by Morris or by Moy and Prasad.

The theorem was already known for $GL(n, F)$ and lead to a complete classification of the irreducible R-representations of $GL(n, F)$ [10, 11]. But the proof could not transfer to a general reductive group and the proof that we give here is more simple and general.

When the characteristic of R is *banal* there is nothing to do. Indeed, if the pro-order of the profinite group K is invertible in R, the representation $\mathrm{ind}_K^G \sigma$ of G is projective and 1) results from an easy lemma in algebra [10, I.6.3]. Let us denote by Z the center of G. The group N contains Z and N/Z is profinite. If the pro-order of N/Z is invertible in R, the category of R-representations of N where Z acts by a given character is semi-simple, and $\mathrm{End}_{RG} \mathrm{ind}_N^G \Lambda \simeq R$ implies the irreducibility of $\mathrm{ind}_N^G \Lambda$.

In the general case the representation $\mathrm{ind}_K^G \sigma$ is not projective, but 1) is true if [11]:

1') $\mathrm{ind}_K^G \sigma$ is almost projective.

We denote by $\mathrm{Irr}_R G$ the set of irreducible R-representations of G, modulo isomorphism. The property 2) is true when (4.2):

2') a) $\mathrm{End}_{RG} \mathrm{ind}_N^G \Lambda \simeq R$,

2') b) if Λ is contained in $\pi|_N$ then Λ is a quotient of $\pi|_N$, for any $\pi \in \mathrm{Irr}_R G$.

We denote by U the pro-p-radical of K. The quotient K/U is a finite reductive group. In the $GL(n, F)$-case, we consider the Bushnell-Kutzko representations $\eta \in \mathrm{Irr}_R(U), \kappa \in \mathrm{Irr}_R(K)$ (see the definition in the paragraph 8). In the level 0 case we suppose that η, κ are trivial representations so that the notation becomes uniform. We will compute the functor

$$\sigma \to \sigma' : \ \mathrm{Mod}_R K/U \to \mathrm{Mod}_R K/U$$

such that the η-isotypic part of $\mathrm{ind}_K^G \kappa \otimes \sigma$ is isomorphic to $\kappa \otimes \sigma'$. This functor is the analogue of a well known functor: the parabolic induction followed by the parabolic restriction in a reductive group. Parabolic groups are replaced by parahoric subgroups or by Bushnell-Kutzko groups and the unipotent radical by the pro-p-radical but: the representation of the pro-p-radical is not trivial in the Bushnell-Kutzko case. We get an analogue of a classical formula for parabolics originally due to Harish-Chandra for finite reductive groups and generalised by

Casselman for p-adic groups. The precise results are given in the paragraphs 6, 7. The following proposition is a corollary of this computation.

The group N is attached to a *maximal* parahoric or Bushnell-Kutzko group K. In the level 0 case, the group N is the G-normalizer $N_G(K)$ of a maximal K and Λ is an irreducible representation of $N_G(K)$ which contains σ. In the $GL(n,F)$-case, there exists an extension E/F with $E^* \subset GL(n,F)$ which normalises η and such that $N = KE^*$ with K maximal, and Λ is an extension of σ to KE^*.

Proposition 1.2. *(for any K) The action of K on the η-isotypic part of $\operatorname{ind}_K^G \sigma$ is isomorphic to $\kappa \otimes W$ where W is a direct sum of irreducible representations conjugate to σ.*

The action of N on the η-isotypic part of $\operatorname{ind}_N^G \Lambda$ is isomorphic to Λ.

The properties 1') and 2') follow from the proposition. The theorem is proved.

The same method applies for a reductive finite group G and gives a new proof of the almost projectivity of any R-representation of G parabolically induced from a cuspidal irreducible representation [4]. This is done in the paragraph 5.

It is pleasure to thank Alberto Arabia for his work which lead to the simple criterium of almost projectivity of $\operatorname{ind}_K^G \sigma$, which is basic in our proof.

2. Almost-Projectivity

The notion of almost projectivity was introduced by Dipper for finite groups and is a particular case of the more general notion of quasi-projectivity.

Let R be a field and let A be an R-algebra. We consider the category $\operatorname{Mod}_{tf}(A)$ (resp. $\operatorname{Mod}_{tf}^o(A)$) of unital finite type left (resp. right) A-modules. A left module is called a module.

Definition 2.1. *A finite type unital A-module Q is called* **quasi-projective** [11, A.3] *when for any two morphisms*

$$Q \overset{\pi}{\underset{\alpha}{\rightrightarrows}} V$$

in $\operatorname{Mod}_{tf}(A)$ with π surjective, there exists $\beta \in \operatorname{End}_A Q$ such that $\alpha = \pi \circ \beta$.

It is called **almost projective** *when there exists a surjective morphism $\pi \colon P \to Q$ from a projective finite type unital A-module P such that for any morphism $\alpha \in \operatorname{Hom}_A(P,Q)$, there exists $\beta \in \operatorname{End}_A Q$ with $\alpha = \beta \circ \pi$.*

An almost projective module is quasi-projective [11, Proposition 7]. The fundamental property of quasi-projective modules is the following:

Main Property. **(Arabia** [11, Appendice th. 10]**)** *When Q is quasi-projective, the functor*

$$\mathrm{Hom}_A(Q,-)\colon \mathrm{Mod}_{tf}(A) \to \mathrm{Mod}^o_{ft}(\mathrm{End}_A Q)$$

induces a bijection between

a) *the isomorphism classes of simple A-modules V such that* $\mathrm{Hom}_A(Q,V) \neq 0$ *and*
b) *the isomorphism classes of simple right* $\mathrm{End}_A Q$*-modules.*

This functor is not in general an equivalence of category.

3. A Simple Criterium of Almost Projectivity

Let G be a locally profinite group, K an open compact subgroup of G, R a field and σ an irreducible smooth R-representation of K which admits a projective cover in $\mathrm{Mod}_R K$.

We denote by

$\mathrm{Mod}_R(G)$ the category of smooth R-representations of G.

V_σ the σ-isotypic part of $V \in \mathrm{Mod}_R(G)$, i.e. the biggest RK-submodule of V which is isomorphic to a direct sum $\oplus^I \sigma$.

The functor of compact induction $\mathrm{ind}_K^G\colon \mathrm{Mod}_R(K) \to \mathrm{Mod}_R(G)$ is exact when G has an R-Haar measure, has a right ajoint (the restriction from G to K denoted by $\pi \to \pi|_K$), respects projectivity and the property of beeing of finite type [10, I.5.10, I.5.7, I.5.9].

Lemma 3.1. *Suppose that the R-representation of G*

$$Q = \mathrm{ind}_K^G \sigma$$

admits a K-equivariant direct decomposition $Q = Q_\sigma \oplus Q^\sigma$ and no subquotient of Q^σ isomorphic to σ. Then Q is almost-projective.

This simple lemma which is basic for us was found by Arabia when G is a finite group.

Proof. Let $f\colon P_\sigma \to \sigma$ be a projective cover in $\mathrm{Mod}_R K$. We define

$$P := \mathrm{ind}_K^G P_\sigma, \quad \pi := \mathrm{ind}_K^G f\,.$$

The isomorphism of adjunction,

$$\mathrm{Hom}_{RG}(\mathrm{ind}_K^G P_\sigma, \mathrm{ind}_K^G \sigma) \simeq \mathrm{Hom}_{RK}(P_\sigma, \mathrm{ind}_K^G \sigma)$$

is given by restriction to the RK-submodule $\underline{P}_\sigma$ isomorphic to P_σ of functions with support in K in the canonical model of $\mathrm{ind}_K^G P_\sigma$. The image of $\mathrm{ind}_K^G f$ under the isomorphism of adjunction is $f\colon \underline{P}_\sigma \to \underline{\sigma}$ (where $\underline{\sigma} \simeq \sigma$ is the space of functions with support in K in the canonical model of $\mathrm{ind}_K^G \sigma$).

The hypothesis on $\operatorname{ind}_K^G \sigma$ and the definition of a projective cover imply that the map

$$\gamma \to \gamma \circ f\colon \operatorname{Hom}_{RK}(\sigma, \operatorname{ind}_K^G \sigma) \to \operatorname{Hom}_{RK}(P_\sigma, \operatorname{ind}_K^G \sigma)$$

is an isomorphism. With the isomorphisms of right adjunction of ind_K^G we obtain that the map

$$\beta \to \beta \circ \pi\colon \operatorname{End}_{RG} \operatorname{ind}_K^G \sigma \to \operatorname{Hom}_{RG}(\operatorname{ind}_K^G P_\sigma, \operatorname{ind}_K^G \sigma)$$

is an isomorphism. We deduce that $\operatorname{ind}_K^G \sigma$ is almost projective. □

Exercise 3.2. *If $Q = \operatorname{ind}_K^G \sigma$ satisfies the lemma* (3.1)*, then Q is quasi-projective.*

Let $\alpha, \pi\colon \operatorname{ind}_K^G \sigma \to V$ be two morphisms in $\operatorname{Mod}_R G$ with π surjective. We look for $\beta \in \operatorname{End}_{RG} \operatorname{ind}_K^G \sigma$ such that $\alpha = \pi \circ \beta$. The lemma implies that there exists a simple RK-submodule W' of Q_σ such that $\pi(W') = \alpha(\underline{\sigma})$. Let $\beta'\colon \underline{\sigma} \to \operatorname{ind}_K^G \sigma$ be the RK-equivariant morphism with image W' with $\pi \circ \beta' = \alpha|_{\underline{\sigma}}$, and $\beta \in \operatorname{End}_{RG} \operatorname{ind}_K^G \sigma$ the image of β' by adjunction. Then $\alpha = \pi \circ \beta$. □

4. A Simple Criterium for Irreducibility

We replace the property "compact" for K by "compact mod center", we denote by $\operatorname{Ind}_K^G\colon \operatorname{Mod}_R K \to \operatorname{Mod}_R G$ the induction without condition on the support. This functor has a left adjoint (the restriction $\pi \to \pi|_K$ from G to K) [10, I.5.7].

Lemma 4.1. *Let $\Lambda \in \operatorname{Irr}_R K$. When the space $\operatorname{End}_{RG}(\operatorname{ind}_K^G \Lambda)$ is finite dimensional, it is equal to $\operatorname{Hom}_{RG}(\operatorname{ind}_K^G \Lambda, \operatorname{Ind}_K^G \Lambda)$.*

Proof. Use the Mackey decomposition and the adjunction. □

Lemma 4.2. *The R-representation $\operatorname{ind}_K^G \Lambda$ is irreducible when*

a) $\operatorname{End}_{RG}(\operatorname{ind}_K^G \Lambda) = R$.
b) *If Λ is contained in $\pi|_K$ then Λ is also a quotient of $\pi|_K$, for any $\pi \in \operatorname{Irr}_R G$.*

Proof. Suppose that a) and b) are true. Let $\pi \in \operatorname{Irr}_R G$ be a quotient of $\operatorname{ind}_K^G \Lambda$. By adjunction $\Lambda \subset \pi|_K$ and by b), Λ is a quotient of $\pi|_K$. By adjunction $\pi \subset \operatorname{Ind}_K^G \Lambda$. Hence there is a morphism $\operatorname{ind}_K^G \Lambda \to \operatorname{Ind}_K^G \Lambda$ with image π. By a) and (4.1) $\operatorname{ind}_K^G \Lambda = \pi$. Hence $\operatorname{ind}_K^G \Lambda$ is irreducible. □

5. Finite Reductive Group

G is the group of rational points of a reductive connected group over a finite field of characteristic p,

$P = MU$ is a parabolic subgroup of G with unipotent radical U, and Levi subgroup M,

$\mathrm{Irr}_R(G)$ is the set of irreducible R-representations of G modulo isomorphism,

$\mathrm{Cusp}_R(G) \subset \mathrm{Irr}_R(G)$ is the subset of cuspidal representations,

$\sigma \in \mathrm{Cusp}_R(M)$ is identified with a representation of P trivial on U. We denote by

$N_G(M)$ the G-normalizer of M,
$W(M) = N_G(M)/M$,
${}^g\sigma_P(?) = \sigma_P(g^{-1}?g) \in \mathrm{Cusp}_R(gPg^{-1})$ for $g \in G$,
${}^w\sigma = {}^g\sigma$ with $g \in N_G(M)$ above $w \in W(M)$,
$W(M,\sigma) := \{w \in W(M),\ {}^w\sigma \simeq \sigma\}$,
$V \to V^U :\ \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, M$ the functor of U-invariant vectors.

Proposition 5.1. ([5]) *We have*

$$(\mathrm{ind}_P^G\, \sigma)^U \simeq \oplus_{w \in W(M)}\, {}^w\sigma\,.$$

With (3.1) we get a new proof of the almost projectivity of $\mathrm{ind}_P^G\, \sigma$ [4, (2.3)]:

Corollary 5.2. $\mathrm{ind}_P^G\, \sigma$ *satisfies the simple criterium* (3.1) *of almost-projectivity.*

Proof. As U is the p-radical of P, we have a direct P-equivariant decomposition

$$\mathrm{ind}_P^G\, \sigma = (\mathrm{ind}_P^G\, \sigma)^U \oplus W' \simeq \oplus^{W(M,\sigma)}\sigma \oplus W$$

where W' has no non zero U-fixed vector, and W has no subquotient isomorphic to σ. □

6. Morris Types of Level 0

F is a local non archimedean field of residual field $\mathbf{F}_q$ with q elements and characteristic p,

G is the group of rational points of a reductive connected group $\mathbf{G}$ over F,

$\mathcal{P}$, $\mathcal{Q}$ are two parahoric subgroups of G [2, 5.2.4], with their canonical exact sequence.

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{P} \xrightarrow{f_{\mathcal{P}}} \mathcal{P}(q) \longrightarrow 1\,,$$

$$1 \longrightarrow \mathcal{V} \longrightarrow \mathcal{Q} \xrightarrow{f_{\mathcal{Q}}} \mathcal{Q}(q) \longrightarrow 1\,.$$

The kernels $\mathcal{U}$, $\mathcal{V}$ are pro-p groups, the quotients $\mathcal{P}(q)$, $\mathcal{Q}(q)$ are the groups of rational points of connected reductive finite groups over $\mathbf{F}_q$. We denote by $f_{\mathcal{P}}^*\colon \mathrm{Mod}_R\, P(q) \to \mathrm{Mod}_R\, \mathcal{P}$ the inflation along $f_{\mathcal{P}}$, and $f_{\mathcal{P}}^{*-1}$ the inverse (on the representations of $\mathcal{P}$ trivial on $\mathcal{U}$). We do not repeat the definitions of the paragraph 5 which extend trivially.

Definition 6.1. *We call* **parahoric induction** *the functor of compact induction:*

$$\mathrm{ind}_{\mathcal{P}(q)}^G = \mathrm{ind}_{\mathcal{P}}^G \circ f_{\mathcal{P}}^*\colon\ \mathrm{Mod}_R\, \mathcal{P}(q) \to \mathrm{Mod}_R\, G$$

where $\mathrm{ind}_{\mathcal{P}}^G$ *is the compact induction and* **parahoric restriction** *the functor*

$$\mathrm{res}_{\mathcal{P}(q)}^G = f_{\mathcal{P}}^{*-1} \circ \mathrm{res}_{\mathcal{P}}^G\colon\ \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, \mathcal{P}(q)\,,$$

(where $\mathrm{res}_{\mathcal{P}}^G V = V^{\mathcal{U}} \in \mathrm{Mod}_R\, \mathcal{P}$*).*

We will compute the composite functor

$$T_{\mathcal{Q}(q),\mathcal{P}(q)}^G = \mathrm{res}_{\mathcal{Q}(q)}^G \circ \mathrm{ind}_{\mathcal{P}(q)}^G\colon\ \mathrm{Mod}_R\, \mathcal{P}(q) \to \mathrm{Mod}_R\, \mathcal{Q}(q)\,.$$

Let

T a maximal split torus of G (more precisely the group of rational points of this torus),

$N = N_G(T)$ the G-normalizer or T (the N of the introduction is no more used),

A the apartment in the semi-simple Bruhat-Tits building of G defined by T,

$x, y \in A$ such that $\mathcal{P}$, $\mathcal{Q}$ are the (connected) parahorics defined by x, y. This is a restriction on $\mathcal{P}$, $\mathcal{Q}$.

We denote $G_x = \mathcal{P}$, $G_y = \mathcal{Q}$, $G_x^1 = \mathcal{U}$, $G_y^1 = \mathcal{V}$, $G_x(q) = \mathcal{P}(q)$, $G_y(q) = \mathcal{Q}(q)$, $f_x = f_{\mathcal{P}}$, $f_x^* = f_{\mathcal{P}}^*$.

Lemma 6.2. *Let z be a point in the building, G_z the corresponding parahoric subgroup of G and $f_x\colon G_x \to G_x(q)$ the canonical surjection. Then $f_x(G_x \cap G_z)$ is a parabolic subgroup of $G_x(q)$ with unipotent radical $f_x(G_x \cap G_z^1)$.*

Proof. ([6]) We reduce easily to the case treated by Morris where $x, x' \in A$ are in the closure of a chamber and $z = nx'$ for $n \in N$, using the following properties:

a) $gG_xg^{-1} = G_{gx}$ for $g \in G$ [9, 2.1], [2, 5.2.4], [1, 6.2.10],

b) the Bruhat decomposition

$$G = G_o N G_o$$

where G_o is the Iwahori subgroup fixing a point o in a chamber $\mathcal{C}$ of A [1, 7.2.6, 7.3.4], [8, page 105],

c) $G_x \supset G_z$ when x is contained in the closure of the facet containing z [9, 5.2.4],

d) given a point a and a chamber $\mathcal{C}$ of closure $\overline{\mathcal{C}}$ in the building, there exists $g \in G$ such that $ga \in \overline{\mathcal{C}}$ [9, 1.7, 2.1].

By d) we can suppose $z = g'x'$ where $x, x' \in \overline{\mathcal{C}}$ for a chamber $\mathcal{C}$ of A and $g' \in G$. By b) we can write $g' = b'nb$ where $b', b \in G_o$ where $o \in \mathcal{C}$ and $n \in N$. By c) $G_o \subset G_x \cap G_{x'}$ hence $bx' = x', b'^{-1}x = x$, $z = b'nx'$. With a) we get $G_x \cap G_z = b'(G_x \cap G_{nx'})b'^{-1}$. The lemma for $G_x \cap G_{nx'}$ [6] implies the lemma for $G_x \cap G_z$. □

We denote

$$(G_x \cap G_z)(q) = f_x(G_x \cap G_z)/f_x(G_x \cap G_z^1) = (G_x \cap G_z)/(G_x \cap G_z^1)(G_x^1 \cap G_z)\,.$$

The groups $G_x(q)$ are isomorphic when x belongs to an orbit of G in the building (property a)) and the conjugation by $g \in G$ induces an equivalence of categories

$$\sigma \to {}^g\sigma\colon\ \mathrm{Mod}_R\, G_x(q) \to \mathrm{Mod}_R\, G_{gx}(q)$$

and more generally an equivalence of categories

$$\mathrm{Mod}_R(G_x \cap G_z)(q) \to \mathrm{Mod}_R(G_{gx} \cap G_{gz})(q)$$

because $\mathrm{Mod}_R(G_x \cap G_z)(q)$ is identified via f_x to the category of R-representations of $G_x \cap G_z$ trivial on $G_x \cap G_z^1$ and on $G_x^1 \cap G_z$, and $\mathrm{Mod}_R(G_{gx} \cap G_{gz})(q)$ is identified via f_{gz} to the category of R-representations of $G_{gx} \cap G_{gz}$ trivial on $G_{gx} \cap G_{gz}^1$ and on $G_{gx}^1 \cap G_{gz}$.

Definition 6.3. *The functor*

$$F^G_{\mathcal{Q}(q)g\mathcal{P}(q)}\colon\ \mathrm{Mod}_R\,\mathcal{P}(q) \to \mathrm{Mod}_R(\mathcal{P} \cap g^{-1}\mathcal{Q}g)_{\mathcal{P}}(q) \to$$
$$\to \mathrm{Mod}_R(g\mathcal{P}g^{-1} \cap \mathcal{Q})_{\mathcal{Q}}(q) \to \mathrm{Mod}_R\,\mathcal{Q}(q)$$

is the composite of

a) *the parabolic restriction along $f_{\mathcal{P}}(\mathcal{P} \cap g^{-1}\mathcal{Q}g)$, i.e. the $f_{\mathcal{P}}(\mathcal{P} \cap g^{-1}\mathcal{V}g)$-invariants*
b) *the conjugation by g*
c) *the parabolic induction along $f_{\mathcal{Q}}(g\mathcal{P}g^{-1} \cap \mathcal{Q})$.*

Proposition 6.4. $T^G_{\mathcal{Q}(q),\mathcal{P}(q)} \simeq \oplus_{g \in \mathcal{Q}\backslash G/\mathcal{P}} F^G_{\mathcal{Q}(q)g\mathcal{P}(q)}$.

Proof. By the Mackey formula, we have a $\mathcal{Q}$-equivariant decomposition

$$\mathrm{ind}^G_{\mathcal{P}(q)}\,\sigma \simeq \oplus_{g \in \mathcal{Q}\backslash G/\mathcal{P}}\, \mathrm{ind}^{\mathcal{Q}}_{\mathcal{Q}\cap g\mathcal{P}g^{-1}}(f^*_{\mathcal{P}}\sigma)(g^{-1}?g)\,.$$

Taking the $\mathcal{V}$ invariants we get the proposition. □

When the functor $F^G_{\mathcal{Q}(q)g\mathcal{P}(q)}$ does not vanish on $\mathrm{Cusp}_R\,\mathcal{P}(q)$, then $f_{\mathcal{P}}(\mathcal{P} \cap g^{-1}\mathcal{V}g) = \{1\}$ i.e.

$$(\mathcal{P} \cap g^{-1}\mathcal{Q}g)(q) = \mathcal{P}(q)\,. \qquad (1)$$

When there exists $\sigma \in \mathrm{Mod}_R\,\mathcal{P}(q)$ such that $F^G_{\mathcal{Q}(q)g\mathcal{P}(q)}(\sigma)$ admits a cuspidal non zero representation as a sub-module or as a quotient, then $f_{\mathcal{Q}}(g\mathcal{U}g^{-1} \cap \mathcal{Q}) = \{1\}$ i.e.

$$(g\mathcal{P}g^{-1} \cap \mathcal{Q})(q) = \mathcal{Q}(q)\,. \qquad (2)$$

The equations (1) and (2) are not independent.

Lemma 6.5. *Let x, z be two points in the building with corresponding parahoric subgroups G_x, G_z. If $(G_x \cap G_z)(q) = G_x(q)$ then the three following properties are equivalent:*

(i) $(G_x \cap G_z)(q) = G_z(q)$,
(ii) *the order of $G_z(q)$ is less or equal to the order of $G_x(q)$,*
(iii) $G_x(q) \simeq G_z(q) \simeq (G_x \cap G_z)/(G^1_x \cap G^1_z)$.

If x is a vertex, then $f_x(G_x \cap G_z) = G_x(q)$ is equivalent to $z = x$.

Proof. $(G_x \cap G_z)(q) = G_x(q)$ is equivalent to $G_x \cap G^1_z = G^1_x \cap G^1_z$. When this holds, we have

$$G_x(q) \simeq (G_x \cap G_z)/(G^1_x \cap G^1_z) \simeq f_z(G_x \cap G_z) \subset G_z(q)\,.$$

We deduce that the three properties are equivalent.

The last part of the lemma follows from the following facts:

Let $\mathcal{C}$ be a chamber of A and let $\Delta = \{\alpha_o, \ldots, \alpha_n\}$ be the basis of the affine roots of G associated to $\mathcal{C}$ [9, 1.8]. For x in the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ the set of α_i with $\alpha_i(x) = 0$ is a proper subset $\Delta_x \subset \Delta$, and the group W_x generated by all reflexions s_α for $\alpha \in \Delta_x$ is finite.

– Let $x, y \in \overline{\mathcal{C}}$ and $n \in N$. Suppose that the image w of n in the affine Weyl group W is of minimal length in $W_x w W_y$. Then $(G_x \cap G_{ny})(q) = G_x(q)$ implies $w\Delta_y \supset \Delta_x$ [6].
– $x \in \overline{\mathcal{C}}$ is a vertex if and only if Δ_x has n elements. Then $y \in \overline{\mathcal{C}}$ is equal to x is and only if $\Delta_x = \Delta_y$. If $y \neq x$ then there is no element $w \in W$ such that $w\Delta_y \supset \Delta_x$. Otherwise $s_\alpha \Delta_y = \Delta_x$ where $\Delta = \Delta_y \cup \{\alpha\}$ but $\alpha \in \Delta_x$ and $\alpha \notin s_\alpha \Delta$.

□

The lemma shows that on an orbit of G in the building the relation

$$(G_x \cap G_z)(q) = G_x(q)$$

is a symetric relation $x \sim z$ because the groups $G_z(q) \simeq G_x(q)$ are isomorphic on an orbit. The set of $g \in G$ with $x \sim gx$ is a disjoint union of double classes $G_x n G_x$ for n in some set of representatives $N(x) \subset N$. For $n \in N(x)$ with image $w \in W$ and $\sigma \in \mathrm{Mod}_R\, G_x(q)$ the isomorphism class of ${}^n\sigma$ depends only on w and we denote by ${}^n\sigma ={}^w \sigma$ and $W(x)$ the image of $N(x)$ in N. We define $W(x,\sigma) := \{w \in W(x),\ {}^w\sigma \simeq \sigma\}$. When $\mathcal{P} = G_x$ we replace x by $\mathcal{P}$ in the notation. We deduce from (6.5) and (6.4):

Corollary 6.6. *Let $\sigma \in \mathrm{Cusp}_R\, \mathcal{P}(q)$. The $\mathcal{U}$-invariants vectors of $\mathrm{ind}^G_{\mathcal{P}}\, \sigma$ is isomorphic to*

$$\oplus_{w \in W(\mathcal{P})} {}^w\sigma\,.$$

As in (5.2) this implies the existence of a $\mathcal{P}$-equivariant decomposition

$$\operatorname{ind}_{\mathcal{P}}^{G} \sigma = \oplus^{W(\mathcal{P},\sigma)} \sigma \oplus W$$

where W has no subquotient isomorphic to σ. Hence $\operatorname{ind}_{\mathcal{P}}^{G} \sigma$ satisfies the simple criterium of almost projectivity (3.1).

7. Cuspidal Representations of Level 0

We suppose that x is a vertex (not necessarily special). The parahoric subgroup G_x is *maximal* among the parahoric subgroups of G, the G-normalizer P_x which is the fixator of x, is an open compact mod center subgroup of G, and is the set of $g \in G$ such $(G_x \cap G_{gx})(q) = G_x(q)$ by (6.5). Let $\Lambda \in \operatorname{Irr}_R P_x$ trivial on G_x^1 and with restriction to G_x identified by f_x^* to a cuspidal representation of $G_x(q)$.

Proposition 7.1. $(\operatorname{ind}_{P_x}^{G} \Lambda)^{G_x^1}$ *is an R-representation of P_x isomorphic to Λ.*

Proof. The functor

$$\Lambda \to (\operatorname{ind}_{P_x}^{G} \Lambda)^{G_x^1} \colon \operatorname{Mod}_R P_x \to \operatorname{Mod}_R P_x/G_x^1$$

is a direct sum

$$(\operatorname{ind}_{P_x}^{G} \Lambda)^{G_x^1} = \oplus_{g \in P_x \backslash G / P_x} F_g^G(\Lambda)$$

of functors $F_g^G \colon \operatorname{Mod}_R P_x \to \operatorname{Mod}_R P_x/G_x^1$ composite of

- the invariants by $P_x \cap g^{-1} G_x^1 g$
- the conjugation by g
- the induction from $(P_x \cap g P_x g^{-1})/(G_x^1 \cap g P_x g^{-1})$ to P_x/G_x^1.

The cuspidality of Λ implies that if the $P_x \cap g^{-1} G_x^1 g^{-1}$-invariants vectors are not 0 then $G_x \cap g^{-1} G_x^1 g^{-1} = G_x^1 \cap g^{-1} G_x^1 g^{-1}$, i.e. $g \in P_x$ by (6.5). Then $(\operatorname{ind}_{P_x}^{G} \Lambda)^{G_x^1} = F_1^G(\Lambda) = \Lambda$ and the proposition is proved. □

The simple criterium for irreduciblity (4.2) is easily deduced from this proposition. By adjunction (7.1) implies

$$\operatorname{End}_{RG}(\operatorname{ind}_{P_x}^{G} \Lambda) = R$$

and if $\pi \in \operatorname{Irr}_R G$ is a quotient of $\operatorname{ind}_{P_x}^{G} \Lambda$ then by adjunction $\Lambda \subset \pi|_{P_x}$ and (7.1) implies

$$\pi^{G_x^1} \simeq \Lambda\,.$$

In $\operatorname{Mod}_R P_x$, $\pi^{G_x^1}$ is a direct factor of $\pi|_{P_x}$ and $\operatorname{ind}_{P_x}^{G} \Lambda$ satifies the simple criterium for irreduciblity (4.2).

8. Bushnell-Kutzko Simple Types in $GL(n, F)$

We suppose $G = GL_F(V) \simeq GL(n, F)$ for an F-vector space V of dimension n.

We fix $\beta \in GL_F(V)$ such that

- the algebra $E = F(\beta)$ is a field
- $k_F(\beta) < 0$ [3, 1.4.5, 1.4.13, 1.4.15, 2.4.1]. We will not use $k_F(\beta)$ and we do not recall its definition.

We denote by q_E the number of elements of the residual field of E and by $B^* = GL_E(V)$ the centralizer of E^* in $GL_F(V)$. If $d[E : F] = n$ then $B^* \simeq GL(d, E)$.

We consider the Bushnell Kutzko group $J = J(\beta, \mathcal{P})$ associated to a "defining sequence" for β and a parahoric subgroup $\mathcal{P}$ in B^* [3, 2.4.2, 3.1.8, 3.1.14]. The group does not depend on the defining sequence [3, 3.1.9 (v)]. The Bushnell-Kutzko groups, called the BK-groups, have the following properties [3, 1.6.1]:

Lemma 8.1. *Let $\mathcal{P}$ be a parahoric subgroup of B^* with the canonical exact sequence*

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{P} \xrightarrow{f_{\mathcal{P}}} \mathcal{P}(q_E) \longrightarrow 1, .$$

The BK-group $J = J(\beta, \mathcal{P})$ is an open compact subgroup of G normalized by E^* with pro-p radical J^1, and

$$J = J^1\mathcal{P}, \quad J \cap B^* = \mathcal{P}, \quad J^1 \cap B^* = \mathcal{U}\,. \tag{3}$$

The canonical surjection f_J given by the lemma

$$1 \longrightarrow J^1 \longrightarrow J \xrightarrow{f_J} \mathcal{P}(q_E) \longrightarrow 1$$

has the property

$$f_J(J \cap K) = f_{\mathcal{P}}(\mathcal{P} \cap \mathcal{Q}) \tag{4}$$

because $f_J(J \cap K) = f_I(J \cap K \cap B^*)$ and $J \cap K \cap B^* = \mathcal{P} \cap \mathcal{Q}$, $f_J(\mathcal{P} \cap \mathcal{Q}) = f_{\mathcal{P}}(\mathcal{P} \cap \mathcal{Q})$. By (4), the properties of parahoric subgroups of B^* seen in the paragraphs 6 and 7 transfer to the BK-groups associated to β.

We consider the irreducible Bushnell Kutzko (or BK) representations $\eta_J \in \mathrm{Irr}_R\, J^1$, $\kappa_J \in \mathrm{Irr}_R\, J$ attached to a fixed endo-class Θ [3, 5.1.8, 5.2.1 and 5.2.2, 2 4.3]. We do not recall the definitions. The BK-representations satisfy:

$$\textit{The restriction of } \kappa_J \textit{ to } J^1 \textit{ is } \eta_J \textit{ and } \eta_J,\, \kappa_J \textit{ are normalised by } E^*\,. \tag{5}$$

When η_J is fixed there is some choice for κ_J but only by multiplication by a character trivial on J^1 and normalized by E^*. We use the definitions of the paragraph 6 for the parahoric subgroup $\mathcal{P}$ of B^*.

Definition 8.2. *We consider the functors:*

a) *The* **κ_J-inflation** *functor*

$$\sigma \to \kappa_J \otimes f_{\mathcal{P}}^* \sigma\colon\ \mathrm{Mod}_R\, \mathcal{P}(q_E) \to \mathrm{Mod}_R\, J$$

which induces an equivalence of categories between $Mod_R\mathcal{P}(q_E)$ and the η_J-isotypic R-representations of J.

b) *The* **compact κ_J-induction** *functor:*

$$\mathrm{ind}_{\kappa_J}^G\colon\ \mathrm{Mod}_R\,\mathcal{P}(q_E)\to \mathrm{Mod}_R\,G$$

given by the κ_J-inflation followed by the compact induction $\mathrm{ind}_J^G\colon \mathrm{Mod}_R\,J\to\mathrm{Mod}_R\,G$.

c) *The* **κ_J-restriction** *functor*

$$\mathrm{res}_{\kappa_J}^G\colon\ \mathrm{Mod}_R\,G\to \mathrm{Mod}_R\,\mathcal{P}(q_E)\qquad \pi\to\sigma$$

given by the η_J-isotypic part $\pi\to\pi_{\eta_J}=\kappa_J\otimes f_{\mathcal{P}}^*\sigma\colon \mathrm{Mod}_R\,G\to\mathrm{Mod}_R\,J$ followed by the inverse of the κ_J-inflation.

d) *When K is the BK-group $J(\beta,\mathcal{Q})$ attached to another parahoric $\mathcal{Q}$ of B^*, the functor*

$$T_{\kappa_K,\kappa_J}^G=\mathrm{res}_{\kappa_K}^G\circ\mathrm{ind}_{\kappa_J}^G\colon\ \mathrm{Mod}_R\,\mathcal{P}(q_E)\to\mathrm{Mod}_R\,\mathcal{Q}(q_E)\,.$$

We will prove (8.5) that the functor T_{κ_K,κ_J}^G is equal to the functor $T^{B^*}_{\mathcal{Q}(q_E),\mathcal{P}(q_E)}$ associated to the parahoric subgroups $\mathcal{P},\mathcal{Q}$ of B^* (6.1) and already described (6.3), (6.4).

Remark 8.3. *For $g\in G$, we have $J(g\beta g^{-1},g\mathcal{P}g^{-1})=gJ(\beta,\mathcal{P})g^{-1}$.*

It is not immediately apparent that the elaborate definition of J is G-equivariant. I suppose that the construction of η_J and κ_J is also G-equivariant [3, 3.5, 5.7] but I didnt check the details for the representations.

Let $\mathcal{L}=(L_i)_{i\in\mathbf{Z}}$ be a strictly decreasing periodic lattice chain of O_E-modules in V such that $\mathcal{P}=GL^o(\mathcal{L})$ is the set of $f\in GL_E(V)$ with $f(L_i)\subset L_i$ for all $i\in\mathbf{Z}$. We consider

- the period e of $\mathcal{L}$ (the smallest integer n such that $L_{i+n}=p_EL_i$ for all $i\in\mathbf{Z}$),
- the $\mathcal{L}$-valuation v of β (the biggest integer n such that $\beta L_i\subset L_{i+n}$ for all $i\in\mathbf{Z}$),
- the herditary order $\mathrm{End}^o_{O_F}\mathcal{L}$ of $\mathrm{End}_F(V)$ associated to $\mathcal{L}$ seen as a chain of O_F-modules (the F-endomorphisms f such that $f(L_i)\subset L_i$ for all $i\in\mathbf{Z}$),
- $s\colon \mathrm{End}_F V\to\mathrm{End}_E V$ the tame corestriction map relative to E/F [3, 1.3.3].

We take $g\in G$ and we replace $(\beta,\mathcal{L})$ by $(g\beta g^{-1},g\mathcal{L})$. It is easy to see what happens to the various objects and numbers introduced above. First we see that $(\mathcal{P},\mathrm{End}^o_F\mathcal{L})$ is replaced by $(g\mathcal{P}g^{-1},g(\mathrm{End}^o_F\mathcal{L})g^{-1})$, the period e and the valuation v do not change. Then looking at the definition of $k_F(\beta)$ [3, 1.3.5], we see that $k_F(\beta)$ does not change, and finally if $c_g\colon \mathrm{End}_E(V)\to\mathrm{End}_{gEg^{-1}}(V)$ is the natural isomorphism $f(x)\to gf(g^{-1}xg)g^{-1}$, we see on the definition [3, 1.3.3] that $c_g\circ s$ is a tame corestriction map relative to gEg^{-1}/F.

These remarks imply that a defining sequence $(\mathcal{A}_i, n, r_i, \gamma_i)$ for $(\mathcal{A}, n, r, \beta)$ gives a defining sequence $(g\mathcal{A}_i g^{-1}, n, r_i, g\gamma_i g^{-1})$ for $(g\mathcal{A}g^{-1}, n, r, g\beta g^{-1})$ with the definitions [3, 2.4.2]. We deduce from [3, 3.1.8] that $J(g\beta g^{-1}, g\mathcal{P}g^{-1}) = gJ(\beta, \mathcal{P})g^{-1}$. □

The three κ_J-functors (8.2) do not determine the group J neither the representation κ_J. In fact we will not keep (J, κ_J).

Let $\mathcal{P}_{\max}$ be a maximal parahoric subgroup of B^* and let $\mathcal{P}_{\min}$ be a minimal parahoric subgroup of B^* contained in $\mathcal{P}_{\max}$. We suppose $\mathcal{P}_{\min} \subset \mathcal{P} \subset \mathcal{P}_{\max}$. Let $\eta_{\max} \in \mathrm{Irr}_R\, J^1_{\max}$, $\kappa_{\max} \in \mathrm{Irr}_R\, J_{\max}$ be the BK-representations associated to (β, Θ). We consider

$$J' = J^1_{\max}\mathcal{P}, \quad {J'}^1 = J^1_{\max}\mathcal{U}, \quad \eta_{J'} = \kappa_{\max}|_{{J'}^1}, \quad \kappa_{J'} = \kappa_{\max}|_{J'}\,.$$

It is clear that (3) hence (4), (5) are satisfied for $(J', {J'}^1, \eta_{J'}, \kappa_{J'})$.

With the notation of the remark 8.3, we can suppose that $\mathcal{L}_{\max} = (L_o p_E^{\mathbf{Z}})$ is a O_E-lattice chain in $\mathrm{End}_E V$ such that $\mathcal{P}_{\max} = GL^o_E(\mathcal{L}_{\max})$ is the set of $g \in GL_E(V)$ with $gL_o \subset L_o$. We denote $GL^1_F(\mathcal{L}_{\max})$ the set of $g \in GL_F(V)$ with $gL_o p_E^i \subset L_o p_E^{i+1}$ for all $i \in \mathbf{Z}$. The open compact subgroup $A = GL^1_F(\mathcal{L}_{\max})\mathcal{P}$ of G has a pro-p-radical $A^1 = GL^1_F(\mathcal{L}_{\max})\mathcal{U}$ and satisfies (3). By construction J, J' are contained in A and $J^1 = A^1 \cap J$, ${J'}^1 = A^1 \cap J'$. We recall [3, 5.2.5] that the R-representations of A^1

$$\eta_A := \mathrm{ind}_{{J'}^1}^{A^1} \eta_{J'} \simeq \mathrm{ind}_J^{A^1} \eta_J$$

are isomorphic and irreducible, and one may suppose (or we twist κ_J by a character normalised by E^*) that the R-representations of A

$$\kappa_A := \mathrm{ind}_{J'}^A \kappa_{J'} \simeq \mathrm{ind}_J^A \kappa_J$$

are isomorphic and irreducible. This implies:

Lemma 8.4. *The κ_A, κ_J, $\kappa_{J'}$-functors are equal, the compact BK-induction and BK-restriction functors associated to κ_A, κ_J, $\kappa_{J'}$ are equal.*

Hence the functor $T^G_{\kappa_J,\kappa_K} = T^G_{\kappa_{J'},\kappa_{K'}}$ can be computed using $\kappa_{J'}$, $\kappa_{K'}$. By the Mackey formula, we have a K'-equivariant decomposition (where we write σ instead of $f^*_{\mathcal{P}}\sigma$)

$$\mathrm{ind}_{J'}^G \kappa_{J'} \otimes \sigma \simeq \oplus_{g \in K'\backslash G/J'}\, \mathrm{ind}^{J^1_{\max}\mathcal{Q}}_{J^1_{\max}\mathcal{Q} \cap g J^1_{\max}\mathcal{P} g^{-1}} (\kappa_{\max} \otimes \sigma)(g^{-1}?g)\,.$$

We compute the $\eta_{K'}$-isotypic part, i.e. the $\kappa_{\max}|_{J^1_{\max}\mathcal{U}}$-isotypic part. We recall [3, 5.1.8 page 160, 5.2.7 page 170]:

$$\dim_R \mathrm{Hom}_{J^1_{\max} \cap g J^1_{\max} g^{-1}}(\eta_{\max}, {}^g\eta_{\max}) = \dim_R \mathrm{Hom}_{J_{\max} \cap g J_{\max} g^{-1}}(\kappa_{\max}, {}^g\kappa_{\max})$$

is equal to 1 *for* $g \in J^1_{\max} B^* J^1_{\max}$, *and is equal to* 0 *when* $g \notin J^1_{\max} B^* J^1_{\max}$.

The terms in $g \notin J^1_{\max} B^* J^1_{\max}$ give no contribution to the $\eta_{K'}$-isotypic part of $\mathrm{ind}_{J'}^G \kappa_{J'} \otimes \sigma$ and

$$\mathcal{Q}\backslash B^*/\mathcal{P} = J^1_{\max}\mathcal{Q}\backslash J^1_{\max} B^* J^1_{\max}/J^1_{\max}\mathcal{P}\,.$$

Using the property (4) for $f_{J'}(J' \cap gK'g^{-1})$ when $g \in B^*$ we deduce with the same proof and notations of (6.3) and (6.4):

Proposition 8.5. $T^G_{\kappa_K,\kappa_J} \simeq \oplus_{g \in \mathcal{Q}\backslash B^*/\mathcal{P}} F^{B^*}_{\mathcal{Q}(q_E)g\mathcal{P}(q_E)} \simeq T^{B^*}_{\mathcal{Q}(q_E),\mathcal{P}(q_E)}$.

With (6.6) we get:

Corollary 8.6. *When* $\sigma \in \mathrm{Cusp}_R\, \mathcal{P}(q_E)$*, then* $\mathrm{res}^G_{\kappa_J}(\mathrm{ind}^G_J\, \kappa_J \otimes \sigma) = \oplus_{g \in W(\mathcal{P})}\, {}^g\sigma$.

As J^1 is a pro-p-group, the restriction of $\mathrm{ind}^G_J\, \kappa_J \otimes \sigma$ to J^1 is semi-simple, and its η_J-isotypic part is a direct factor. We deduce as in (5.2) that $\mathrm{ind}^G_J\, \kappa \otimes \sigma$ satisfies the simple criterium of almost projectivity (3.1).

The representation $\kappa_{\max}$ extends to $J_{\max}E^*$ by Clifford theory. An R-representation $\Lambda \in \mathrm{Irr}_R\, J_{\max}E^*$ which extends $\kappa \otimes \sigma$ with $\sigma \in \mathrm{Cusp}_R\, \mathcal{P}_{\max}(q_E)$ is called a maximal extended Bushnell-Kutzko type. We prove as in the level 0 case (7.1):

Proposition 8.7. *The* $\eta_{\max}$*-isotypic part of* $\mathrm{ind}^G_{J_{\max}E^*}\, \Lambda$ *is an* R*-representation of* $J_{\max}E^*$ *isomorphic to* Λ.

As in the level 0 case we deduce from (8.7) that $\mathrm{ind}^G_{J_{\max}E^*}\, \Lambda$ satisfies the simple criterium for is irreducibility (4.2).

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