Optimal Portfolios under a Value at Risk Constraint

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Abstract. Recently, financial institutions discovered that portfolios with a limited Value at Risk often showed returns that were close to the VaR and had large losses in the exceptional cases where losses exceeded VaR. In this paper we consider the construction of portfolios with options that maximize expected return with a restriction on the Value at Risk. These theoretically optimal portfolios indeed have the properties as experienced by financial institutions and illustrate that maximizing under a VaR-constraint is very dangerous. We also show that if one considers market prices of options there will be an even higher impetus to go for gambling portfolios.

1. Introduction

Over the last few years Value-at-Risk has become one of the standard instruments for measuring risk for banks and other financial institutions. Regulators such as the Bank for International Settlements recommend VaR-measures to determine capital adequacy requirements. Some institutions use the VaR-measure together with expected return to compare the attractiveness of different activities. Most research has focussed on methods to properly determine the VaR, where attention has been paid to the following issues: large portfolios, portfolios with derivatives, nonnormality of asset returns and description of the tails of distributions by extreme value theory. Furthermore, the concept of marginal VaR has been introduced to determine which parts of a portfolio have the highest impact on its VaR. However, there has only been very limited research on the determination of optimal portfolios in a VaR-framework. One of the few exceptions is a paper by Ahn, Boudoukh, Richardson and Whitelaw [1], who determine the portfolio with the lowest VaR, that can be created by buying put options with a restriction on the option premium paid. Furthermore, Litterman [4, 5] describes assets that optimally reduce the VaR of a portfolio based on marginal VaR. These are called best hedges. In this paper we study a more general problem, where we try to determine the portfolio with the highest expected return under a restriction on the VaR. As stated before this combination of expected return and VaR is popular by some institutions to determine optimal portfolios. In most cases there are restrictions on the (number of) assets or positions that can be used in a portfolio. We will not invoke these kinds of restrictions and even allow trading in Arrow-Debreu securities. Within this general framework we will show that optimal portfolios are quite risky although they have a limited Value at Risk. They also bet on certain specific events to realize a high expected return. We are fully aware of the fact that Arrow-Debreu securities are not traded, but will argue that close substitutes can be created. Also, if only plain vanilla options are used to create an optimal portfolio our results will help in understanding the specific characteristics of these optimal portfolios. Our ideas are described with a simple 7-period Cox, Ross and Rubinstein [3] binomial model. We take a 7-period model since it is the simplest model within which we can demonstrate the intuition behind our results. Of course it can be extended to binomial models with more periods and to the continuous time Black-Scholes world as in Basak and Shapiro [2], but this only complicates the mathematics without providing much additional insight. Although our model is simple, according to Roth [7] it is believed to be able to explain some of the recent disasters in the financial world. Our results are based on theoretical option prices but we argue that if we consider market prices, which often include a smirk in implied option prices, our results will even be stronger.

2. The Optimization Model in Discrete Time

In this section we derive the basic result concerning portfolio strategies that maximize expected return under a value at risk constraint. If we could only select assets where returns are normally distributed, also all portfolios have normally distributed returns and a restriction on the Value at Risk would be equivalent to a restriction on the standard deviation. Now, maximizing expected return with a constraint on the standard deviation leads to the well known Markowitz mean variance optimization. In this paper we will focus on the extra possibilities created in the market by options. These have definitely non normal distributions and thus the mean variance framework does not apply. Hence, we focus on the case with only one basic asset and the complications come from the introduction of options. We assume that the investor wants to invest in the market index and uses (exotic) options to enhance the expected return of his portfolio and at the same time fulfill the VaR-constraint. The Value at Risk constraint is at the 99% level and the investor is not allowed to loose more than 10% of the initial portfolio value over the next ten days at this probability level.

Assume that the investor holds 1000 USD. We describe the movements of the index over the Value at Risk typical period by a non-recombining binomial tree with 7 periods. Each period the index can go up by a factor u with probability 0.5 or go down by a factor d with the same probability. u and d should reflect the expected return and volatility of the index over a period in the binomial tree. For ease of exposition we assume that the riskless interest rate is equal to zero. This is not essential for our arguments. However, we also assume that investments in stocks come with a risk premium, i.e. the expected return of an investment in stocks is larger than zero. This implies that

$$0.5u + 0.5d - 1 > 0\,. \tag{1}$$

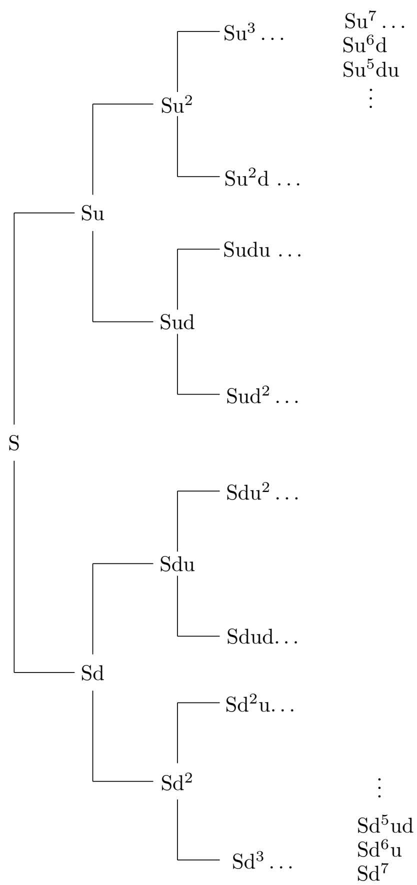

FIGURE 1. Non recombining tree for price of stock index for 7 periods.

This assumption is essential for our further arguments but is very natural in a world with risk averse investors. The left handside of equation (1) is known as the equity risk premium per period. If the initial index level is equal to S we get the binomial tree as given in figure 1. In fact, we might also use a recombining binomial tree, but it is easier to illustrate our arguments with the non-recombining binomial tree. We assume one can trade in stocks and riskless bonds and hence it follows from the completeness of the market that for each final state of the binomial tree one can find a dynamic portfolio strategy that pays off 1 as this final state occurs and 0 in all other states. These are the payoffs of the well-known

Arrow-Debreu securities. Since the payoffs can be constructed we assume that these Arrow-Debreu securities are also traded and the portfolio manager might use these securities to compose his optimal portfolio. It follows from the binomial Cox, Ross and Rubinstein option-pricing model that the cost of such an Arrow-Debreu security is given by

$$C_j = p^j(1-p)^{7-j} \tag{2}$$

if at the final node where the payoff is equal to 1, the value of the index is equal to $Su^j d^{7-j}$, where

$$p = (1-d)/(u-d) < 0.5 \tag{3}$$

since $0.5u + 0.5d > 1$.

Since $p < 0.5$ the costs C_j are a decreasing function of j. Hence, the state where the index goes down 7 times has the highest Arrow-Debreu price, while the state with 7 upswings has the lowest price. Hence, if the fund manager can spend his $1000 on Arrow-Debreu securities, he simply specifies desired payoffs in all final states of the world such that the total costs, i.e. the sum of the Arrow-Debreu prices multiplied by the desired pay off for each state, is equal to $1000. Since all states in the tree are equally probably, the fund manager has to specify the desired pay off in such a way that in at most one case, i.e. 1 out of 128, he does not meet the VaR-restriction final value of $900. Furthermore, he will like to maximize his expected return. So the fund manager will select the state with 7 down turns as the state in which he will not meet the $900 level, since as remarked before, this is the most expensive state. For all other states he will buy 900 Arrow-Debreu securities. The total costs are

$$C = 900 \times \sum_{j=1}^{7} \binom{7}{j} p^j (1-p)^{7-j} = 900 \times \left(1 - (1-p)^7\right). \tag{4}$$

How much will he spend on the Arrow-Debreu security for the lowest state? Since this is the most expensive Arrow-Debreu security, he will not buy any Arrow-Debreu security for this state since it does not influence his Value at Risk. In fact, given the high price of this security, he might even take a short position. We assume that this is not allowed, but it is a potential danger in using the Value at Risk concept. What will the fund manager do with the remaining amount equal to $1000 - 900(1-(1-p)^7) = 100 + 900(1-p)^7$? Since, he has already secured his Value at Risk and all states are equally probable and have the same final payoffs, he will spend his money on the cheapest Arrow-Debreu securities, i.e. the state with 7 upswings. This will give him the highest expected return. Hence, he will buy an extra

$$[100 + 900(1-p)^7]/p^7 \tag{5}$$

of these Arrow-Debreu securities. In table 1 we give the total payoff in the 7 upswings state for different values of the annualized volatility σ and the annualized equity risk premium μ of the index, where we set $u = e^{(\mu-\sigma^2/2)T/7+\sigma\sqrt{T/7}}$ and $d = e^{(\mu-\sigma^2/2)T/7-\sigma\sqrt{T/7}}$ and $T = 0.04$ for a ten trading days Value at Risk period.

	σ			
μ	0.15	0.2	0.25	0.3
0.02	15677.0	15398.7	15234.7	15126.5
0.04	16854.4	16252.5	15904.1	15677.0
0.06	18143.0	17165.9	16610.3	16252.5
0.08	19555.1	18142.9	17355.7	16854.3

Table 1. Payoffs of the optimal portfolio if the stock price goes up for 7 consecutive days.

Hence, the portfolio that maximizes expected return under a Value at Risk constraint will end up with a probability of 126/128 at the level of 900, will end up worthless with a probability of $1/128 \leq 0.01$ and with the same probability will end up with the very high values given in table 1. Given the Value at Risk constraint it is a gambling portfolio since, it has a high probability of a loss and a very low probability of receiving 15 to 20 times the probable loss. It is not surprising that with a higher equity risk premium the payoff in the 7 upswing state increases. If the underlying index is more volatile one has to offer some upside potential in order to remain within the Value at Risk limits. Hence the payoff in the extreme case decreases with an increasing volatility.

Our optimal portfolio is not path dependent, since the money invested in particular Arrow-Debreu securities only depends on the final value of the index. Hence, we can combine all Arrow-Debreu securities with the same final underlying stock price into one Arrow-Debreu security. Therefore only these securities have to be traded. We can also derive this result directly by considering a recombining tree in stead off a non recombining tree. In this case, there are only 8 Arrow-Debreu securities, one for each $0 \leq j \leq 7$. Each state is specified by the number of upswings j. The probability of state j is equal to $\left(\begin{array}{c} n \\ j \end{array} \right) (0.5)^{n-j}/(0.5)^j$, while the Arrow-Debreu price is given by $\left(\begin{array}{c} n \\ j \end{array} \right) p^j(1-p)^{n-j}$. Hence, the price per unit of probability is given by

$$p^j(1-p)^{n-j}/(0.5)^n \tag{6}$$

which again is a decreasing function of j. Also in this setting the same portfolio will result that maximizes expected return given the Value at Risk constraint. Of course, one might argue that a 7 period binomial tree is not very realistic and more periods or a continuous time model are needed to describe future portfolio values. If we would use more than 7 periods the fund manager can do exactly the same, i.e. invest nothing for the lowest state, guarantee the Value at Risk level in all other states and use the remaining money to buy Arrow-Debreu securities, for the highest state. He might even select a few of the next lowest states to give a final pay off equal to 0 as long as the total probability of all states with 0 values does not exceed 1%. He uses the proceeds of this cost reduction to buy more Arrow-Debreu securities in the highest states.The continuous geometric Brownian motion can be seen as a limiting case.

3. Relation with Markets in the Real World

One might argue that the Arrow-Debreu securities, as used in this paper, are not traded in financial markets. This is certainly true for the path dependent Arrow-Debreu securities. However, payoffs of none path dependent Arrow-Debreu securities are offered by so called digital options. These over-the-counter traded digital options, payoff one unit of currency if the underlying value ends in a certain range $[X_2, X_3]$. Since, these digital options are only traded over the counter they might be quite expensive compared with their theoretical no-arbitrage price. However, the payoffs of these options can be approximated by combinations of standard options. For example, for exercise prices $X_1 < X_2 < X_3 < X_4$ with $X_2 - X_1 = X_4 - X_3 = 1/n$, one can buy n calls with exercise prices X_1 and n calls with exercise price X_4 and short the same number of call options with exercise prices X_2 and X_3.

Furthermore, our approach sheds light on the shape of optimal portfolios if one is only allowed to buy and sell standard options with a limited number of different exercise prices. The optimal portfolios have a strong tendency to have long positions in out of the money calls and short positions in out of the money puts. Especially, the tendency for out-of-the-money calls is also described in Oldenkamp [6].

In constructing the optimal portfolio we used theoretical Arrow-Debreu prices, which is equivalent to using theoretical option prices. One might wonder how the optimal portfolio will look like if one uses market prices, since it is well known that market prices deviate from theoretical prices. A way to measure these deviations is to consider implied volatilities, based on market prices, where options with high implied volatilities are relatively expensive compared with the theoretical prices and those with low implied volatilities are cheap. In equity derivatives markets one usually observes a "smirk" pattern, which means that out-of-the money calls have a low implied volatility and out-of-the money puts have a high implied volatility. This means that the Arrow-Debreu securities for the high final stock prices are priced cheaply in the market compared with the theoretical values and for low final stock prices, the Arrow-Debreu securities are expensive. Hence, based on market prices there will be an even stronger impetus to go for the gambling portfolios.

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