Simple Coisotropic Projections and Caustics

Vladimir Zakalyukin

Abstract. A generalization of the Arnold's simple Lagrangian singularities is presented. Consider a fibration of a symplectic space by coisotropic fibers. A generic Lagrangian submanifold can meet certain fibers non-transversally. We call the set of corresponding points from the base of the fibration —the coisotropic caustic. It turn out that the classification of simple local singularities of coisotropic caustics is related to all Coxeter crystallographic groups generated by reflections A, B, C, D, E, F (except G_2).

Similar answers arise in the parallel theory of Legendre singularities in contact spaces.

1. Introduction

Various applications of singularity theory in geometry and physics involve symplectic geometry and in particular Arnold's construction [1] of Lagrangian projections.

A bundle with a symplectic total space and Lagrangian fibers is called a Lagrangian bundle. The main examples are: the cotangent bundle and the foliation by common level sets of integrals of a completely integrable Hamiltonian system. The restriction to an immersed Lagrangian submanifold L of the bundle projection is called a Lagrangian mapping. Its critical value locus (in other words, the base point set of nontransversal intersections of L with the corresponding fibers) is called caustic.

The simple stable classes of caustic germs are related to A, D, E simple Lie groups.

Here we describe a rather natural generalization of this construction corresponding to the case of noncompletely integrable Hamiltonian systems.

Consider a collection of independent functions $h_1, \ldots, h_k \quad k \leq n$ defined on a symplectic space (M^{2n}, ω), which are pairwise in involution. Their common level sets C_c^{2n-k}, $c \in \mathbf{R}^k$ are coisotropic (at each point the tangent space to C_c contains its symplectic orthogonal) and foliate M. Each fiber is itself foliated by characteristic (isotropic submanifolds of dimension k being the integral submanifolds of the distributions spanned by h_i Hamiltonian vector fields). The space of these characteristics is a symplectic (reduced) space of dimension $2(n-k)$.

The Lagrangian submanifold $L \subset M^{2n}$ can meet certain fiber C_c nontransversally. The isotropic variety $L \cap C_c$ in this case projects to a Lagrangian subvariety (which in general is singular) in the corresponding space of characteristics. The set of such values of $c \in \mathbf{R}^k$ is called a *coisotropic caustic* of L.

Our result is the classification of simple (having no continuous invariants) stable coisotropic caustics. The underlying equivalence relation is provided by the group of symplectomorphisms of the ambient space, which preserve the coisotropic fibration.

The answer was unexpected. Simple stable coisotropic caustics occur to be diffeomorphic (with one exception) to irregular orbit hypersurfaces of Weil groups of A, B, C, D, E, F types.

The same list of normal forms remains for simple stable (with respect to the group of contactomorphisms) projections of Legendre submanifolds along coisotropic fibration of a contact space.

The proofs (see [6]) are based on the classification of singularities of the contact of Lagrangian submanifold with only one coisotropic subspace [5].

We determine also the range of nice dimensions n and k, for which simple stable coisotropic singularities are dense. In particular, if $k = 1$ then generic Morse nontransversal intersections of Lagrangian submanifold with the level hypersurfaces of regular Hamiltonian function are stable.

All constructions are local, and the initial objects are supposed to be C^∞-smooth.

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2. Definitions

Let $h\colon \mathbf{R}^{2n} \to \mathbf{R}^k$, $k \le n$ be a germ (at the origin) of a fibration with coisotropic fibers C_c, $c \in \mathbf{R}^k$ of the standard symplectic space $(\mathbf{R}^{2n}, \omega)$. The Darboux theorem implies that any two such fibrations (of equal dimensions) are symplectomorphic.

Consider a pair (L^n, h), where L is a germ of Lagrangian submanifold of $(\mathbf{R}^{2n}, \omega)$. Two pairs will be called *equivalent*, if one of them can be transformed into the other by some local symplectomorphism of $\mathbf{R}^{2n}$.

A pair is called *stable* if its orbit of the (pseudo) group of equivalencies is an open subset (in the space of pairs equipped with the appropriate topology).

A pair is called *simple*, if germs of L and h have such representatives that the germs of their small deformations at any nearby point define a pair, which is equivalent to one of the finite list of normal forms.

Denote by h_0 a distinguished coordinate coisotropic fibration defined in Darboux coordinates $\mathbf{R}^{2n} = \{(x, y, u, v)\}$, $\omega = dy \wedge dx + dv \wedge du$, $x = (x_1, \dots, x_k) \in \mathbf{R}^k$, $y = (y_1, \dots, y_k) \in \mathbf{R}^k$, $u = (u_1, \dots, u_{n-k}) \in \mathbf{R}^{n-k}$, $v = (v_1, \dots, v_{n-k}) \in \mathbf{R}^{n-k}$ by the projection $h_0\colon (x, y, u, v) \mapsto y$.

The associated isotropic characteristics are the subspaces parallel to x coordinate subspace and the reduction mapping is the projection $\rho_c\colon C_c \to \mathbf{R}^{2(n-k)}$, $\rho_c\colon (x, c, u, v) \mapsto (u, v)$.

Denote by Γ_k the (pseudo)group of symplectomorphism germs at the origin of the standard symplectic space $\mathbf{R}^{2n}$, which commute with the projection h_0 (map fibers to fibers). It contains the germs of the products of symplectomorphisms of $(x, y), dy \wedge dx$ space, preserving Lagrangian fibration $(y, x) \mapsto y$, and symplectomorphisms of $(u, v), dv \wedge du$ space.

If k is odd the group is not connected. In this case we denote by Γ_k^+ its connected component of the identity, whose elements preserve the orientation of the fiber $C_0 = \{y = 0\}$ The group Γ_k (for odd k) is isomorphic to the semidirect product of Γ_k^+ and of $\mathbf{Z}_2$, generated by reflection

$$I\colon (x_1, \dots , x_k, y_1, \dots , y_k, u, v) \mapsto (-x_1, \dots , x_k, -y_1, \dots , y_k, u, v)\,,$$

which changes the orientation of C_0.

Let r be the rank of the projection h_0 restricted to the tangent space T_0L at the origin of the Lagrangian submanifold L.

There exists a Lagrange coordinate subspace L_* transversal to T_0L having r-dimensional intersection with x isotropic coordinate subspace. A permutations of subset $(1, \dots , k)$ of indices and a symplectic permutation of (u, v) coordinates space induce a symplectomorphism from Γ_k^+. Hence, without loss of genericity, suppose that this is the subspace $L_* = \{x_{r+1} = \cdots = x_k = y_1 = \cdots = y_r = 0, u = 0\}$. Then the Lagrangian germ L is defined by a (generating) function S in the variables $x_{r+1}, \dots , x_k, y_1 \dots , y_r,\ u$ by standard formulas

$$L = \left\{ (x, y, u, v) \,\middle|\, y_j = \frac{\partial S}{\partial x_j},\ j = r+1, \dots , k;\ x_i = -\frac{\partial S}{\partial y_i} \quad j = 1, \dots , r;\ v = \frac{\partial S}{\partial u} \right\}.$$

Such a Lagrangian germ, its generating function and its variables will be called (r, k)-*adapted.*

If $r = k$ (that is L^n is transversal to C_0 at the origin), then the intersection $L \cap C_0$ is transversal to characteristic fibers of C and ρ_c is nonsingular.

All such transversal pairs are equivalent to each other. Really, in this case function S depends only on y and u and the symplectomorphism

$$(x, y, u, v) \mapsto \left(x + \frac{\partial S}{\partial y},\ y,\ v - \frac{\partial S}{\partial u},\ u \right)$$

belongs to Γ_k^+ and maps L to the coordinate Lagrangian submanifold L_* defined by the zero generating function.

Denote by G_k the subgroup of symplectomorphisms, which preserve only one distinguished coisotropic fiber C_0.

3. Classification Theorems

If $k = n$ we are in classical Lagrangian projection setting without any v, u coordinates. The simple stable pairs are classified by generating functions S in x and y only, which are versal (with respect to the R^+ group of right diffeomorphisms and additions with constants) deformations with y parameters of functions in x having A, D, E simple singularities.

Suppose now $k < n$.

Theorem 3.1. **i.** *Any simple stable nontransversal pair has corank $k-r$ equal to 1.*
ii. *Any simple stable nontransversal germ (L^n, h) is equivalent to a pair germ (L_S, h_0), where Lagrangian submanifold L_S is defined in $(k-1,k)$-adapted Darboux coordinates by the generating function (with $k-r=1$) $S(y_1, \dots, y_{k-1}, t, u)$ (here t stands for x_k) of one of the following forms:*

(In the following $Q(u_1, \dots, u_k)$ denotes a nondegenerate quadratic form $Q = \pm u_1^2 \pm \cdots \pm u_k^2$.)

(1) *$S = t^3 + tf(y,u)$, where $f(y,u)$ is a restricted versal deformation with parameters y of one of the simple singularities of functions in u:*

$$A_m: \quad k \geq m \geq 1 \quad f(u) = \pm u_1^{m+1} + Q(u_2, \dots, u_{n-k}) + y_1 u_1 + \cdots + y_{m-1} u_1^{m-1}$$

(for even m the singularities with $\pm$ signs are equivalent);

$$D_m: \quad k \geq m \geq 4 \quad f(y,u) = u_1^2 u_2 \pm u_2^{m-1} + Q(u_3, \dots, u_{n-k}) + y_1 u_2^1 + \cdots + y_{m-2} u_2^{m-2} + y_{m-1} u_1$$

(for odd m the singularities with $\pm$ signs are equivalent);

$$E_6: \quad k \geq 5 \quad f(y,q) = u_1^3 \pm u_2^4 + Q(u_3, \dots, u_{n-k}) + y_1 u_1 + y_2 u_2 + y_3 u_1 u_2 + y_4 u_2^2 + y_5 u_1 u_2^2\,;$$

$$E_7: \quad k \geq 6 \quad f(q) = u_1^3 + u_1 u_2^3 + Q(u_3, \dots, u_{n-k}) + y_1 u_1 + y_2 u_2 + y_3 u_1 u_2 + y_4 u_2^2 + y_5 u_1^2 + y_6 u_1^2 u_2\,;$$

$$E_8: \quad k \geq 7 \quad f(q) = u_1^3 + u_2^5 + Q(u_3, \dots, u_{n-k}) + y_1 u_1 + y_2 u_2 + y_3 u_1 u_2 + y_4 u_2^2 + y_5 u_1 u_2^2 + y_6 u_2^3 + y_7 u_2^3 u_1\,.$$

(2**)** *Classes, corresponding to boundary singularities*

$$C_m: \quad k \geq m \geq 2 \quad S = t^{2m+1} + y_{m-1} t^{2m-1} + \cdots + y_1 t^3 + t^2 u_1 + tQ(u_2, \dots, u_{n-k})\,;$$

$$B_m: \quad k \geq m \geq 2 \quad S = t^3 u_2 + t^2 u_1 + t(\pm u_2^m + Q(u_3, \dots, u_{n-k}) + y_1 u_2^1 + \cdots + y_{m-1} u_2^{m-1})\,;$$

$$F_4: \quad k \geq 4 \quad S = t^5 + y_3 t^3 + t^2 u_1 + t(u_2^3 + Q(u_3, \dots, u_{n-k}) + y_2 u_2^2 + y_1 u_2)\,.$$

(3**)** *Exceptional class*

$$U_{n-k+2}: \quad k \geq n-k+2 \quad S = \pm t^4 + t^2(y_1 u_1 + \cdots + y_{n-k} u_{n-k} + y_{n-k+1}) + tQ(u_1, \dots, u_{n-1})\,.$$

Remark 3.2. *Classes C_k, F_4 have an alternative equivalent form (when $n-k > 2$), $S = t^3 u_2 + t^2 u_1 + tf$, where f is a versal deformation (with parameters y) of the* *corresponding simple boundary singularity of functions in $u_2, \dots, u_{n-k}$ with the hypersurface $u_2 = 0$ as the boundary. In particular, B_2 is equivalent to C_2.*

Remark 3.3. *The adjacency table of these simple classes differs from that of boundary singularities by the term $U_{n-k+2} \to B_2$ only.*

Generically the coisotropic caustic of a pair consists of two components: one (Σ_s) is formed by those values of $y = (y_1, \dots, y_k)$, for which the corresponding reduced Lagrangian variety $\rho(L \cap C_y)$ has singular points, and the other (Σ_i) —is formed by those values of y, for which the corresponding reduced Lagrangian variety $\rho(L \cap C_y)$ has nontransversal intersections of smooth branches. The straightforward calculations imply the following

Proposition 3.4. *The hypersurface (Σ_s) in y space for A, D, E classes from theorem 3.1 is defined by the equations $f(u, y) = y_k, \frac{\partial f}{\partial u} = 0$, for the corresponding family $f = f(u, y_1, \dots, y_{k-1})$. Thus it is the irregular orbit hypersurface in the space of orbits of one of the A, D, E Coxeter finite groups, generated by reflections.*

The hypersurface (Σ_i) for these classes is a cylinder (with the line generator parallel to y_k axis) over the ordinary A, D, E caustic in $y_1, \dots, y_{k-1}$ space, defined by equations

$$\frac{\partial f}{\partial u} = 0, \quad \det\left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right) = 0\,.$$

Remark 3.5. *Using methods of* [4, 3] *one can verify that the Lagrangian projections defined by the fibration $(u, v) \mapsto u$ of reduced singular Lagrangian varieties $\rho(L \cap C_y)$ for A, D, E classes of theorem 3.1 are stable in the sense of* [3] *(with respect to perturbations of simplectic structure and Lagrangian projections) for any y.*

Proposition 3.6. *The coisotropic caustics of normal forms B, C, F are diffeomorphic to the bifurcation sets of zeros (alternatively called wave fronts) of the corresponding boundary singularities, that is to the irregular orbit hypersurface Σ of the corresponding reflection group.*

Amazingly, one irreducible component of Σ coincide with Σ_s and the other —with Σ_i.

For example, the curve Σ_s of the normal form $C_2 : S = t^5 + y_1 t^3 + t^2 u_1$ (here we put $k = 2, n = 4$) is the set of parameters y_1, y_2 where zero is a multiple root of the polynomial $P(t) = 5t^4 + 3y_1 t^2 + 2tu_1 - u_2$, while the curve Σ_i is determined by the condition that $P(t)$ has an arbitrary multiple root provided that $u_1 = 0$.

Thus the coisotropic caustic in the (y_1, y_2)-plane is the union of a line and a half of a parabola tangent to this line (that is the C_2 bifurcation diagram).

Similar answers arise in the parallel problem of classifying the simple stable pairs of Legendre submanifold germs and coisotropic contact fibrations with respect to the group of contactomorphisms of contact space.

Let (K^{2n+1}, α) be a contact space. A submanifold $M \subset K$ is called *coisotropic* if at each point $m \in M$ the subspace $T_m M$ is transversal to the contact distribution

$A_m = \{v \in T_mK : \alpha(v) = 0\}$ and subspace $ST_mM \subset A_m$ of vectors skew-orthogonal to $A_m \cap T_mM$ with respect to non-degenerate two-form $\beta = d\alpha\,|_A$ (β is defined up to a multiplication by a non-zero factor) belong to $A_m \cap T_mM$ (therefore ST_mM determines an integrable distribution on M).

A fibration $\rho\colon K \to E$ with isotropic fibers is called *isotropic*. Locally all isotropic fibrations are contactomorphic to standard one $\rho\colon (z, x, y, u, v,) \mapsto (z, y)$, $z \in \mathbf{R}$, $\alpha = dz - ydx - udv$.

A pair consisting of a germ of Legendre submanifold $\mathcal{L} \in K$ and a coisotropic fibration is contactomorphic to a pair of the standard fibration and a Legendre germ $\mathcal{L} = \{z, (x, y, u, v)|z = S, (x, y, u, v) \in L\}$ determined in appropriate adapted coordinates by a generating function S of the associated Lagrangian submanifold $L \subset \mathbf{R}^{2n}$.

The contact counterparts of definitions of simple stable pairs are straightforward.

Theorem 3.7. *Non-transversal contact pair $\mathcal{L}, \rho$ is simple and stable if it is contactomorphic to a pair determined by one of the generating functions from theorem 3.1.*

The following theorem characterizes the range of nice dimensions (n, k), for which any generic pair of Lagrangian submanifold and collection of functions in involution has stable and simple singularities.

However, involutive collections of functions $h_1, \dots, h_k$ on $\mathbf{R}^{2n}$ form a subset with very complicated singularities in the space of collections of arbitrary functions, if at some point the rank of the mapping h is less than $k - 1$. In this case it is not clear what does "generic" mean. To avoid this difficulty consider only collections (called *1-generic*) without such points.

Theorem 3.8. *For open and dense subset in the space (equipped with the fine Whitney topology) of pairs formed by a proper Lagrangian submanifold in $\mathbf{R}^2n$ and a 1-generic coisotropic mapping h the germs of pairs (L, h) are stable at each point if and only if $n = k$ and $k < 6$, or $n > k$ and $k < 4$ except for the pair $(k, n) = (3, 4)$.*

Corollary 3.9. *In the range of nice dimensions $k < n$ only the following classes generically appear: for $k = 1 : A_1$ (at isolated points); for $k = 2 : A_1$ (on curves), $A_2, B_2 \approx C_2$ (at isolated points) and for $k = 3, n > 4 : A_1$ (on surfaces), $A_2, B_2 \approx C_2$ (on curves), B_3, C_3 (at isolated points).*

4. Infinitesimal Stability

Time-depending family of local symplectomorphisms from Γ_k are phase flows of time depending Hamilton vector fields, whose $-\frac{\partial H}{\partial x}\frac{\partial}{\partial y}$ components, may depend only on y. Similarly for G_k-families these components vanish on C_0. This observation proves the following

Lemma 4.1. i. *Infinitesimal transformations from Γ_k^+ are defined by Hamiltonian vector fields with Hamiltonian functions H of the form*

$$H = \sum_{i=1}^{k} x_i H_i(y) + H_0(y,u,v)$$

with certain smooth functions H_i and H_0.

In particular, this lemma implies, that Γ_k-symplectomorphisms preserve a well-defined affine structure on isotropic characteristic fibers.

The image of an (r,k)-adapted Lagrangian germ under a symplectomorphism φ from a neighbourhood of the identity in Γ_k^+ or in G_k^+ remains transversal to L_*. Thus φ acts on the adapted generating families.

Let for (r,k)-adapted coordinates $p = (p_1,\dots,p_{k-r}) = (y_{r+1},\dots,y_k)$, $q = (q_1,\dots,q_{k-r}) = (x_{r+1},\dots,x_k)$, $w = (w_1,\dots,w_r) = (y_1,\dots,y_r)$ and $z = (z_1,\dots z_r) = (x_1,\dots,x_r)$.

Lemma 4.2. i. *The tangent space $T_S\Gamma$ to the Γ_k^+-orbit of generating function S consists of all functions of the form*

$$\tilde{S} = \sum_{i=1}^{k-r} q_i H_i \left(\frac{\partial S}{\partial q}, w\right) + \sum_{j=1}^{r} \frac{\partial S}{\partial w} H_j \left(\frac{\partial S}{\partial q}, w\right) + H_0 \left(\frac{\partial S}{\partial q}, w, u, \frac{\partial S}{\partial u}\right),$$

with smooth functions H_i, H_j and H_0.

ii. *The tangent space $T_S G$ to the G_k^+-orbit of generating function S consists of all functions of the form*

$$\tilde{S} = \sum_{i=1}^{k-r} \frac{\partial S}{\partial q_i} H_i(q,w,u) + \sum_{j=1}^{r} w_j H_j(q,w,u) + H_0\left(\frac{\partial S}{\partial u}, u\right),$$

with smooth functions H_i, H_j and H_0.

Remark 4.3. *In the contact case the tangent space to the orbit of the group of contactomorphisms is determined by the set of contact Hamiltonians. The counterparts of the formulas from this lemma are similar: only each summand can contain the functions S itself as an extra argument.*

The (r,k)-adapted Lagrangian germ $L(S)$ with generating function germ $S(q,w,u)$ is called *infinitesimally stable* if it tangent space $T_S\Gamma$ coincides with the total space $C^\infty(q,w,u)$ of germs of smooth functions in q, w and u. The infinitesimally stable germ is stable.

Lemma 4.4. *Germ S is infinitesimally stable if and only if any function $\tilde{S}$ in x, y, u variables has a decomposition*

$$\tilde{S} = \sum_{i=1}^{k-r} \frac{\partial S}{\partial q_i} H_i(q,y,u) + \sum_{j=1}^{k} \frac{\partial S}{\partial y_j} H_j(y,u) + H_0\left(y, \frac{\partial S}{\partial u}, u\right).$$

Similarly to the classical result of J.Mather on right-left equivalence of smooth mappings, the classes, which are stable with respect to the group Γ_k, are deformations transversal to the orbits of the bigger group G_k.

Lemma 4.5. *If classes of* $\frac{\partial(S(q,w,u)-q_1p_1-\cdots-q_{k-r}p_{k-r})}{\partial y}\Big|_{y=o}$ *span* $\mathcal{O}$*, then the germ* S *is infinitesimally stable (and therefore stable).*

Remark 4.6. *The transversality to the* G_k *orbit persists under the transformations from* Γ_k*. Thus any stable adapted germ* S *of* (r,k)*-finite type is infinitesimally stable. Otherwise it would be not equivalent to nearby germs which are transversal to the corresponding* G_k*-orbit.*

Since Γ_k-simple germ ought to be G_k simple it rests to classify simple G_k orbits (see[5, 6]).

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Moscow Aviation Institute,
125871, Moscow, Russia
E-mail address: `vladimir@zakal.mccme.ru`